MA 138
Calculus II with Life Science Applications
THIRD MIDTERM

Spring 2023
04/11/2023

Name: Answer Key

Sect. #: ____________

Do not remove this answer page — you will return the whole exam. No books or notes may be used. Use the backs of the question papers for scratch paper. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The **first part of the exam** consists of 10 multiple choice questions, each worth 5 points. Record your answers on this page by filling in the box corresponding to the correct answer. For example, if (a) is correct, you must write

■ [b] [c] [d] [e]

Do not circle answers on this page, but please do circle the letter of each correct response in the body of the exam. It is your responsibility to make it CLEAR which response has been chosen. You will not get credit unless the correct answer has been marked on **both** this page **and** in the body of the exam.

The **second part of the exam** consists of five open-response questions and one bonus question. When answering these questions, check your answers when possible. Clearly indicate your answer and the reasoning used to arrive at that answer. *Unsupported answers may receive NO credit.*

1. [a] [b] ■ [d] [e]

2. [a] ■ [c] [d] [e]

3. [a] [b] [c] ■ [e]

4. [a] [b] [c] [d] ■

5. [a] [b] [c] ■ [e]

6. [a] [b] ■ [d] [e]

7. [a] ■ [c] [d] [e]

8. [a] ■ [c] [d] [e]

9. [a] ■ [c] [d] [e]

10. ■ [b] [c] [d] [e]

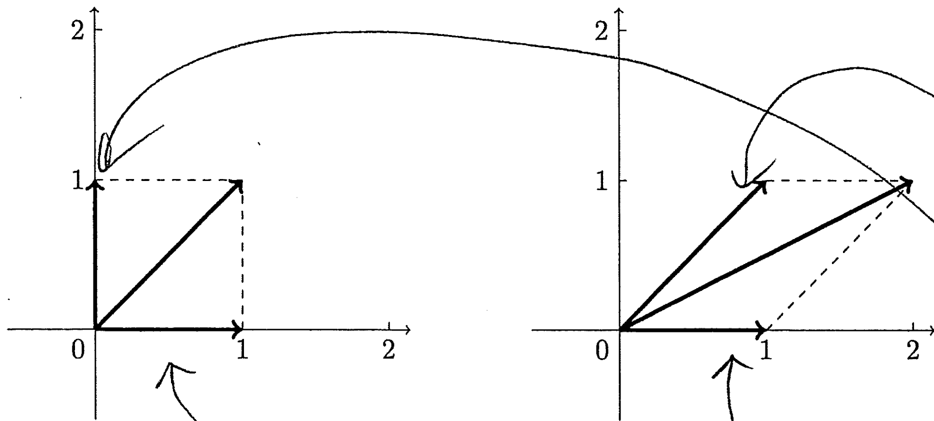
GOOD LUCK!

QUESTION	SCORE	OUT OF
Multiple Choice		50 pts
11.		10 pts
12.		10 pts
13.		10 pts
14.		10 pts
15.		10 pts
Bonus.		10 pts
TOTAL		100 pts

Please make sure to list the correct section number on the front page of your exam. In case you forgot your section number, consult the following table:

Sections #	Lecturer	Time/Location
001–004	Alberto Corso	MWF 10:00 am - 10:50 am, CB 110
Section #	**Recitation Instructor**	**Time/Location**
001	Kathryn Hechtel	TR 08:00 am - 08:50 am, CB 307
002	Kathryn Hechtel	TR 09:00 am - 09:50 am, CB 307
003	Davis Deaton	TR 10:00 am - 10:50 am, CB 307
004	Davis Deaton	TR 11:00 am - 11:50 am, CB 307

1. Suppose a linear map $T(\mathbf{x})$ transforms the unit square depicted on the left into the shape on the right.



Which of the following is the 2×2 matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for any 2×1 vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$?

Possibilities:

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

(c) $\boxed{A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}$

(d) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(e) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

Observe that the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has the property that

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

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2. Let $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$ be a 3×3 matrix. Consider the following four vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

Which of the following statement is correct?

