Name: Answer Key

MA 138
Calculus II with Life Science Applications
FINAL EXAM

Spring 2023
05/03/2023

Sect. #: ____________

Do not remove this answer page — you will return the whole exam. No books or notes may be used. Use the backs of the question papers for scratch paper. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The **first part of the exam** consists of 10 multiple choice questions, each worth 5 points. Record your answers on this page by filling in the box corresponding to the correct answer. For example, if (b) is correct, you must write

a ■ c d e

Do not circle answers on this page, but please do circle the letter of each correct response in the body of the exam. It is your responsibility to make it CLEAR which response has been chosen. You will not get credit unless the correct answer has been marked on **both** this page **and** in the body of the exam.

The **second part of the exam** consists of five open-response questions and one bonus question. When answering these questions, check your answers when possible. Clearly indicate your answer and the reasoning used to arrive at that answer. *Unsupported answers may receive NO credit.*

1. ■ b c d e

2. a b c ■ e

3. ■ b c d e

4. a ■ c d e

5. a b c d ■

6. a b ■ d e

7. a ■ c d e

8. a b c d ■

9. a b c d ■

10. a b c ■ e

GOOD LUCK!

QUESTION	SCORE	OUT OF
Multiple Choice		50 pts
11.		10 pts
12.		10 pts
13.		10 pts
14.		10 pts
15.		10 pts
Bonus.		10 pts
TOTAL		100 pts

Please make sure to list the correct section number on the front page of your exam. In case you forgot your section number, consult the following table:

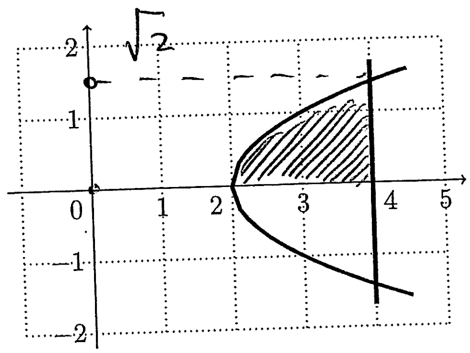
Sections #	Lecturer	Time/Location
001–004	Alberto Corso	MWF 10:00 am - 10:50 am, CB 110
Section #	**Recitation Instructor**	**Time/Location**
001	Kathryn Hechtel	TR 08:00 am - 08:50 am, CB 307
002	Kathryn Hechtel	TR 09:00 am - 09:50 am, CB 307
003	Davis Deaton	TR 10:00 am - 10:50 am, CB 307
004	Davis Deaton	TR 11:00 am - 11:50 am, CB 307

1. Let R be the region in the first quadrant bounded by the x-axis, the graph of $x = y^2 + 2$, and the line $x = 4$. Which of the following integrals gives the area of R?

the intersect points between $x=4$ and $x = y^2+2$ are

$\Longleftrightarrow 4 = y^2 + 2 \Longleftrightarrow y^2 = 2$

$\Longleftrightarrow y = \pm\sqrt{2}$



Possibilities:

(a) $\boxed{\displaystyle\int_0^{\sqrt{2}} \left[4 - (y^2+2)\right] dy}$

(b) $\displaystyle\int_0^{\sqrt{2}} \left[(y^2+2) - 4\right] dy$

(c) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \left[4 - (y^2+2)\right] dy$

(d) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \left[(y^2+2) - 4\right] dy$

(e) $\displaystyle\int_2^{4} \left[4 - (y^2+2)\right] dy$

Hence the area in the first quadrant is given by

$$\int_0^{\sqrt{2}} \left[4 - (y^2+2)\right] dy$$

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2. Suppose I (correctly) find the following expression after using integration by parts once:

$$x^2 \sin(x) - \int 2x \sin(x)\, dx.$$

Which of the following integrals could I have been trying to evaluate?

