**MA 138**
**Calculus II with Life Science Applications**
SECOND MIDTERM

**Spring 2023**
03/07/2023

**Name:** Answer Key

**Sect. #:** ______

Do not remove this answer page — you will return the whole exam. No books or notes may be used. Use the backs of the question papers for scratch paper. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The **first part of the exam** consists of 10 multiple choice questions, each worth 5 points. Record your answers on this page by filling in the box corresponding to the correct answer. For example, if (a) is correct, you must write

■ b c d e

Do not circle answers on this page, but please do circle the letter of each correct response in the body of the exam. It is your responsibility to make it CLEAR which response has been chosen. You will not get credit unless the correct answer has been marked on **both** this page **and** in the body of the exam.

The **second part of the exam** consists of five open-response questions and one bonus question. When answering these questions, check your answers when possible. Clearly indicate your answer and the reasoning used to arrive at that answer. *Unsupported answers may receive NO credit.*

**1.** a b ■ d e

**2.** a b ■ d e

**3.** a ■ c d e

**4.** ■ b c d e

**5.** a b c ■ e

**6.** a b c ■ e

**7.** a b c d ■

**8.** ■ b c ■ e

**9.** a ■ c d e

**10.** a b c ■ e

**GOOD LUCK!**

| QUESTION | SCORE | OUT OF |
|---|---|---|
| **Multiple Choice** | | 50 pts |
| **11.** | | 10 pts |
| **12.** | | 10 pts |
| **13.** | | 10 pts |
| **14.** | | 10 pts |
| **15.** | | 10 pts |
| **Bonus.** | | 10 pts |
| **TOTAL** | | 100 pts |

Please make sure to list the correct section number on the front page of your exam. In case you forgot your section number, consult the following table:

| Sections # | Lecturer | Time/Location |
| --- | --- | --- |
| **001–004** | Alberto Corso | MWF 10:00 am - 10:50 am, CB 110 |
| **Section #** | **Recitation Instructor** | **Time/Location** |
| **001** | Kathryn Hechtel | TR 08:00 am - 08:50 am, CB 307 |
| **002** | Kathryn Hechtel | TR 09:00 am - 09:50 am, CB 307 |
| **003** | Davis Deaton | TR 10:00 am - 10:50 am, CB 307 |
| **004** | Davis Deaton | TR 11:00 am - 11:50 am, CB 307 |

**1.** Which of the following differential equations is separable?

$\frac{dy}{dx} = \frac{2y}{x^2+1}$ is the only separable differential equation

$$\boxed{\frac{1}{y}\,dy = \frac{2\,dx}{x^2+1}}$$

then integrate to get the solution

**Possibilities:**

(a) $\frac{dy}{dx} = \frac{2x+xy^2}{x+x^2y}$

(b) $\frac{dy}{dx} = \frac{x+y}{2y}$

(c) $\boxed{\frac{dy}{dx} = \frac{2y}{x^2+1}}$

(d) $\frac{dy}{dx} = \frac{y^2}{2x-3y}$

(e) None of the above

**2.** The solution of the initial value problem

$$\frac{dy}{dx} = 3x^2y^2 \qquad \text{with} \qquad y(1) = 1$$

is

$\int \frac{1}{y^2}\,dy = \int 3x^2\,dx$ after separating the variables

$-\frac{1}{y} = x^3 + C$ using $y(1) = 1$ we obtain $-1 = 1^3 + C$ so $\boxed{C = -2}$

Thus $-\frac{1}{y} = x^3 - 2$ or

$\frac{1}{y} = 2 - x^3$ or $\boxed{y = \frac{1}{2-x^3}}$

**Possibilities:**

(a) $y = \frac{1}{2-x}$

(b) $y = \frac{1}{2-x^2}$

(c) $\boxed{y = \frac{1}{2-x^3}}$

(d) $y = \frac{1}{2-x^4}$

(e) $y = \frac{1}{2-x^5}$

3. The equation $y^2 = cx$ is the general solution of

- One could solve the 4 D.E. and compare the answers
- One could solve for $y$, $y = \sqrt{cx}$ and do the derivative and verify which D.E. is satisfied
- Easier, use the chain rule and implicit differentiation on $\boxed{y^2 = cx}$

(a) $\frac{dy}{dx} = \frac{2y}{x}$

→ $\boxed{\text{(b) } \frac{dy}{dx} = \frac{y}{2x}}$

(c) $\frac{dy}{dx} = \frac{2x}{y}$

(d) $\frac{dy}{dx} = \frac{x}{2y}$

(e) None of the above.

