MA 138
Calculus II with Life Science Applications
THIRD MIDTERM

Spring 2024
04/09/2024

Name: Answer Key

Sect. #: ______________

Do not remove this answer page — you will return the whole exam. No books or notes may be used. Use the backs of the question papers for scratch paper. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The **first part of the exam** consists of 10 multiple choice questions, each worth 5 points. Record your answers on this page by filling in the box corresponding to the correct answer. For example, if (a) is correct, you must write

$\blacksquare$ b c d e

Do not circle answers on this page, but please do circle the letter of each correct response in the body of the exam. It is your responsibility to make it CLEAR which response has been chosen. You will not get credit unless the correct answer has been marked on **both** this page **and** in the body of the exam.

The **second part of the exam** consists of five open-response questions and one bonus question. When answering these questions, check your answers when possible. Clearly indicate your answer and the reasoning used to arrive at that answer. *Unsupported answers may receive NO credit.*

1. a b c d $\blacksquare$

2. a b $\blacksquare$ d e

3. a $\blacksquare$ c d e

4. $\blacksquare$ b c d e

5. a b $\blacksquare$ d e

6. $\blacksquare$ b c d e

7. a b $\blacksquare$ d e

8. a b c $\blacksquare$ e

9. a $\blacksquare$ c d e

10. a b c $\blacksquare$ e

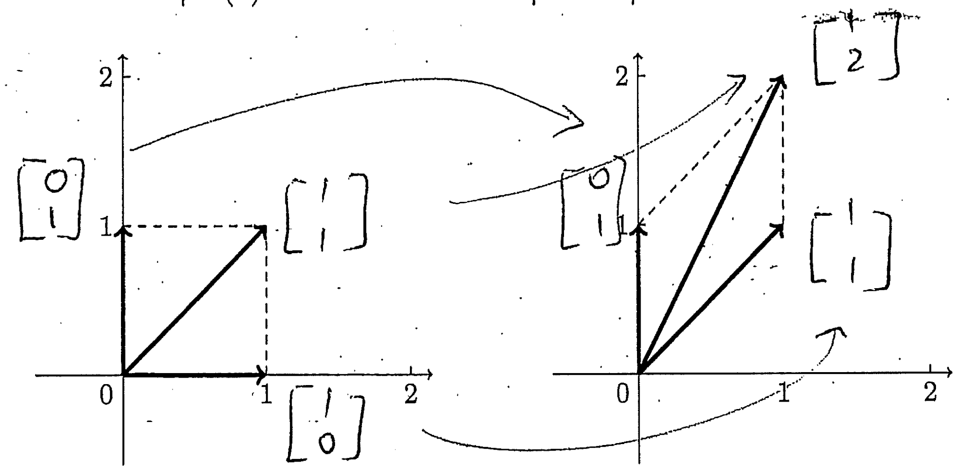
GOOD LUCK!

QUESTION	SCORE	OUT OF
Multiple Choice		50 pts
11.		10 pts
12.		10 pts
13.		10 pts
14.		10 pts
15.		10 pts
Bonus.		10 pts
TOTAL		100 pts

Please make sure to list the correct section number on the front page of your exam. In case you forgot your section number, consult the following table:

Sections #	Lecturer	Time/Location
001-004	Alberto Corso	MWF 10:00 am - 10:50 am, CB 212
Section #	**Recitation Instructor**	**Time/Location**
001	Ian Robinson	TR 09:00 am - 09:50 am, CB 307
002	Ian Robinson	TR 10:00 am - 10:50 am, CB 307

1. Suppose a linear map $T(\mathbf{x})$ transforms the unit square depicted on the left into the shape on the right.



Which of the following is the 2×2 matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for any 2×1 vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$?

Possibilities:

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

note that

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

2. Let A be the 3×3 matrix

$$A = \begin{bmatrix} 9 & 14 & -2 \\ -8 & -13 & 2 \\ -16 & -28 & 5 \end{bmatrix}$$

Only one of the following vectors is an eigenvector for A. Which one? And what is its eigenvalue?