(a) $\mathbf{v}_1$ and $\mathbf{v}_3$ are eigenvectors of A

(b) $\boxed{\mathbf{v}_1 \text{ and } \mathbf{v}_2 \text{ are eigenvectors of } A}$

(c) $\mathbf{v}_2$ and $\mathbf{v}_3$ are eigenvectors of A

(d) $\mathbf{v}_3$ and $\mathbf{v}_4$ are eigenvectors of A

(e) None of the above

Check that

$$A\underline{v}_1 = 2\underline{v}_1$$

$$A\underline{v}_2 = 2\underline{v}_2$$

3. The eigenvalues of the matrix $M = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$ are

$$\det \begin{bmatrix} 0-\lambda & 1 \\ -2 & 3-\lambda \end{bmatrix} = 0 \iff$$

$$(-\lambda)(3-\lambda) + 2 = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0$$

$$\therefore \lambda_1 = 1 \ \& \ \lambda_2 = 2$$

Possibilities:

(a) There are no eigenvalues

(b) $\lambda_1 = -1$ and $\lambda_2 = -2$

(c) $\lambda_1 = -1$ and $\lambda_2 = 2$

$\boxed{\text{(d) } \lambda_1 = 1 \text{ and } \lambda_2 = 2}$

(e) $\lambda_1 = 1$ and $\lambda_2 = -2$

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4. Find the value of k so that the matrix $\begin{bmatrix} 2 & 1 \\ 5 & k \end{bmatrix}$ has 0 as an eigenvalue.

Since the product of the eigenvalues λ_1 and λ_2 is the determinant of the matrix, the question is equivalent to asking that

$$\det \begin{bmatrix} 2 & 1 \\ 5 & k \end{bmatrix} = 0 \quad \text{or} \quad 2k - 5 = 0$$

$$\text{or} \quad \boxed{k = 2.5}$$

Possibilities:

(a) -2.5

(b) 0

(c) 0.4

(d) 1

$\boxed{\text{(e) None of the above}}$

5. There is a correlation between the amount of oxygen dissolved in a body of water and the depth of the water: oxygen is most abundant near the surface with lesser amounts at deeper levels. Suppose that the amount of dissolved oxygen is measured at several depths in a lake and the following data are collected:

depth (ft)	dissolved oxygen (mg/L)
2	10.5
10	6.0
20	0.5

We expect a linear relationship between the depth (d) and the dissolved oxygen (O_2), that is:

$$O_2 = md + b$$

for appropriate values of m and b. Which of the following systems should we solve in order to find the least-squares solution to this linear problem?

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 10 & 1 \\ 20 & 1 \end{bmatrix}}_{A} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 10.5 \\ 6 \\ 0.5 \end{bmatrix}$$

That's the system in matrix form. Multiply by A^T on both sides to get

$$\begin{bmatrix} 504 & 32 \\ 32 & 3 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 91 \\ 17 \end{bmatrix}$$

Possibilities:

(a) $\begin{bmatrix} 3 & 17 \\ 17 & 146.5 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 32 \\ 91 \end{bmatrix}$

(b) $\begin{bmatrix} 146.5 & 17 \\ 17 & 3 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 91 \\ 32 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 32 \\ 32 & 504 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 17 \\ 91 \end{bmatrix}$

(d) $\boxed{\begin{bmatrix} 504 & 32 \\ 32 & 3 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 91 \\ 17 \end{bmatrix}}$

(e) $\begin{bmatrix} 5 & 21 & 41 \\ 21 & 101 & 201 \\ 41 & 201 & 401 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 91 \\ 17 \end{bmatrix}$

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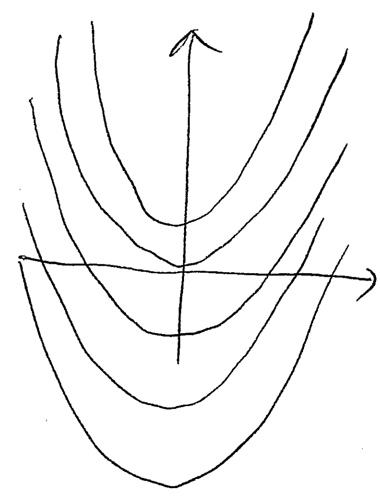
6. Describe the level curves of $f(x, y) = \sqrt{1 + x^2 - y}$.

$$f(x,y) = c \iff \sqrt{1+x^2-y} = c$$

$$\iff 1 + x^2 - y = c^2 \iff$$

$$y = x^2 + \underbrace{1 - c^2}_{\text{Constant}}$$



Possibilities:

(a) The level curves are circles

(b) The level curves are straight lines passing through the origin

(c) $\boxed{\text{The level curves are parabolas with vertices on the } y\text{-axis}}$

(d) The level curves are parabolas with vertices on the x-axis

(e) None of the above

7. Compute the partial derivative of the function

$$f(x,y,z) = e^{1-x\cos y} + ze^{-1/(1+y^2)}$$

with respect to x at the point $(1, 0, \pi)$.