$\frac{d}{dx}(y^2) = \frac{d}{dx}(cx)$

$\therefore 2y\frac{dy}{dx} = c$ so $\frac{dy}{dx} = \frac{c}{2y}$

But $c = y^2/x$ so when you substitute $\frac{dy}{dx} = \frac{y^2/x}{2y}$

we get $\boxed{\frac{dy}{dx} = \frac{y}{2x}}$

4. Two different strains of bacteria sometimes feed on chemicals excreted by one another, i.e. strain $A$ feeds on chemical produced by strain $B$ and vice versa. This phenomenon is referred to as *cross-feeding*. Suppose that the amount of strain $A$ bacteria is governed by

$$\frac{dP}{dt} = P(1-P)(3P-6). = g(P)$$

Using the analytic approach we see that the stable equilibrium point(s) is (are):

$\hat{P}_1 = 0$, $\hat{P}_2 = 1$, $\hat{P}_3 = 2$

now $g(P) = P(1-P)(3P-6) = (P-P^2)(3P-6)$

$= -3P^3 + 9P^2 - 6P$

$g'(P) = -9P^2 + 18P - 6$

$g'(0) = -6 < 0$ $\quad g'(1) = 3$

$g'(2) = -6 < 0$

**Possibilities:**

→ $\boxed{\text{(a) } \widehat{P}_1 = 0 \text{ and } \widehat{P}_2 = 2}$

(b) $\widehat{P} = 1$

(c) $\widehat{P} = 0$

(d) $\widehat{P} = 3$

(e) $\widehat{P}_1 = 0$ and $\widehat{P}_2 = 6$

**5.** Consider the following differential equations labeled **A.** through **D.**

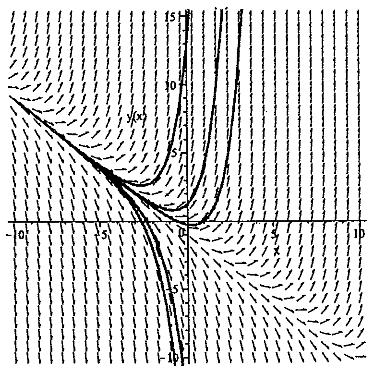
**A.** $y' = 2 - y$ $\longleftrightarrow$ III

**B.** $y' = x + y$ $\longleftrightarrow$ I

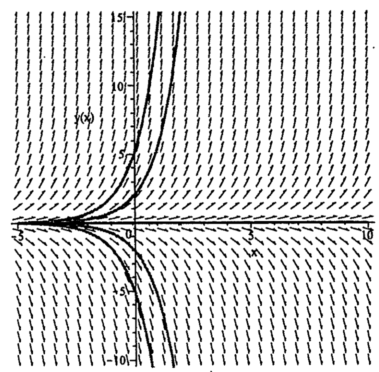
**C.** $y' = y$ $\longleftrightarrow$ II

**D.** $y' = \dfrac{10}{x}$ $\longleftrightarrow$ IV

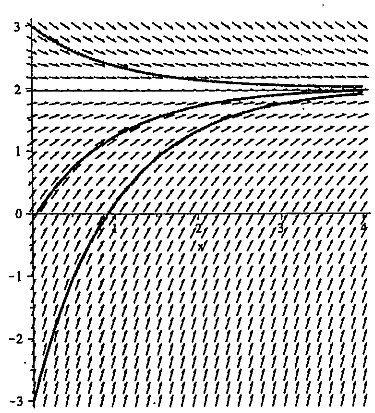
and consider the following slope (or direction) fields labeled I. through IV.



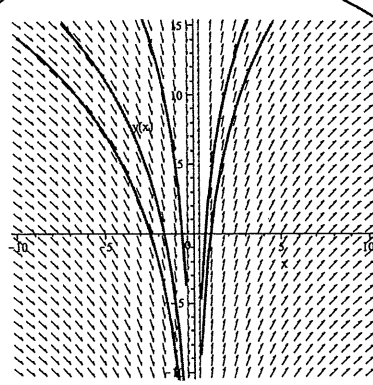
slope field I.



slope field II.



slope field III.



slope field IV.

note that $y' = \frac{10}{x}$ corresponds to $y = 10\ln|x| + c$

Which slope field corresponds to differential equation **D.**?