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

Which of the following statement is correct?

(a) $\mathbf{v}_1$ is an eigenvector of A, with eigenvalue 1

(b) $\mathbf{v}_1$ is an eigenvector of A, with eigenvalue -1

(c) $\mathbf{v}_2$ is an eigenvector of A, with eigenvalue 1

(d) $\mathbf{v}_2$ is an eigenvector of A, with eigenvalue -1

(e) $\mathbf{v}_3$ is an eigenvector of A, with eigenvalue 1

$$A \cdot \underline{v}_1 = \begin{bmatrix} 4 \\ -4 \\ -9 \end{bmatrix}$$

$$\boxed{A \underline{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \underline{v}_2}$$

$$A \underline{v}_3 = \begin{bmatrix} 17 \\ -15 \\ \end{bmatrix}$$

3. Let B be the 2×2 matrix

$$B = \begin{bmatrix} 6 & 3 \\ -3 & -4 \end{bmatrix}$$

What are the eigenvalues of B?

The eigenvalues are the roots of the characteristic polynomial, which is $\det \begin{bmatrix} 6-\lambda & 3 \\ -3 & -4-\lambda \end{bmatrix} = 0$

$(6-\lambda)(-4-\lambda) - (3)(-3) = 0$

OR

$\lambda^2 - 2\lambda - 15 = 0$

$(\lambda - 5)(\lambda + 3) = 0$

$\lambda_1 = 5$ & $\lambda_2 = -3$

Possibilities:

(a) The eigenvalues of B are 5 and 3

(b) $\boxed{\text{The eigenvalues of } B \text{ are } 5 \text{ and } -3}$

(c) The eigenvalues of B are -6 and 4

(d) The eigenvalues of B are 6 and -4

(e) The eigenvalues of B are 6 and -3

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4. The 2×2 matrix A has eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with corresponding eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1/7$. Choose the most accurate way to complete the sentence

"The vector $A^{1000} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ will be closest to ..."

(**Hint:** it may help to know that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{v}_1 - 2\mathbf{v}_2$.)

Given the hint, $A^{1000} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = A^{1000} \left(\begin{bmatrix} 3 \\ 5 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$

$= A^{1000} \begin{bmatrix} 3 \\ 5 \end{bmatrix} - 2 A^{1000} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= 1^{1000} \begin{bmatrix} 3 \\ 5 \end{bmatrix} - 2 \frac{1}{7^{1000}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 5 \end{bmatrix} - \frac{2}{7^{1000}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \approx \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

very small

Possibilities:

(a) $\boxed{\text{the vector } \mathbf{v}_1}$

(b) the vector $\mathbf{v}_2$

(c) a very large multiple of the vector $\mathbf{v}_1$

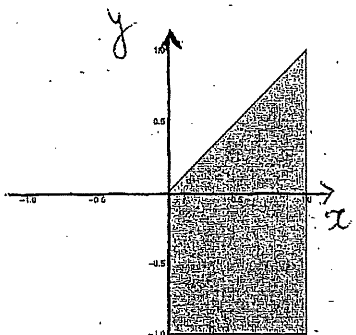
(d) a very large multiple of the vector $\mathbf{v}_2$

(e) the vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

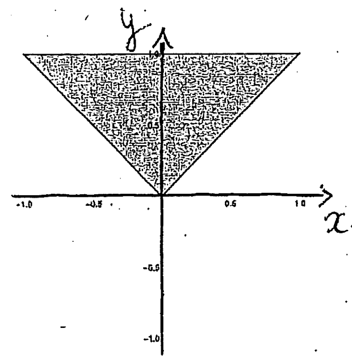
The domain of f is the region when $x \geq 0$ ~~and $y - x > 0$ OR $y > x$~~