Possibilities:

(a) $-1/e$

(b) $\boxed{-1}$

(c) 0

(d) π/e

(e) π

$$\frac{\partial f}{\partial x} = e^{1-x\cos(y)} \cdot (-\cos(y))$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,0,\pi)} = e^{1-1\cdot\cos(0)} \cdot (-\cos(0)) = -1$$

8. (Heat Index) On a hot day, extreme humidity makes us think the temperature is higher than it really is, whereas in very dry air we perceive the temperature to be lower than the thermometer indicates. The National Weather Service has devised the *heat index* to describe the combined effects of temperature and humidity. The heat index I is the perceived air temperature when the actual temperature is T and the relative humidity is H. So I is a function of T and H and we can write $I = f(T, H)$. The following table of values of I is an excerpt from a table compiled by the National Weather Service.

		Relative humidity H (%)								
		50	55	60	65	70	75	80	85	90
Actual temperature T (°F)	90	96	98	100	103	106	109	112	115	119
	92	100	103	105	108	112	115	119	123	128
	94	104	107	111	114	118	122	127	132	137
	96	109	113	116	121	125	130	135	141	146
	98	114	118	123	127	133	138	144	150	157
	100	119	124	129	135	141	147	154	161	168

Estimate the rate of change of the heat index with respect to the temperature, that is $\frac{\partial f}{\partial T}$, when the temperature is 96°F and the humidity is 70%

(a) -3.75

(b) $\boxed{3.75}$

(c) 0.9

(d) -0.9

(e) 0

$$\frac{\partial f}{\partial T} \approx \frac{f(96+h,70) - f(96,70)}{h}$$

eg. $\dfrac{f(94,70) - f(96,70)}{-2} \approx 3.5$

$\dfrac{f(98,70) - f(96,70)}{2} \approx 4$

9. Which of the following is an equation for the tangent plane to

$$f(x, y) = x^2 + 3xy + y^2$$

when $x_0 = -1$ and $y_0 = 2$?

$f(-1,2) = (-1)^2 + 3(-1)(2) + 2^2 = 1 - 6 + 4 = -1$

$f_x(x,y) = 2x + 3y \quad f_x(-1,2) = 2(-1) + 3(2) = 4$

$f_y(x,y) = 3x + 2y \quad f_y(-1,2) = 3(-1) + 2(2) = 1$

equation of the tg plane

$z - (-1) = 4(x - (-1)) + 1 \cdot (y - 2)$

or $z = -1 + 4x + 4 + y - 2$

or $\boxed{z = 4x + y + 1}$

Possibilities:

(a) $z = 8x + 7y + 11$

(b) $\boxed{z = 4x + y + 1}$

(c) $z = 4x + y - 7$

(d) $z = x + 4y - 1$

(e) $z = 4x + y + 3$

10. Let $f(x, y) = (x - y)^3 + 2xy + x^2 - y$. Find the linear approximation $L(x, y)$ near the point $(1, 2)$.

$f(1,2) = (1-2)^3 + 2(1)(2) + 1^2 - 2 = 2$

$f_x = 3(x-y)^2 \cdot (1) + 2y + 2x \qquad f_x(1,2) = 9$

$f_y = 3(x-y)^2(-1) + 2x - 1 \qquad f_y(1,2) = -2$

$\boxed{L(x,y) = 2 + 9(x-1) - 2(y-2)}$

Possibilities:

(a) $\boxed{L(x, y) = 2 + 9(x - 1) - 2(y - 2)}$

(b) $L(x, y) = -2 + 9(x + 1) - 2(y + 2)$

(c) $L(x, y) = -2 + 9(x - 1) - 2(y - 2)$

(d) $L(x, y) = 2 + 9(x + 1) - 2(y + 2)$

(e) $L(x, y) = 2 - 9(x - 1) + 2(y - 2)$

11. Consider the matrix $A = \begin{bmatrix} 2 & -6 \\ 0 & -1 \end{bmatrix}$.

We can show that A has eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(a) Find the corresponding eigenvalues λ_1 and λ_2 of A.

$$\begin{bmatrix} 2 & -6 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \text{so } \lambda_1 = 2$$

$$\begin{bmatrix} 2 & -6 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4-6 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = (-1)\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{so } \lambda_2 = -1$$

(b) Find coefficients c_1 and c_2 so that $\begin{bmatrix} 11 \\ 4 \end{bmatrix} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

We need to solve $\begin{bmatrix} 11 \\ 4 \end{bmatrix} = c_1\begin{bmatrix} -1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ or

$$\begin{cases} -c_1 + 2c_2 = 11 \\ c_2 = 4 \end{cases}$$

so that $c_2 = 4$ and $-c_1 + 2(4) = 11 \rightsquigarrow c_1 = -3$

$$\therefore \boxed{\begin{bmatrix} 11 \\ 4 \end{bmatrix} = -3\begin{bmatrix} -1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

(c) Use your above responses to evaluate $A^{13}\begin{bmatrix} 11 \\ 4 \end{bmatrix}$.

Thus $A^{13}\begin{bmatrix} 11 \\ 4 \end{bmatrix} = A^{13}\left(-3\begin{bmatrix} -1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) =$

$$= -3A^{13}\begin{bmatrix} -1 \\ 0 \end{bmatrix} + 4A^{13}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = -3(2)^{13}\begin{bmatrix} -1 \\ 0 \end{bmatrix} + 4(-1)^{13}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot 2^{13} - 8 \\ -4 \end{bmatrix} = \begin{bmatrix} 24{,}568 \\ -4 \end{bmatrix}$$

pts: /10

12. Ibuprofen is an over-the-counter anti-inflammatory drug often taken to treat headaches and other pain-related symptoms. Suppose that we expect a linear relationship of the form $y = at + b$, where y denotes the amount of the drug in an individual's system t hours after taking the recommended dosage. Through experimentation, we obtain the following data of drug concentration versus time elapsed.

time (hr)	concentration (ppm)
1	8
2	6
3	3
4	1

(*a*) Find a matrix equation that corresponds to the initial linear system that we wish to solve.

$$8 = a\cdot 1 + b$$
$$6 = a\cdot 2 + b$$
$$3 = a\cdot 3 + b$$
$$1 = a\cdot 4 + b$$

or in matrix form $\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 3 \\ 1 \end{bmatrix}$

(*b*) Find the matrix equation whose solution is the least-squares solution to the system from part (*a*).

Multiply both sides of the above equation by A^T

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 3 \\ 1 \end{bmatrix}$$

to obtain

$$\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 33 \\ 18 \end{bmatrix}$$

(*c*) Find the least-squares model of drug absorption that corresponds to this scenario.

Multiply by the inverse of A^TA to obtain $\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}$

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 4 & -10 \\ -10 & 30 \end{bmatrix} \begin{bmatrix} 33 \\ 18 \end{bmatrix} = \begin{bmatrix} -12/5 \\ \frac{21}{2} \end{bmatrix}$$

$$\boxed{y = -\frac{12}{5}t + \frac{21}{2}}$$

pts: /10

13. (*a*) (5 pts) Commensalism is a biological relationship between two organisms where one (usually smaller) organism obtains food or other benefits from the other while the (usually larger) organism is unaffected. For example, the remora is a small fish that attaches itself to larger fish for locomotion as well as to feed on the host's leftovers. Suppose that the growth rate of a remora population is given by the function

$$f(x,y) = \ln(y - x^2)$$

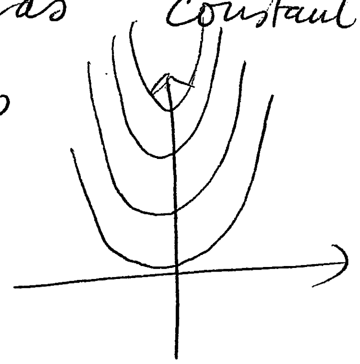
where x is the current number of remora and y is the number of viable host fish in the same region.