(a) $\displaystyle\int x^2 \sin(x)\, dx$

(b) $\displaystyle\int 2x \cos(x)\, dx$

(c) $\displaystyle\int 2x \sin(x)\, dx$

(d) $\boxed{\displaystyle\int x^2 \cos(x)\, dx}$

(e) None of the above

3. The improper integral $\int_e^{\infty} \frac{1}{x \ln x}\, dx$

Use the substitution $\boxed{u = \ln x}$ so that $\boxed{du = \frac{1}{x} dx}$. Hence we need to compute

$$= \int_{1}^{+\infty} \frac{1}{u}\, du = \lim_{b \to \infty} \int_1^b \frac{1}{u}\, du =$$

Possibilities:

$$= \lim_{b \to \infty} \ln|u| \Big|_1^b =$$

$$= \lim_{b \to \infty} \left[\ln(b) - \ln(1) \right] = \infty$$

diverges

$\boxed{\text{(a) diverges}}$

(b) converges and equals $\frac{\pi}{2}$

(c) converges and equals 2

(d) converges and equals $\sqrt{5}$

(e) converges and equals 4

---

4. The equation $y^3 = cx$ is the general solution of

By implicit differentiation $\frac{d}{dx}[y^3] = \frac{d}{dx}[cx]$

$$\Longleftrightarrow 3y^2 \frac{dy}{dx} = c$$

substitute now $c = y^3/x$.

To get

$$\frac{dy}{dx} = \frac{y^3/x}{3y^2} = \frac{y}{3x}$$

(a) $\frac{dy}{dx} = \frac{3y}{x}$

$\boxed{\text{(b) } \frac{dy}{dx} = \frac{y}{3x}}$

(c) $\frac{dy}{dx} = \frac{3x}{y}$

(d) $\frac{dy}{dx} = \frac{x}{3y}$

(e) None of the above.

Note: you can also substitute $y = \sqrt[3]{cx}$ in

5. Two different strains of bacteria sometimes feed on chemicals excreted by one another, i.e. strain A feeds on chemical produced by strain B and vice versa. This phenomenon is referred to as *cross-feeding.* Suppose that the amount of strain A bacteria is governed by

$$\frac{dP}{dt} = P(1-P)(3P-6). = g(P)$$

Using the analytic approach we see that the stable equilibrium point(s) is (are):

Possibilities:

(a) $\widehat{P} = 0$

(b) $\widehat{P} = 1$

(c) $\widehat{P} = 3$

(d) $\widehat{P}_1 = 0$ and $\widehat{P}_2 = 6$

(e) $\boxed{\widehat{P}_1 = 0 \text{ and } \widehat{P}_2 = 2}$

$g(P) = P(1-P)(3P-6) = -3P^3 + 9P^2 - 6P$

There are 3 equilibria; $\hat{P} = 0, 1, 2$

$g'(P) = -9P^2 + 18P - 6$

$\underbrace{g'(0) = -6}_{\text{stable}}$; $g'(1) = 3$; $\underbrace{g'(2) = -6}_{\text{stable}}$

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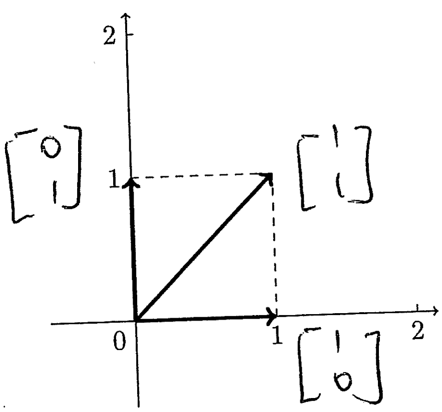
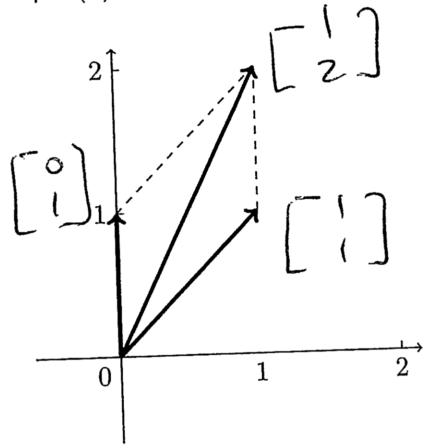
6. Vertical Shear Transformation: Suppose a linear map $T(\mathbf{x})$ transforms the unit square depicted on the left into the shape on the right.





Which of the following is the 2×2 matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for any 2×1 vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$?

Possibilities:

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(c) $\boxed{A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}$

(d) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(e) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

note that

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

7. A medical statistician wanted to examine the relationship between the amount of sunshine (x) in hours, and incidence of skin cancer (y). He expects a linear relationship between these quantities; that is: $y = mx + b$ for some constants m and b.

As an experiment he found the number of skin cancer cases detected per 100,000 of population and the average daily sunshine in six counties around the country. These data are shown below.