5. Which of the following shaded regions shows (part of) the domain of the function

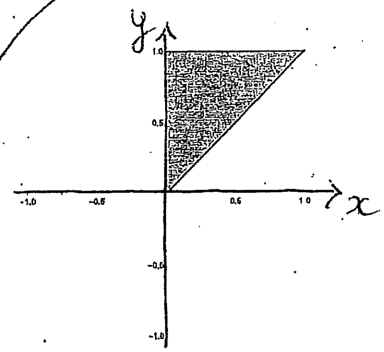
$$f(x,y) = \sqrt{x}\ln(y-x)$$



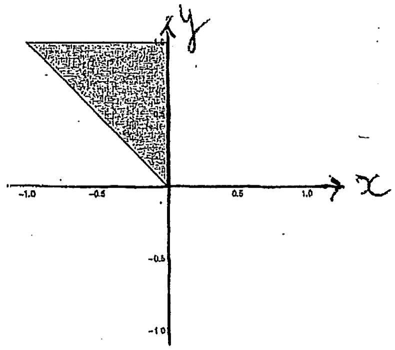
(a)



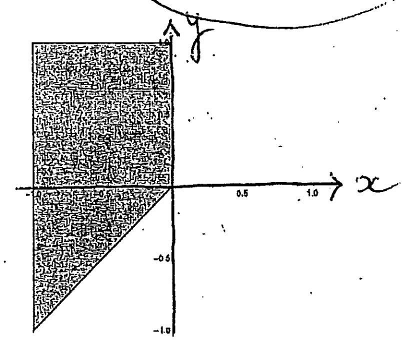
(b)



(c)



(d)



(e)

This is the region on the right of the y-axis and above the line $y = x$. Thus (C)

6. Choose the most accurate description of the level curves of the function

$$f(x,y) = \frac{x-y}{x+y}$$

Choose a constant c and setup the equation $f(x,y) = c$. Thus $\frac{x-y}{x+y} = c$ OR $x - y = c(x+y)$

$x - cx = y + cy$

OR $y = \underbrace{\frac{1-c}{1+c}}_{m} x$

(a) $\boxed{\text{The level curves are lines through the origin}}$

(b) The level curves are circles centered at the origin

(c) The level curves are hyperbolas

(d) The level curves are parabolas that open in the positive y direction

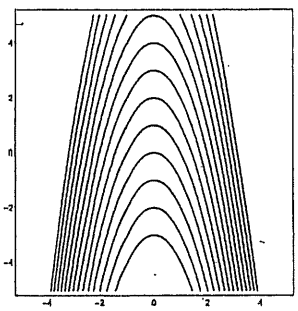
(e) The level curves are parabolas that open in the positive x direction

this is the equation of a line through

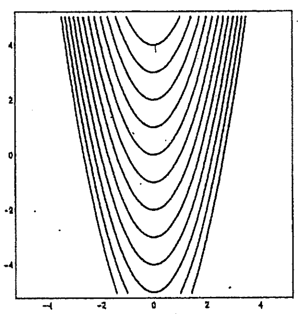
Choose a constant c and setup the equation $f(x,y) = y^2 + x = c$. This can be written as $\boxed{x = -y^2 + c}$

7. Which of these pictures shows the level curves for the function

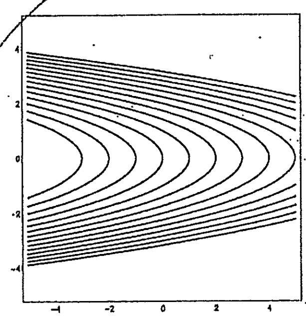
$$f(x, y) = y^2 + x$$



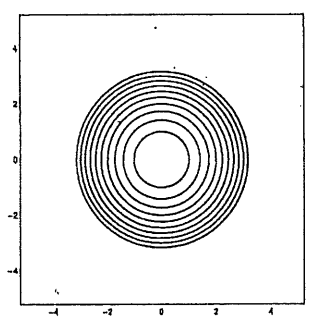
(a)