Find a simplified formula for the level curves $f(x,y) = c$ and describe these curves.

$c = \ln(y-x^2) \iff e^{c} = e^{\ln(y-x^2)}$

$\iff e^c = y - x^2$ or $\boxed{y = x^2 + \boxed{e^c}}$ positive constant

thus the level curves are parabolas with vertex on the positive y-axis



(*b*) (5 pts) Does the following limit

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{x^2y}{x^4+y^2}$$

exist? Explain.

Consider paths of the form $y = mx^2$.
Along these paths the limit becomes

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx^2}} \frac{x^2y}{x^4+y^2} = \lim_{\substack{x\to 0\\ y=mx}} \frac{x^2(mx^2)}{x^4+(mx^2)^2} =$$

$$= \lim_{\substack{x\to 0\\ y=mx^2}} \frac{m\cancel{x^4}}{\cancel{x^4}(1+m^2)} = \frac{m}{1+m^2}$$

since the limit depends on the choice of m

Limit D.N.E.

pts: /10

14. Compute the partial derivatives g_x and g_y for the function

$$g(x,y) = \frac{3x}{2x-4y^2}.$$

$$\frac{\partial g}{\partial x} = \frac{3(2x-4y^2)-3x(2)}{(2x-4y^2)^2}$$

$$= \frac{\cancel{6x}-12y^2-\cancel{6x}}{(2x-4y^2)^2} = \boxed{\frac{-12y^2}{(2x-4y^2)^2}}$$

For $\frac{\partial g}{\partial y}$ consider $g(x,y) = 3x(2x-4y^2)^{-1}$

So by the power chain rule

$$\frac{\partial g}{\partial y} = 3x(-1)(2x-4y^2)^{-2}\cdot(-8y)$$

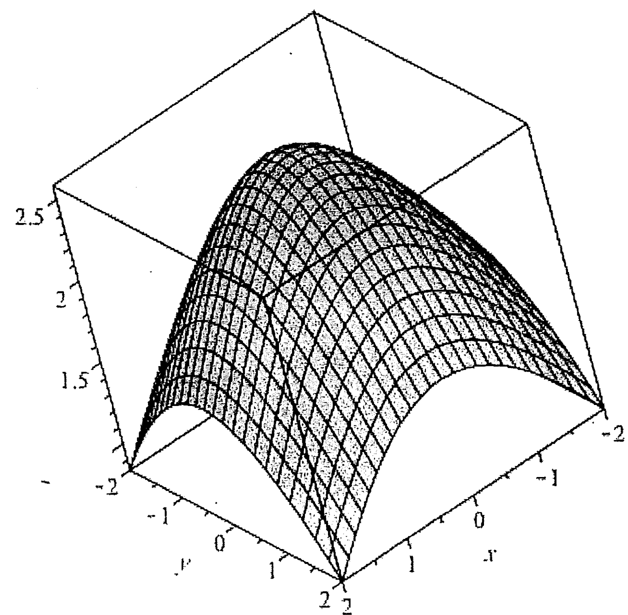
$$= \boxed{\frac{24xy}{(2x-4y^2)^2}}$$

pts: /10

15. Consider the function

$$f(x, y) = 4 - \sqrt{x^2 + y^2 + 2}$$

whose graph is given in the picture on the right.



(*a*) Find the z-coordinate z_0 of the point P on the graph of the function $z = f(x, y)$ with x-coordinate $x_0 = 1$ and y-coordinate $y_0 = 1$.

$$z_0 = f(1,1) = 4 - \sqrt{1^2 + 1^2 + 2}$$
$$= 4 - \sqrt{4} = 2$$

(*b*) Write the equation of the tangent plane to the graph of the function $z = f(x, y)$ at the point P, as above, with coordinates $x_0 = 1$ and $y_0 = 1$.

