MA 138
Calculus II with Life Science Applications
SECOND MIDTERM

Spring 2024
03/05/2024

Name: Answer Key

Sect. #: ____________

Do not remove this answer page — you will return the whole exam. No books or notes may be used. Use the backs of the question papers for scratch paper. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The **first part of the exam** consists of 10 multiple choice questions, each worth 5 points. Record your answers on this page by filling in the box corresponding to the correct answer. For example, if (a) is correct, you must write

■a □b □c □d □e

Do not circle answers on this page, but please do circle the letter of each correct response in the body of the exam. It is your responsibility to make it CLEAR which response has been chosen. You will not get credit unless the correct answer has been marked on **both** this page **and** in the body of the exam.

The **second part of the exam** consists of five open-response questions and one bonus question. When answering these questions, check your answers when possible. Clearly indicate your answer and the reasoning used to arrive at that answer. *Unsupported answers may receive NO credit.*

1. □a ■b □c □d □e

2. □a □b ■c □d □e

3. □a □b □c ■d □e

4. □a ■b □c □d □e

5. □a □b □c ■d □e

6. ■a □b □c □d □e

7. □a □b ■c □d □e

8. ■a □b □c □d ■e

9. □a □b ■c □d □e

10. ■a □b □c □d □e

GOOD LUCK!

QUESTION	SCORE	OUT OF
Multiple Choice		50 pts
11.		10 pts
12.		10 pts
13.		10 pts
14.		10 pts
15.		10 pts
Bonus.		10 pts
TOTAL		100 pts

Please make sure to list the correct section number on the front page of your exam. In case you forgot your section number, consult the following table:

Sections #	Lecturer	Time/Location
001-004	Alberto Corso	MWF 10:00 am - 10:50 am, CB 212
Section #	**Recitation Instructor**	**Time/Location**
001	Ian Robinson	TR 09:00 am - 09:50 am, CB 307
002	Ian Robinson	TR 10:00 am - 10:50 am, CB 307

1. Suppose $y = f(x)$ is the solution to the differential equation

$$\frac{dy}{dx} = 3x + y$$

that satisfies the initial value condition $f(1) = 2$. What is the equation of the tangent line to $y = f(x)$ at the point $(1, 2)$?

the slope of the tg. line at (1,2) is

$$m = \left.\frac{dy}{dx}\right|_{(1,2)} = 3(1) + 2 = 5$$

so the equat. is

Possibilities:

(a) $y = 5x$

(b) $\boxed{y = 5x - 3}$

(c) $y = 5x - 1$

(d) $y = 2x + 1$

(e) $y = 2x - 1$

$$(y-2) = 5(x-1)$$

or

$$y = 5x \underbrace{-5+2}_{-3}$$

2. What is the general solution to the differential equation $\frac{dy}{dx} = 2xy$? Use C for the constant.

$$\frac{1}{y}\,dy = 2x\,dx$$

so

$$\int \frac{1}{y}\,dy = \int 2x\,dx \qquad \ln|y| = x^2 + A$$

or

$$|y| = e^{x^2 + A}$$

(a) $y = (xy)^2 + C$

(b) $y = C + e^{x^2}$

(c) $\boxed{y = Ce^{x^2}}$

(d) $y = Ce^x$

(e) $x = Ce^{y^2}$

$$y = \underbrace{(\pm e^A)}_{C} e^{x^2}$$

$$y = Ce^{x^2}$$

3. Which of the following differential equations are separable?

$$\textbf{I. } \frac{dy}{dx} = \frac{x+1}{y+1} \qquad \textbf{II. } \frac{dy}{dx} = xy + x \qquad \textbf{III. } \frac{dy}{dx} = e^{x+y}$$

$$\boxed{(y+1)\,dy = (x+1)\,dx}$$

$$\frac{dy}{dx} = x(y+1)$$

so

$$\boxed{\frac{1}{y+1}\,dy = x\,dx}$$

$$\frac{dy}{dx} = e^x e^y$$

so

$$\frac{1}{e^y}\,dy = e^x\,dx$$

or

$$\boxed{e^{-y}\,dy = e^x\,dx}$$

Possibilities:

(a) Only **I.**

(b) Only **I.** and **II.**

(c) Only **I.** and **III.**

(d) All three equations are separable.

(e) None of the equations are separable.