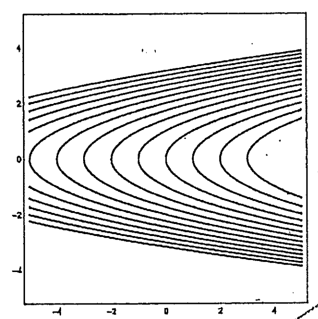
(b)



(c)



(d)



(e)

this is the equation of a parabola that opens to the left. So the answer is (c)

8. Evaluate the limit

$$\lim_{(x,y)\to(1,1)} \frac{x^3 + y^3}{x^4 + xy + y^2}$$

Rational functions are continuous whenever they are defined. This function is defined at (1,1) so it is continuous. This means that the limit can be evaluated by simply evaluating at (1,1)

Possibilities:

(a) 1

(b) -1

(c) 0

$\boxed{\text{(d)}\ \ 2/3}$

(e) The limit does not exist

$$\lim_{(x,y)\to(1,1)} \frac{x^3+y^3}{x^4+xy+y^2} = \frac{1^3+1^3}{1^4+1\cdot1+1^2} = \frac{2}{3}$$

9. Consider the function $g(x,y) = x^5 + xy + y^5$. What is the value of

$$\frac{\partial^2 g}{\partial x \partial y}$$

at the point $(2,3)$?

$$\frac{\partial g}{\partial y} = 0 + x + 5y^4$$

Hence $\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right)$

$$= \frac{\partial}{\partial x}\left(x + 5y^4\right) = 1$$

The point $(2,3)$ is irrelevant!

Possibilities:

(a) 0

$\boxed{\text{(b) } 1}$

(c) -1

(d) 2

(e) -2

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10. Let $f(x,y) = \sqrt{8 - 3x^2 - 4y^2}$. Find the linear approximation $L(x,y)$ to this function at the point $x_0 = 1$ and $y_0 = 1$.

$$f_x = \frac{\partial f}{\partial x} = \frac{1}{2\sqrt{8-3x^2-4y^2}} \cdot (-6x) = \frac{-3x}{\sqrt{8-3x^2-4y^2}}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{1}{2\sqrt{8-3x^2-4y^2}} (-8y) = \frac{-4y}{\sqrt{8-3x^2-4y^2}}$$

Possibilities:

$f(1,1) = 1$; $f_x(1,1) = -3$

and $f_y(1,1) = -4$

Thus $L(x,y) = 1 - 3(x-1) - 4(y-1)$

$= -3x - 4y + 8$

(a) $L(x,y) = -3x - 4y + 1$

(b) $L(x,y) = -4x - 3y + 1$

(c) $L(x,y) = -6x - 8y + 8$

$\boxed{\text{(d) } L(x,y) = -3x - 4y + 8}$

(e) $L(x,y) = -6x - 8y + 15$

11. It is given that the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 4$, with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

(*a*) Find values c_1 and c_2 such that $\begin{bmatrix} 4 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

We need to solve $\begin{cases} c_1 + 2c_2 = 4 \\ -c_1 + 3c_2 = 1 \end{cases}$

OR $\underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{3+2} \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3/5 & -2/5 \\ 1/5 & 1/5 \end{bmatrix}$$

thus $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3/5 & -2/5 \\ 1/5 & 1/5 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(*b*) Use part (*a*) and the properties of eigenvalues and eigenvectors to compute $A^{10} \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

thus $\begin{bmatrix} 4 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$A^{10} \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix} = A^{10} \cdot \left(2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right)$$

$$= 2 A^{10} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + A^{10} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= 2 (-1)^{10} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 4^{10} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \cdot 4^{10} + 2 \\ 3 \cdot 4^{10} - 2 \end{bmatrix}$$