(a) slope field I.

(b) slope field II.

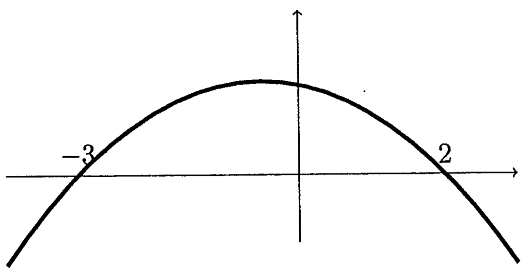
(c) slope field III.

$\boxed{\text{(d) slope field IV.}}$

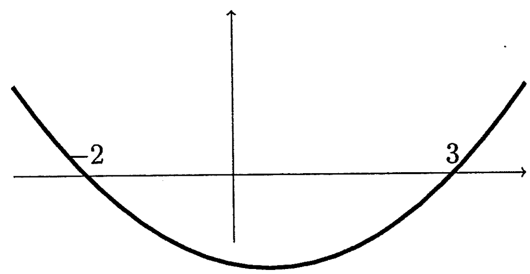
(e) None of the above

**6.** A phase line for an autonomous differential equation $\dfrac{dy}{dx} = f(y)$ is shown below

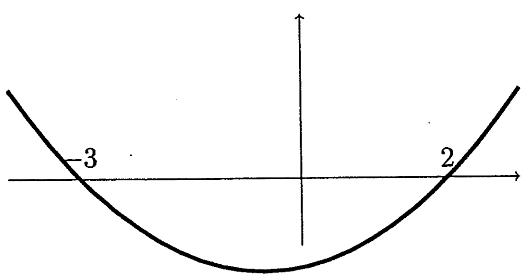

Which of the following graphs most closely matches the graph corresponding to the differential equation? **Note:** $y$ and $f(y)$ are plotted on the horizontal and vertical axes, respectively



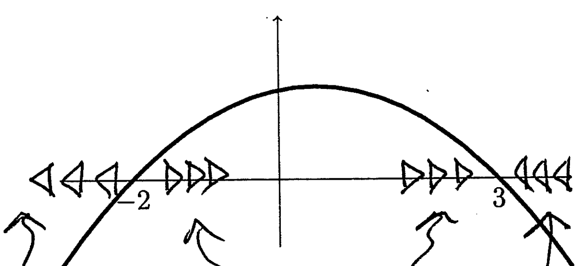

**Graph A**




**Graph B**




**Graph C**




**Graph D**

as $\dfrac{dy}{dx} < 0$ we have that $y = y(x)$ is decreasing

as $\dfrac{dy}{dx} > 0$ we have that $y = y(x)$ is increas

**Possibilities:**

(a) Graph A

(b) Graph B

(c) Graph C

$\boxed{\text{(d) Graph D}}$

(e) None of the above

**7.** The augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

can be put into the row-reduced form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

using just two row operations. Which row operations will accomplish this?
(**Note:** $R_i$ denotes the entries on the $i$-th row)

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 7 \end{array}\right] \rightsquigarrow \begin{array}{c}\boxed{R_2-R_3}\\ \text{produces}\end{array} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 7 \end{array}\right] \rightsquigarrow \begin{array}{c}\boxed{R_1+2R_2}\\ \text{produces}\end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

**Possibilities:**

(a) First replace $R_2$ with $R_2 - R_3$, then replace $R_1$ with $R_1 - 2R_2$

(b) First replace $R_1$ with $R_1 + 2R_2$, then replace $R_2$ with $R_2 - R_3$

(c) First replace $R_3$ with $\frac{1}{7}R_3$, then replace $R_2$ with $\frac{1}{3}R_2$

(d) First replace $R_2$ with $R_3 - R_2$, then replace $R_1$ with $R_2 + 2R_1$

(e) First replace $R_2$ with $R_2 - R_3$, then replace $R_1$ with $R_1 + 2R_2$

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**8.** Suppose the augmented matrix of a linear system is row-equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & k & h \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Which of the following must be true? (**Note:** More than one answer may be correct.)