Average Daily Sunshine	5	7	7	8	4	3
Skin Cancer per 100,000	7	11	12	15	7	5

Use the least squares method to estimate the number of skin cancer cases per 100,000 people who live in a county that gets 6 hours of sunshine on average.

$$\underbrace{\begin{bmatrix} 5 & 1 \\ 7 & 1 \\ 7 & 1 \\ 8 & 1 \\ 4 & 1 \\ 3 & 1 \end{bmatrix}}_{A} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 12 \\ 15 \\ 7 \\ 5 \end{bmatrix}$$

Multiply both side by A^T. We get

$$\begin{bmatrix} 212 & 34 \\ 34 & 34 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 350 \\ 57 \end{bmatrix}$$

Solve to get $y = \frac{54}{29}x - \frac{61}{58}$

$y(6) = 10.1207$

Possibilities:

(a) 9.60351

$\boxed{\text{(b) } 10.12069}$

(c) 10.90437

(d) 11.02057

(e) None of the above

8. **Holling 's disk equation:** Holling (1959) derived an expression for the number of prey items P_e eaten by a predator during an interval T as a function of prey density N and the handling time T_h of each prey item:

$$P_e = P_e(N, T, T_h) = \frac{aNT}{1 + aT_hN}.$$

Here, a is a positive constant called the *predator attack rate*.

Find $\frac{\partial P_e}{\partial N}$.

$$\frac{\partial P_e}{\partial N} = \frac{aT(1+aT_hN) - aNT(aT_h)}{(1+aT_hN)^2}$$

$$= \frac{aT + \cancel{a^2TT_hN} - \cancel{a^2TT_hN}}{(1+aT_hN)^2}$$

$$= \frac{aT}{(1+aT_hN)^2}$$

Possibilities:

(a) $\frac{aN}{1+aT_hN}$

(b) $\frac{-a^2N^2T}{(1+aT_hN)^2}$

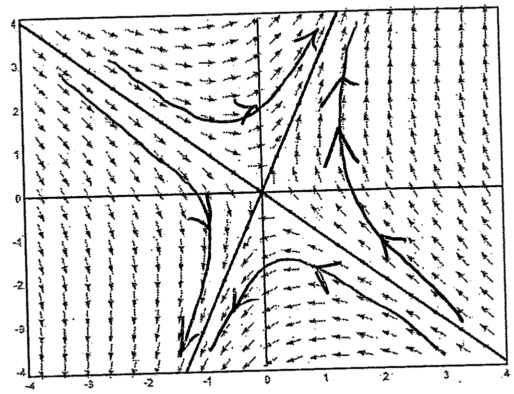
(c) $\frac{aT}{aT_h}$

(d) $\frac{aNT(aT_h) - aT(1+aT_hN)}{1+aT_hN}$

$\boxed{\text{(e) } \frac{aT}{(1+aT_hN)^2}}$

9. Consider a system of linear autonomous differential equations of the form $\frac{d\mathrm{x}}{dt} = A\mathrm{x}$, where $\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and A is a 2×2 matrix with constant entries.

The direction field of the above system is shown on the right-hand side, and the lines through the eigenvectors of A are graphed as well.



Which of the following could be the characteristic polynomial of the matrix A?

(**Hint:** analyze the eigenvalues for each characteristic polynomial.)

(a) and (d) have complex eigenvalues
(b) has real eigenvalues both negative
(c) has real eigenvalues both positive

$\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$ has real eigenvalues with opposite signs

Possibilities:

(a) $\lambda^2 - 2\lambda + 6$
(b) $\lambda^2 + 5\lambda + 6$
(c) $\lambda^2 - 5\lambda + 6$
(d) $\lambda^2 + 9$
(e) $\boxed{\lambda^2 - \lambda - 6}$

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10. How many equilibrium points does this non-linear system of differential equations have?

$$\frac{dx}{dt} = (y + x)(y - x)$$
$$\frac{dy}{dt} = (x - 2)(x - 3)$$

We need to solve the system

$$\begin{cases} (y+x)(y-x) = 0 \\ (x-2)(x-3) = 0 \end{cases}$$

From the second equation we get $\hat{x} = 2$ OR $\hat{x} = 3$. Substitute in [illegible] We get $(2,2); (2,-2); (3,3); (3,-3$

Possibilities:

(a) 1
(b) 2
(c) 3
(d) $\boxed{4}$
(e) 5

Solution: $y = \dfrac{4x}{x-2}$

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11. Separate variables and use partial fractions to solve the following differential equation

$$\frac{dy}{dx} = \frac{-2y}{x(x-2)}$$

with initial condition $y(4) = 8$.