---

4. The solution of the initial value problem

$$\frac{dy}{dx} = xy^3 \quad \text{with} \quad y(0) = 2$$

is

$$\int \frac{1}{y^3}\,dy = \int x\,dx$$

$$-\frac{1}{2}\frac{1}{y^2} = \frac{1}{2}x^2 + C$$

$$\frac{1}{y^2} = -x^2 + \underbrace{A}_{-2C}$$

$$\frac{1}{4} = -0^2 + A$$

$$\therefore \quad \frac{1}{y^2} = \frac{1}{4} - x^2$$

OR

$$\frac{1}{y^2} = \frac{1-4x^2}{4}$$

$$y^2 = \frac{4}{1-4x^2}$$

Possibilities:

(a) $y = \sqrt{\dfrac{4}{1-4x}}$

(b) $y = \sqrt{\dfrac{4}{1-4x^2}}$

(c) $y = \sqrt{\dfrac{4}{1-4x^3}}$

(d) $y = \sqrt[4]{\dfrac{16}{1-16x^2}}$

(e) $y = \sqrt[4]{\dfrac{16}{1-16x^3}}$

$g(P) = P(7-8P)(1-P) = \dots = 7P - 15P^2 + 8P^3$

$g'(P) = 7 - 30P + 24P^2$

5. Two different strains of bacteria sometimes feed on chemicals excreted by one another, i.e. strain A feeds on chemical produced by strain B and vice versa. This phenomenon is referred to as *cross-feeding*. Suppose that the amount $P = P(t)$ of strain A bacteria is governed by

$$\frac{dP}{dt} = P(7-8P)(1-P).$$

Using the analytic approach we see that the stable equilibrium point(s) is (are):

$\hat{P} = 0,\ 7/8,\ 1$

Possibilities:

(a) $\widehat{P}_1 = 0$ and $\widehat{P}_2 = 7/8$

(b) $\widehat{P} = 1$

(c) $\widehat{P} = 0$

$\boxed{\text{(d)} \quad \widehat{P} = 7/8}$

(e) $\widehat{P}_1 = 0$ and $\widehat{P}_2 = 1$

$g'(0) = 7 > 0$

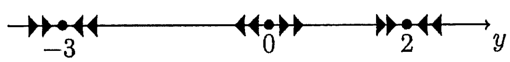
$g'(1) = 1 > 0$

BUT

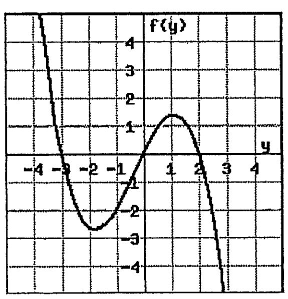
$g'\left(\frac{7}{8}\right) = -0.875$

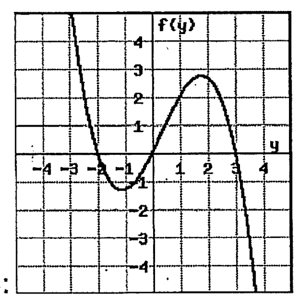
locally stable equil

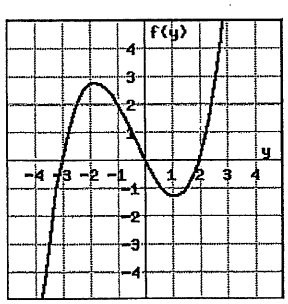
6. A phase line for an autonomous differential equation $\dfrac{dy}{dt} = f(y)$ is shown below:

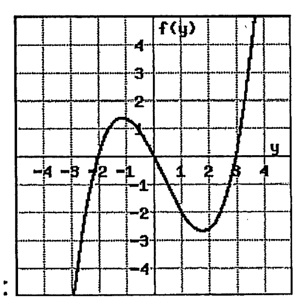


Which graph A-D most closely matches the graph corresponding to the differential equation?

A:


B:


C:


D:


Possibilities:

$\boxed{\text{(a)} \quad \text{Graph A}}$

(b) Graph B

(c) Graph C

(d) Graph D

(e) None of the above

7. The solution of the system of linear equations

$$\begin{cases} 2x + 2y - z = 7 \\ \quad\quad\; y + 3z = -2 \\ \quad\quad\; y + 4z = -3 \end{cases}$$

is given by

(a) $x = 9 \quad y = -5 \quad z = 1$

(b) $x = -5 \quad y = 7 \quad z = -3$

(c) $x = 2 \quad y = 1 \quad z = -1$

(d) $x = -5 \quad y = 1 \quad z = -1$

(e) $x = 1 \quad y = 1 \quad z = -1$

Check

8. Suppose the augmented matrix of a linear system is row-equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & k & h \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Choose all possible correct answers from the list below.