$= \begin{bmatrix} 2{,}907{,}154 \\ 3{,}145{,}726 \end{bmatrix}$ OR

pts: /10

least squares solution $\boxed{P_1 = 1.75 P_0 + 0.625}$

12. Suppose a scientist has four colonies of the same bacteria. She wants to estimate the growth rate of this species of bacteria. To do this, she measures the population of each colony at time $t = 0$ and one day later at time $t = 1$. The data from her measurements are listed below:

Colony	P_0	P_1
A	9	17
B	2	4.5
C	4	9
D	5	7

Find the *least squares approximation* of the form

$$P_1 = m\,P_0 + b\,,$$

that expresses the population P_1 at time $t = 1$ as a function of the population P_0 at $t = 0$.

Substituting the values in $P_1 = mP_0 + b$ we get the system:

$$\begin{aligned} 17 &= 9m + b \\ 4.5 &= 2m + b \\ 9 &= 4m + b \\ 7 &= 5m + b \end{aligned} \qquad \text{OR} \qquad \underbrace{\begin{bmatrix} 9 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix}}_{A} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 17 \\ 4.5 \\ 9 \\ 7 \end{bmatrix}$$

Multiply both sides by A^T to obtain

$$\begin{bmatrix} 9 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 9 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 17 \\ 4.5 \\ 9 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 126 & 20 \\ 20 & 4 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 233 \\ 37.5 \end{bmatrix}$$

multiply both sides by the inverse

$$\begin{bmatrix} m \\ b \end{bmatrix} = \frac{1}{104} \begin{bmatrix} 4 & -20 \\ -20 & 126 \end{bmatrix} \begin{bmatrix} 233 \\ 37.5 \end{bmatrix} \approx \begin{bmatrix} 1.75 \\ 0.625 \end{bmatrix}$$

pts: /10

13. Compute the partial derivatives g_x and g_y for the function

$$g(x,y) = \frac{x^2 y}{x+y^3}.$$

$$\frac{\partial g}{\partial x} = g_x = \frac{2xy(x+y^3) - x^2y(1)}{(x+y^3)^2}$$

$$= \frac{2x^2y + 2xy^4 - x^2y}{(x+y^3)^2}$$

$$= \boxed{\frac{x^2y + 2xy^4}{(x+y^3)^2}}$$

$$\frac{\partial g}{\partial y} = g_y = \frac{x^2(x+y^3) - x^2y(3y^2)}{(x+y^3)^2}$$

$$= \frac{x^3 + x^2y^3 - 3x^2y^3}{(x+y^3)^2}$$

$$= \boxed{\frac{x^3 - 2x^2y^3}{(x+y^3)^2}}$$

pts: /10

14. The *number of prey encounters per predator* is a function of the prey density N, the time available for searching for prey T and the handling time of each prey item per predator T_h.

Experimentally, it has been observed that for some predators the graph of the above functional response, as a function of the prey density N, is sigmoidal (that is, S-shaped). See the picture on the side.

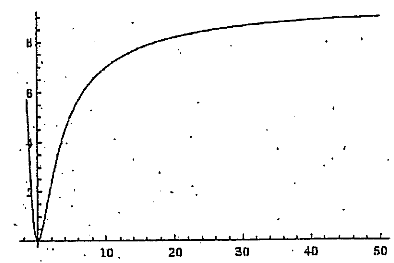

Figure: Graph of $f(N,T,T_h)$ as a function of N when $T = 2.4$ hours, $T_h = 0.2$ hours, $b = 0.8$, and $c = 0.5$. That is, $f(N, 2.4, 0.2) = \frac{1.536N^2}{1+0.5N+0.16N^2}$.

One possible function for the number of prey encounters per predator is

$$f(N,T,T_h) = \frac{b^2N^2T}{1+cN+bT_hN^2},$$

where b and c are positive constants. Notice that T, T_h, and N are also always positive.