We need $f_x(1,1)$ and $f_y(1,1)$

$$f_x(x,y) = 0 - \frac{1}{2}(x^2+y^2+2)^{-1/2} \cdot (2x) = \frac{-x}{\sqrt{x^2+y^2+2}}$$

Similarly

$$f_y(x,y) = \frac{-y}{\sqrt{x^2+y^2+2}}$$

$$\boxed{f_x(1,1) = -\tfrac{1}{2} = f_y(1,1)}$$

$$\boxed{z = 2 - \tfrac{1}{2}(x-1) - \tfrac{1}{2}(y-1)}$$

(*c*) Write the linear approximation, $L(x, y)$, of the function f at the point with $x_0 = 1$ and $y_0 = 1$, as above, and use it to approximate $f(1.1, 0.9)$.
Compare this approximate value to the exact value $f(1.1, 0.9)$.

$$L(x,y) = 2 - \tfrac{1}{2}(x-1) - \tfrac{1}{2}(y-1) = 3 - \tfrac{1}{2}x - \tfrac{1}{2}y$$

$$L(1.1, 0.9) = 2 - \tfrac{1}{2}(0.1) - \tfrac{1}{2}(-0.1) = 2$$

exact value $f(1.1, 0.9) = 1.995$

pts: /10

$(*)\quad J_{t+1}/A_{t+1} \cong \dfrac{c_1 \lambda_1^{t+1}\cdot 10}{c_1 \lambda_1^{t+1}\cdot 1} = \boxed{10}$

Bonus. Consider an animal species that has two life stages: juvenile and adult. Suppose that we count the number of members of this population on a weekly basis. Let J_t denote the number of juveniles at week t and A_t denote number of adults at week t. The relation between the population during two consecutive weeks can reasonably be described as follows

$$\begin{aligned} J_{t+1} &= J_t - mJ_t - gJ_t + fA_t \\ A_{t+1} &= A_t - \mu A_t + gJ_t \end{aligned} \tag{1}$$

where m is the fraction of juveniles that dies, g is the fraction of juveniles that becomes adult, f accounts for the newborns, and μ is the fraction of adults that dies.

(*a*) Write the matrix form of the system of equations in (1) when $m = 0.8$, $g = 0.1$, $f = 10$, and $\mu = 0.9$.

$$\begin{bmatrix} J_{t+1} \\ A_{t+1} \end{bmatrix} = \begin{bmatrix} 1-m-g & f \\ g & 1-\mu \end{bmatrix}\begin{bmatrix} J_t \\ A_t \end{bmatrix} = \begin{bmatrix} 0.1 & 10 \\ 0.1 & 0.1 \end{bmatrix}\begin{bmatrix} J_t \\ A_t \end{bmatrix}$$

(*b*) Find the eigenvalues and the corresponding eigenvectors associated with the matrix found in part (*a*).

The eigenvalues of $\begin{bmatrix} 0.1 & 10 \\ 0.1 & 0.1 \end{bmatrix}$ are given by

$$\det\begin{bmatrix} 0.1-\lambda & 10 \\ 0.1 & 0.1-\lambda \end{bmatrix} = 0 \iff (0.1-\lambda)^2 - 1 = 0$$

OR $(\lambda - 0.1)^2 = 1$ OR $\lambda - 0.1 = \pm 1$

$\lambda_1 = 1.1$ Check that $\underline{v}_1 = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$ is an eigenvector

$\lambda_2 = -0.9$ Check that $v_2 = \begin{bmatrix} -10 \\ 1 \end{bmatrix}$ is an eigenvector

(*c*) Compute the long-term ratio of juveniles versus adults in this animal species.

$$\begin{bmatrix} J_{t+1} \\ A_{t+1} \end{bmatrix} = \begin{bmatrix} 0.1 & 10 \\ 0.1 & 0.1 \end{bmatrix}^{t+1}\begin{bmatrix} J_0 \\ A_0 \end{bmatrix} = \begin{bmatrix} 0.1 & 10 \\ 0.1 & 0.1 \end{bmatrix}^{t+1}(c_1 \underline{v}_1 + c_2 \underline{v}_2)$$ for appropriate c_1 and c_2

$$= c_1 \lambda_1^{t+1} \underline{v}_1 + c_2 \lambda_2^{t+1} \underline{v}_2 \approx c_1 \lambda_1^{t+1} \underline{v}_1 = c_1 (1.1)^{t+1}\begin{bmatrix} 10 \\ 1 \end{bmatrix} \quad (*)$$