(b) is false. If $k=-2$ and $h=0$ then the system has one solution: $x=4$, $y=7$, $z=0$

**Possibilities:**

(a) If $k = 0$ and $h = -2$, the linear system has no solution. ✓

(b) If $k = -2$ and $h = 0$, the linear system has no solution.

(c) If $k = 0$ and $h = 0$, the linear system has a unique solution.

(d) If $k = 0$ and $h = 0$, the linear system has infinitely many solutions. ✓

(e) If $k = -2$ and $h = -2$, the linear system has infinitely many solutions.

(c) is false, because the system has infinitely many solutions

(e) if false because the system has one solution

**9.** Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & -1 \end{bmatrix}.$$

Then the matrix $AA^T$ is:

(Recall that the notation $A^T$ denotes the transpose of the matrix $A$. Our textbook uses the alternative notation $A'$.)

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2\cdot 2 + 1\cdot 1 + 3\cdot 3 & 2\cdot 1 + 1\cdot 0 + 3(-1) \\ 1\cdot 2 + 0\cdot 1 - 1\cdot 3 & 1\cdot 1 + 0\cdot 0 + (-1)(-1) \end{bmatrix}$$

**Possibilities:**

(a) $AA^T = \begin{bmatrix} 6 & -1 \\ -1 & 0 \end{bmatrix}$

(b) $\boxed{AA^T = \begin{bmatrix} 14 & -1 \\ -1 & 2 \end{bmatrix}}$ $= \begin{bmatrix} 14 & -1 \\ -1 & 2 \end{bmatrix}$

(c) $AA^T = \begin{bmatrix} 5 & 2 & 5 \\ 2 & 1 & 3 \\ 5 & 3 & 10 \end{bmatrix}$

(d) $AA^T = \begin{bmatrix} 3 & 3 & 5 \\ 3 & 0 & 3 \\ 5 & 3 & 8 \end{bmatrix}$

(e) None of the above

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**10.** Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

What is the fourth power of the matrix $A$, namely $A^4 = A \cdot A \cdot A \cdot A$?

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$A^4 = A^3 \cdot A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

**Possibilities:**

(a) $A^4 = \begin{bmatrix} 1^4 & 1^4 \\ 1^4 & 0 \end{bmatrix}$

(b) $A^4 = \begin{bmatrix} 4 & 4 \\ 4 & 0 \end{bmatrix}$

(c) $A^4 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

(d) $\boxed{A^4 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}}$

(e) $A^4 = \begin{bmatrix} 8 & 5 \\ 5 & 3 \end{bmatrix}$

**11.** Separate variables and use the partial fraction method to solve the first order differential equation

$$\frac{dy}{dt} = \frac{1}{4}y(y-4)$$

with the following initial condition: $y = 2$ for $t = 0$.

(**Hint:** start from $\frac{4}{y(y-4)}dy = dt$ ...)