Possibilities:

(a) If $k = 0$ and $h = -2$, the linear system has no solution.

(b) If $k = -2$ and $h = 0$, the linear system has no solution.

(c) If $k = 0$ and $h = 0$, the linear system has a unique solution.

(d) If $k = 0$ and $h = 0$, the linear system has no solution.

(e) If $k = -2$ and $h = -2$, the linear system has a unique solution.

9. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & -1 \end{bmatrix}.$$

Then the matrix $A^T A$ is:

(Recall that the notation A^T denotes the transpose of the matrix A. Our textbook uses the alternative notation A'.)

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 3 & -1 \end{bmatrix}}_{A^T} \cdot \underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & -1 \end{bmatrix}}_{A}$$

$$= \begin{bmatrix} 5 & 2 & 5 \\ 2 & 1 & 3 \\ 5 & 3 & 10 \end{bmatrix}$$

Possibilities:

(a) $\begin{bmatrix} 6 & -1 \\ -1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 14 & -1 \\ -1 & 2 \end{bmatrix}$

(c) $\boxed{\begin{bmatrix} 5 & 2 & 5 \\ 2 & 1 & 3 \\ 5 & 3 & 10 \end{bmatrix}}$

(d) $\begin{bmatrix} 3 & 3 & 5 \\ 3 & 0 & 3 \\ 5 & 3 & 8 \end{bmatrix}$

(e) None of the above

---

10. Suppose A is a 2×2 matrix, and its **inverse** is

$$A^{-1} = \begin{bmatrix} 4 & 1 \\ 19 & 5 \end{bmatrix}.$$

Find the solution to the matrix equation:

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\underbrace{A^{-1} A}_{I_2} \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 19 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Possibilities:

(a) $\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}}$

(b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

(d) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$

(e) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -31 \end{bmatrix}$

11. Verify that

$$y = x\sin x - x$$

is a solution of the initial-value problem

$$x\frac{dy}{dx} = y + x^2\cos x \qquad y(\pi/2) = 0.$$

$y = x\sin x - x$ so $\dfrac{dy}{dx} = 1\cdot\sin x + x(\cos x) - 1$

substitute in

$$x\frac{dy}{dx} = y + x^2\cos(x)$$

$$x\left[\sin(x) + x\cos(x) - 1\right] \stackrel{?}{=} \left[x\sin x - x\right] + x^2\cos(x)$$

OR

$$x\sin(x) + x^2\cos(x) - x \stackrel{?}{=} x\sin(x) - x + x^2\cos(x)$$

Yes

pts: /10

$$\boxed{y = \frac{1}{\frac{1}{3}-x^3} - 1 = \frac{3}{1-3x^3} - 1 = \frac{2+3x^3}{1-3x^3}}$$

12. Solve the separable differential equation

$$\frac{dy}{dx} = 3(y+1)^2x^2$$

with initial condition $y(0) = 2$.

$$\frac{1}{(y+1)^2}\,dy = 3x^2\,dx \qquad \text{so that}$$

$$\underbrace{\int \frac{1}{(y+1)^2}\,dy}_{} = \int 3x^2\,dx$$

you can use the substitution $y+1 = u$

so $du = dy$ and $\int \frac{1}{(y+1)^2}\,dy = \int \frac{1}{u^2}\,du$

$$= -\frac{1}{u} + C_1 = -\frac{1}{y+1} + C_1$$

$$\therefore \quad -\frac{1}{y+1} + C_1 = x^3 + C_2$$

or $-\frac{1}{y+1} = x^3 + \cancel{C_2 - C_1}\; C$ at $x=0$, $y=2$

$$\therefore \quad -\frac{1}{3} = 0^3 + C \qquad \therefore \quad C = -\frac{1}{3}$$