(a) Compute $\partial f/\partial N$ and show that it is always positive. How does an increase in the prey density N affect the number of prey encounters per predator?

$$\frac{\partial f}{\partial N} = \frac{2b^2NT(1+cN+bT_hN^2) - b^2N^2T(c+2bT_hN)}{(1+cN+bT_hN^2)^2} = \frac{2b^2NT + cb^2N^2T}{(1+cN+bT_hN^2)^2}$$

$$= \boxed{\frac{b^2NT(2+cN)}{(1+cN+bT_hN)^2}}$$

always positive as quotient of positive quantities

An increase in prey density implies an increase in prey encounters

(b) Compute $\partial f/\partial T$ and show that it is always positive. How does an increase in the time T available for search affect the number of prey encounters per predator?

$$\frac{\partial f}{\partial T} = \boxed{\frac{b^2N^2}{1+cN+bT_hN^2}}$$

it is always positive as it is the quotient of positive quantities

This means that an increase in time T available for search implies an increase in prey encounter

(c) Compute $\partial f/\partial T_h$ and show that it is always negative. How does an increase in the handling time T_h affect the number of prey encounters per predator?

$$f = b^2N^2T(1+cN+bN^2T_h)^{-1} \Longrightarrow$$

$$\frac{\partial f}{\partial T_h} = -b^2N^2T(1+cN+bN^2T_h)^{-2}(bN^2)$$

$$= \boxed{\frac{-b^3N^4T}{(1+cN+bN^2T_h)^2}}$$

always negative because of the negative sign

an increase of the handling time implies a decrease in the number of prey encounters

pts: /10

15. What is the linear approximation to the function

$$f(x,y) = \ln(x^2+y^2)$$

at the point $x_0 = 1$ and $y_0 = 0$? Use your result to approximate the value of $f(1.1, 0.1)$.

(a) $L(x,y) = 2(x-1)$

We can evaluate $f(1,0) = \ln(1^2+0^2) = \underline{\underline{0}}$

Moreover

$$\frac{\partial f}{\partial x} = f_x = \frac{1}{x^2+y^2} \cdot 2x = \frac{2x}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = f_y = \frac{1}{x^2+y^2} \cdot 2y = \frac{2y}{x^2+y^2}$$

$f_x(1,0) = 2$; $f_y(1,0) = 0$ so

(b) $f(1.1, 0.1) \approx L(1.1, 0.1) = 0.2$

$L(x,y) = 0 + 2(x-1) + 0 \cdot (y-0)$

$\therefore \boxed{L(x,y) = 2(x-1)}$

$f(1.1, 0.1) \approx L(1.1, 0.1) = 2(1.1-1) =$

$= \boxed{0.2}$

notice that the exact value of f at $(1.1, 0.1)$ is 0.198850

pts: /10

Bonus. For a square matrix A, an *eigenvector* of A is a non-zero column vector $\mathbf{v}$ that satisfies an equation of the form

$$A\mathbf{v} = \lambda \mathbf{v}$$

for some number λ. The number λ is called the *eigenvalue* corresponding to the eigenvector $\mathbf{v}$. Find a 2×2 matrix A that has eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix},$$

and so that the eigenvalue of $\mathbf{v}_1$ is $\lambda_1 = -1$, and the eigenvalue of $\mathbf{v}_2$ is $\lambda_2 = \frac{1}{2}$.

We need to find a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{OR} \quad \begin{cases} a-b=-1 \\ c-d=1 \end{cases}$$

AND

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \text{OR} \quad \begin{cases} 2a+4b=1 \\ 2c+4d=2 \end{cases}$$

This means that we need to solve the two systems

$$\begin{cases} a-b=-1 \\ 2a+4b=1 \end{cases} \quad \text{and} \quad \begin{cases} c-d=1 \\ 2c+4d=2 \end{cases}$$

check that $a = -\frac{1}{2}$ $b = \frac{1}{2}$

check that $c = 1$ and $d = 0$

thus $A = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$

pts: /10