pts: /10

To find the eigenvector $\underline{v}_1 = \begin{bmatrix} y \\ a \end{bmatrix}$ corresponding to $\lambda_1 = 1.1$ we solve

$$\begin{bmatrix} 0.1 & 10 \\ 0.1 & 0.1 \end{bmatrix} \underbrace{\begin{bmatrix} y \\ a \end{bmatrix}}_{\underline{v}_1} = 1.1 \underbrace{\begin{bmatrix} y \\ a \end{bmatrix}}_{\underline{v}_1}$$

or $\begin{cases} 0.1\,y + 10\,a = 1.1\,y \\ 0.1\,y + 0.1\,a = 1.1\,a \end{cases}$ which reduce to the same equation

$$-1\,y + 10\,a = 0 \quad \text{or} \quad \boxed{y = 10\,a}$$

so we can choose $\underline{v}_1 = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$

To find the eigenvector $\underline{v}_2 = \begin{bmatrix} y \\ a \end{bmatrix}$ corresponding to $\lambda_2 = -0.9$ we solve

$$\begin{bmatrix} 0.1 & 10 \\ 0.1 & 0.1 \end{bmatrix} \underbrace{\begin{bmatrix} y \\ a \end{bmatrix}}_{\underline{v}_2} = -0.9 \underbrace{\begin{bmatrix} y \\ a \end{bmatrix}}_{\underline{v}_2}$$

or $\begin{cases} 0.1\,y + 10\,a = -0.9\,y \\ 0.1\,y + 0.1\,a = -0.9\,a \end{cases}$

$\iff \boxed{y = -10\,a}$ so we can choose $\underline{v}_2 = \begin{bmatrix} -10 \\ 1 \end{bmatrix}$

For part (c) observe that

$$\begin{bmatrix} J_{t+1} \\ A_{t+1} \end{bmatrix} = \begin{bmatrix} 0.1 & 10 \\ 0.1 & 0.1 \end{bmatrix}^{t+1} \cdot \underbrace{\begin{bmatrix} J_0 \\ A_0 \end{bmatrix}}_{\substack{\text{initial distribution} \\ \text{of our animal species}}}$$

Without knowing the initial distribution $\begin{bmatrix} J_0 \\ A_0 \end{bmatrix}$ we may assume that we can write it as $c_1 \begin{bmatrix} 10 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -10 \\ 1 \end{bmatrix}$ for some c_1 and c_2. Hence

$$\begin{bmatrix} J_{t+1} \\ A_{t+1} \end{bmatrix} = \begin{bmatrix} 0.1 & 10 \\ 0.1 & 0.1 \end{bmatrix}^{t+1} \begin{bmatrix} J_0 \\ A_0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0.1 & 10 \\ 0.1 & 0.1 \end{bmatrix}^{t+1} \left(c_1 \underbrace{\begin{bmatrix} 10 \\ 1 \end{bmatrix}}_{\substack{\text{corresponding} \\ \text{to } \lambda_1 = 1.1}} + c_2 \underbrace{\begin{bmatrix} -10 \\ 1 \end{bmatrix}}_{\substack{\text{corresponding} \\ \text{to } \lambda_2 = -0.9}} \right)$$

$$= c_1 (1.1)^{t+1} \begin{bmatrix} 10 \\ 1 \end{bmatrix} + c_2 (-0.9)^{t+1} \begin{bmatrix} -10 \\ 1 \end{bmatrix}$$

Notice that for t very large $(-0.9)^{t+1}$ will be essentially zero.

Thus for large t

$$\begin{bmatrix} J_{t+1} \\ A_{t+1} \end{bmatrix} \approx c_1 (1.1)^{t+1} \begin{bmatrix} 10 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 (1.1)^{t+1} \cdot 10 \\ c_1 (1.1)^{t+1} \end{bmatrix}$$

and

$$\frac{J_{t+1}}{A_{t+1}} \approx \frac{c_1 \cancel{(1.1)^{t+1}} \cdot 10}{c_1 \cancel{(1.1)^{t+1}}} = 10$$

So $$\lim_{t \to \infty} \frac{J_{t+1}}{A_{t+1}} = 10$$