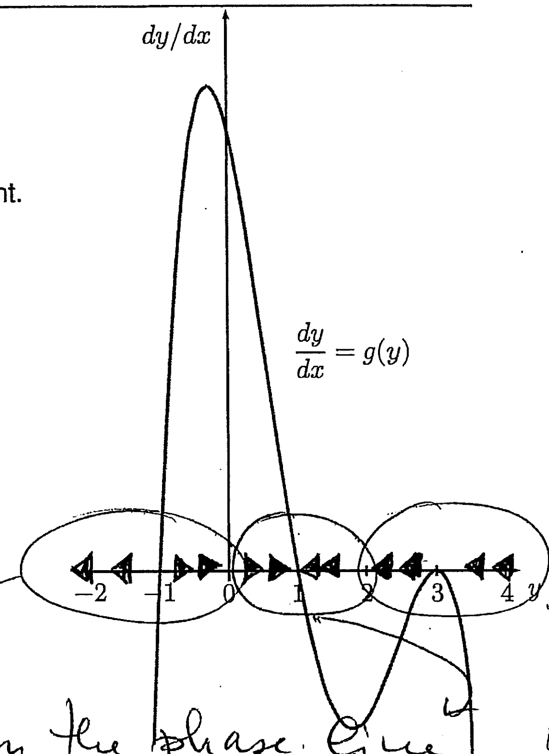
$$\frac{1}{y+1} = -x^3 + \frac{1}{3} \qquad \text{or} \qquad \frac{1}{\frac{1}{3}-x^3} = y+1$$

pts: /10

13. Consider the autonomous differential equation

$$\frac{dy}{dx} = g(y).$$

The graph of dy/dx as a function of y is given on the right.



(a) Use the graph on the right to find the equilibria $\widehat{y}$ of the differential equation.

$\boxed{\hat{y} = -1 \text{ , } 1 \text{ and } 3}$

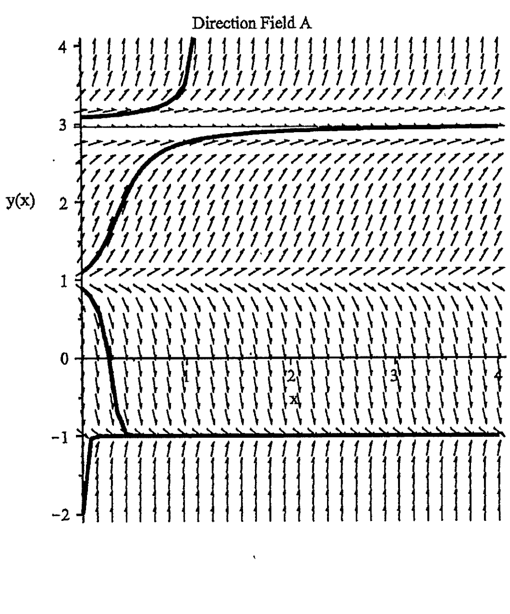
(b) Use the ~~graph on the right and the geometric~~ approach to discuss the stability of the equilibria you found in (a).

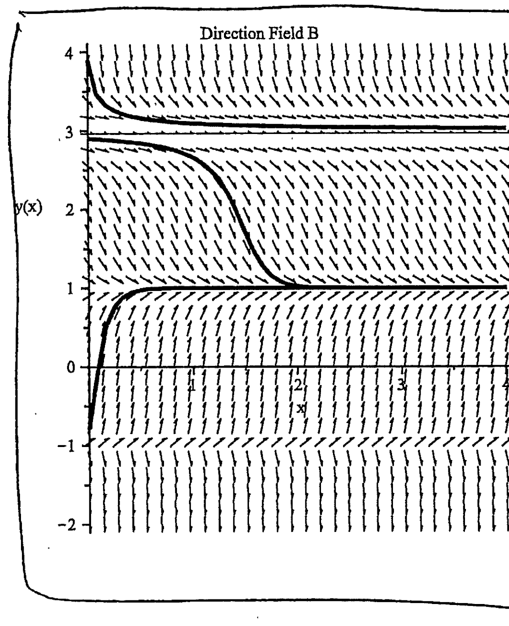
$\hat{y} = -1$ is unstable from the phase line

$\hat{y} = 1$ is locally stable from the phase line

$\hat{y} = 3$ is unstable on the left and locally stable on the right

(c) Using the information found in (a) and (b), which of the following direction fields (phase portraits) matches the given differential equation? Circle the correct one.