We need a partial fraction decomposition for $\frac{4}{y(y-4)}$

We start with $\frac{4}{y(y-4)} = \frac{A}{y} + \frac{B}{y-4}$ OR

$\boxed{4 = A(y-4) + By}$. Using the cover-up method

if $y = 4$ then $4 = A \cdot 0 + B \cdot 4$ or $\boxed{B=1}$

if $y = 0$ then $4 = A(-4) + B \cdot 0$ or $\boxed{A=-1}$

Thus $\boxed{\frac{4}{y(y-4)} = -\frac{1}{y} + \frac{1}{y-4}}$

$\int \frac{1}{y-4}\,dy - \int \frac{1}{y}\,dy = \int 1 \cdot dt$ or

$\ln|y-4| - \ln|y| = t + C$

OR

$\ln\left|\frac{y-4}{y}\right| = t + C$

$y(t) =$ $\frac{4}{1+e^t}$

Find $\lim_{t\to\infty} y(t) =$ 0

pts: /10

From $\ln\left|\frac{y-4}{y}\right| = t + C$

we obtain $e^{\ln\left|\frac{y-4}{y}\right|} = e^{t+C}$

OR $\left|\frac{y-4}{y}\right| = e^{C} \cdot e^{t}$ OR

$\frac{y-4}{y} = \underbrace{\pm e^{C}}_{A} e^{t}$ OR $\frac{y-4}{y} = A e^{t}$

With $y(0) = 2$ we obtain $\frac{2-4}{2} = A \cdot \underbrace{e^{0}}_{1}$

So $\boxed{A = -1}$

Finally from $\frac{y-4}{y} = -e^{t}$

OR $1 - \frac{4}{y} = -e^{t}$ we get $1 + e^{t} = \frac{4}{y}$

OR $\boxed{y = \frac{4}{1+e^{t}}}$

notice $\boxed{\lim_{t\to\infty} \frac{4}{1+e^{t}} = 0}$

**12.** Consider the differential equation

$$\frac{dy}{dx} = \frac{1}{4}y(y-4)$$

that we already studied in Problem **11**.

Find all equilibria $\widehat{y}$ of the above differential equation and discuss the stability of these equilibria using the Stability Criterion ($\equiv$ analytic approach).

In problem #11 we discovered after a lengthy calculation that

$$\lim_{t \to \infty} y(t) = 0$$

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The equilibria of our D.E. are

$$\hat{y}_1 = 0 \quad \text{and} \quad \hat{y}_2 = 4$$

Now $g(y) = \frac{1}{4}y(y-4) = \frac{1}{4}y^2 - y$

So $g'(y) = \frac{1}{2}y - 1$

and $\boxed{g'(0) = -1}$ while $g'(4) = 1$

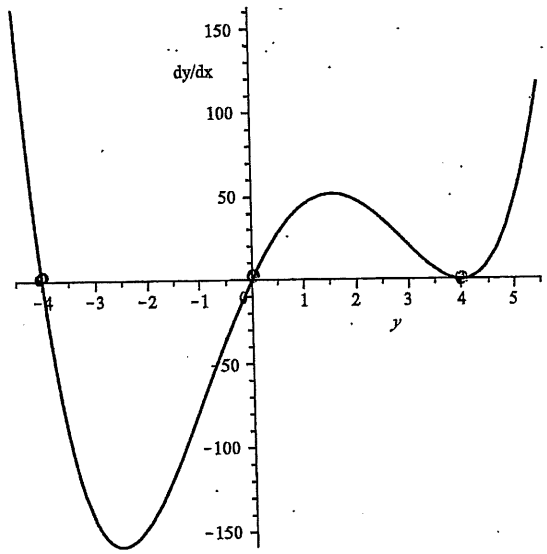
By the stability criterion $\boxed{\hat{y}_1 = 0}$ is locally stable; $\hat{y}_2 = 4$ is unstable

pts: /10

**13.** Consider the differential equation

$$\frac{dy}{dx} = f(y) .$$

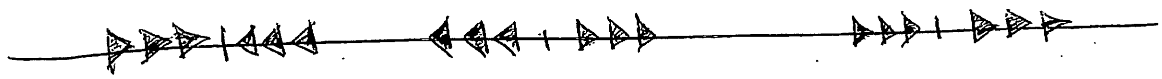
where the graph of $f(y)$ is given below




Find all equilibria $\widehat{y}$ of the above differential equation and discuss the stability of these equilibria using the graphical approach.

The equilibria of the differential equation are $\hat{y} = -4, 0, 4$

Given the graph of $f(y)$, the phase line around those points is



| -4 | 0 | 4 |
|---|---|---|
| locally stable | unstable | stable from left and unstable from right |

pts: /10

**14.** Solve the following system of linear equations

$$\begin{cases} x + 4y + 3z = 8 \\ x + 2y - z = 2, \\ 3x + 8y + z = 12 \end{cases}$$

by writing the corresponding augmented matrix and then by row reducing.

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 8 \\ 1 & 2 & -1 & 2 \\ 3 & 8 & 1 & 12 \end{array}\right] \rightsquigarrow \begin{array}{c} \\ R_2 - R_1 \\ R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 8 \\ 0 & -2 & -4 & -6 \\ 0 & -4 & -8 & -12 \end{array}\right]$$

$$\begin{array}{c} \\ -\frac{1}{2}R_2 \\ -\frac{1}{4}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 8 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{array}\right] \rightsquigarrow \begin{array}{c} \\ \\ R_3 - R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 - 4R_2 \left[\begin{array}{ccc|c} 1 & 0 & -5 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