pts: /10

$x = 9/2 - 5/2\, t \; ; \; y = -19/6 + \frac{13}{2} t \; ; \; z = t$ any real

---

14. Solve the linear system using the Gaussian Elimination Algorithm:

these are infinitely many solutions

$$\begin{cases} 2x + 6y - 8z &= -10 \\ 2x + 5z &= 9 \\ 6x + 6y + 2z &= 8 \end{cases}$$

If the system has infinitely many solutions, you must write all solutions in terms of a parameter $t \in \mathbb{R}$. If it has no solutions, explain why.

$$\left[\begin{array}{ccc|c} 2 & 6 & -8 & -10 \\ 2 & 0 & 5 & 9 \\ 6 & 6 & 2 & 8 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 3 & -4 & -5 \\ 2 & 0 & 5 & 9 \\ 6 & 6 & 2 & 8 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 6R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -4 & -5 \\ 0 & -6 & 13 & 19 \\ 0 & -12 & 26 & 38 \end{array}\right] \longrightarrow \begin{array}{l} \\ \\ R_3 - 2R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -4 & -5 \\ 0 & -6 & 13 & 19 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$-\frac{1}{6}R_2 \left[\begin{array}{ccc|c} 1 & 3 & -4 & -5 \\ 0 & 1 & -\frac{13}{6} & -19/6 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow R_1 - 3R_2 \left[\begin{array}{ccc|c} 1 & 0 & 5/2 & 9/2 \\ 0 & 1 & -\frac{13}{6} & -\frac{19}{6} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

i.e. $\begin{cases} x + \frac{5}{2} z = 9/2 \\ y - \frac{13}{2} z = -19/6 \end{cases}$ OR

$x = 9/2 - 5/2\, z$

$y = -19/6 + \frac{13}{2} z$ and $z = t$ anything

pts: /10

15. Let $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 4 \\ 0 & 0 \\ 1 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$.

(*a*) Compute $2A - 3B$.

$$2\begin{bmatrix} 3 & 2 \\ -1 & 0 \\ -2 & 1 \end{bmatrix} - 3\begin{bmatrix} -1 & 4 \\ 0 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -2 & 0 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} -3 & 12 \\ 0 & 0 \\ 3 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 6+3 & 4-12 \\ -2-0 & 0-0 \\ -4-3 & 2-15 \end{bmatrix} = \begin{bmatrix} 9 & -8 \\ -2 & 0 \\ -7 & -13 \end{bmatrix}$$

(*b*) Find the entry d_{23} of the matrix $D = (d_{ij})_{\substack{1 \le i \le 3 \\ 1 \le j \le 3}}$ where D is the product AC.

d_{23} = row 2, column 3

$$D = A \cdot C = \begin{bmatrix} 3 & 2 \\ \boxed{-1 \quad 0} \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & * & -1 \\ * & * & * \end{bmatrix}$$

(*c*) Which of $\begin{bmatrix} 0 & 1 & \frac{1}{2} \\ -\frac{3}{4} & \frac{1}{2} & -\frac{3}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \\ -\frac{3}{4} & \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$ is the inverse of $\begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix}$?

check that

$$\begin{bmatrix} \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \\ -\frac{3}{4} & \frac{1}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

etc....

pts: /10

Bonus. (1) (5 pts) Find all the values of k for which the system is consistent

$$\begin{cases} 3x + 2y = 0 \\ 6x + ky = -3 \end{cases}$$

$$\left[\begin{array}{cc|c} 3 & 2 & 0 \\ 6 & k & -3 \end{array}\right] \rightsquigarrow \tfrac{1}{3}R_1 \left[\begin{array}{cc|c} 1 & 2/3 & 0 \\ 6 & k & -3 \end{array}\right]$$

$$\rightsquigarrow_{R_2 - 6R_1} \left[\begin{array}{cc|c} 1 & 3/2 & 0 \\ 0 & k-4 & -3 \end{array}\right]$$

The system is consistent if $k-4 \neq 0$ OR $k \neq 4$

(2) (5 pts) Suppose β is a real number. The solution(s) for the system of linear equations corresponding to the following augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & \beta \end{array}\right]$$

is (are): (**Hint:** Your answer will depend on the value of β.)

If $\boxed{\beta \neq 0}$ the system is inconsistent so there are no solutions

If $\boxed{\beta = 0}$ the system is consistent and there are infinitely many solutions

$x = -1 + 2t$; $y = 3 - 5t$; $z = t$ any real

pts: /10