ie. the system reads

$$\begin{cases} x \quad -5z = -4 \\ y + 2z = 3 \end{cases}$$

infinite #

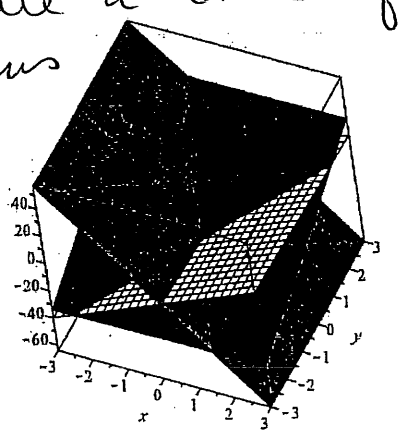
How many solutions does the system have?

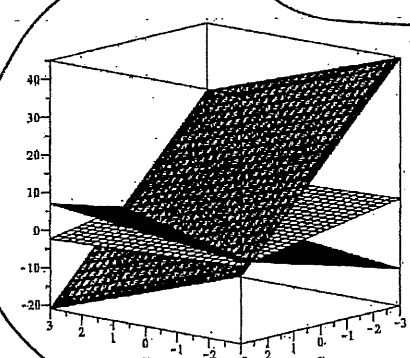
$z$ = can be anything so $x = -4 + 5t$ $\quad y = 3 - 2t$

$z = t$ any real

Which of the two pictures below illustrates the geometric situation described by the given system of linear equations?

there are a line of solutions




pts: /10

**15.** Find the values of $a$ and $b$ that satisfy the following matrix equation:

$$\begin{bmatrix} 1 & 2a \\ 1 & 2 \end{bmatrix} \cdot \left( \begin{bmatrix} 0 & 2 \\ 6-a & 1 \end{bmatrix}^T \right) = \frac{1}{2} \begin{bmatrix} -40 & 2 \\ 8 & b \end{bmatrix}.$$

(Recall that the notation $A^T$ denotes the transpose of the matrix $A$. Our textbook uses the alternative notation $A'$.)

$$\underbrace{\begin{bmatrix} 1 & 2a \\ 1 & 2 \end{bmatrix} \cdot \begin{pmatrix} 0 & 6-a \\ 2 & 1 \end{pmatrix}} = \frac{1}{2} \begin{bmatrix} -40 & 2 \\ 8 & b \end{bmatrix}$$

$$\begin{bmatrix} 4a & 6+a \\ 4 & 8-a \end{bmatrix} = \begin{bmatrix} -20 & 1 \\ 4 & \frac{b}{2} \end{bmatrix}$$

since the matrices are the same the entries must be the same

$4a = -20$ $\quad$ $6 + a = 1$

$4 = 4$ ✓ $\quad$ $8 - a = b/2$

$\boxed{a = -5}$ from the first 2 equations

and $8 - (-5) = \frac{b}{2}$ so $\boxed{b = 26}$

pts: /10

**Bonus.** Consider the system

$$\begin{cases} 8x + 3y = 13 \\ 2x + y = 4. \end{cases}$$

(*a*) Write this system in matrix form $A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, where $A$ is a $2 \times 2$ matrix and $b_1, b_2$ are appropriate numbers.

$$\underbrace{\begin{bmatrix} 8 & 3 \\ 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 4 \end{bmatrix} \qquad (*)$$

(*b*) Solve this system using the inverse of $A$. The inverse of A is

$$A^{-1} = \frac{1}{8 \cdot 1 - 2 \cdot 3} \begin{bmatrix} 1 & -3 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} 1/2 & -3/2 \\ -1 & 4 \end{bmatrix}$$

Hence, if we multiply $(*)$ by $A^{-1}$ on both sides we obtain:

$$\underbrace{\begin{bmatrix} 1/2 & -3/2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 8 & 3 \\ 2 & 1 \end{bmatrix}}_{} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/2 & -3/2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 13 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/2 \cdot 13 - 3/2 \cdot 4 \\ (-1)(13) + 4 \cdot 4 \end{bmatrix}$$

so $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3 \end{bmatrix}$

(*c*) Check your answer.

Indeed

$$\begin{cases} 8(1/2) + 3(3) = 13 \checkmark \\ 2(1/2) + 3 = 4 \checkmark \end{cases}$$

pts: /10