$\dfrac{dy}{dx} = \dfrac{-2y}{x(x-2)} \iff \dfrac{1}{y}\,dy = \dfrac{-2}{x(x-2)}\,dx$

We need to integrate both sides:

$\displaystyle\int \frac{1}{y}\,dy = \int \frac{-2}{x(x-2)}\,dx$

Note that $\dfrac{-2}{x(x-2)} = \dfrac{A}{x} + \dfrac{B}{x-2} = \dfrac{A(x-2)+Bx}{x(x-2)}$

requires $-2 = A(x-2) + Bx$. Set $x=0$ to get $-2 = A(-2)$ OR $A=1$. Set $x=2$ to get $-2 = B(2)$ OR $B=-1$. Thus

$\displaystyle\int \frac{1}{y}\,dy = \int \left(\frac{1}{x} - \frac{1}{x-2}\right)dx \iff$

$\ln|y| = \ln|x| - \ln|x-2| + C \iff$

$\ln|y| = \ln\left|\dfrac{x}{x-2}\right| + C$. Take the exp. of both sides $|y| = \left|\dfrac{x}{x-2}\right| \cdot e^{C}$ OR

$y = \pm e^{C}\,\dfrac{x}{x-2}$; $y = D \cdot \dfrac{x}{x-2}$ is the general solution

pts: /10

$x = 9/2 - \frac{5}{2}t$; $y = -19/6 + \frac{13}{6}t$, $z = t$ any value

12. Solve the linear system using the Gaussian Elimination Algorithm:

$$\begin{cases} 2x + 6y - 8z &= -10 \\ 2x \qquad + 5z &= 9 \\ 6x + 6y + 2z &= 8 \end{cases}$$

If the system has infinitely many solutions, you must write all solutions in terms of a parameter $t \in \mathbb{R}$.
If it has no solutions, ~~explain why~~.

$$\left[\begin{array}{ccc|c} 2 & 6 & -8 & -10 \\ 2 & 0 & 5 & 9 \\ 6 & 6 & 2 & 8 \end{array}\right] \longrightarrow \tfrac{1}{2}R_1 \left[\begin{array}{ccc|c} 1 & 3 & -4 & -5 \\ 2 & 0 & 5 & 9 \\ 6 & 6 & 2 & 8 \end{array}\right]$$

$$\begin{array}{c} \\ R_2-2R_1 \\ R_3-6R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -4 & -5 \\ 0 & -6 & 13 & 19 \\ 0 & -12 & 26 & 38 \end{array}\right] \longrightarrow \begin{array}{c} \\ \\ R_3-2R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -4 & -5 \\ 0 & -6 & 13 & 19 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ -\frac{1}{6}R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -4 & -5 \\ 0 & 1 & -13/6 & -19/6 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{c} R_1-3R_2 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 5/2 & 9/2 \\ 0 & 1 & -13/6 & -19/6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

i.e. $\begin{cases} x \qquad + 5/2\, z = 9/2 \\ \qquad y - 13/6\, z = -19/6 \end{cases}$ OR

$x = 9/2 - 5/2\, z$

$y = -19/6 + \frac{13}{6} z$

$z = t$ = any value

pts: /10

13. Find the eigenvalues and the corresponding eigenvectors of $A = \begin{bmatrix} 4 & -7 \\ 2 & -5 \end{bmatrix}$.

In particular, determine the equations of the lines through the origin in the direction of the eigenvectors and graph the lines together with the eigenvectors in the chart below.

$\det \begin{bmatrix} 4-\lambda & -7 \\ 2 & -5-\lambda \end{bmatrix} = 0 \iff (4-\lambda)(-5-\lambda) + 14 = 0$

$\lambda^2 + \lambda - 6 = 0 \iff (\lambda+3)(\lambda-2) = 0$

so that $\lambda_1 = -3$ and $\lambda_2 = 2$

eigenvector corresponding to $\underline{\lambda_1 = -3}$. We need to solve

$\begin{bmatrix} 4 & -7 \\ 2 & -5 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{v_1} = -3 \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{v_1} \iff \begin{cases} 4x - 7y = -3x \\ 2x - 5y = -3y \end{cases}$

$\iff x - y = 0$ OR $y = x$ so we can choose $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

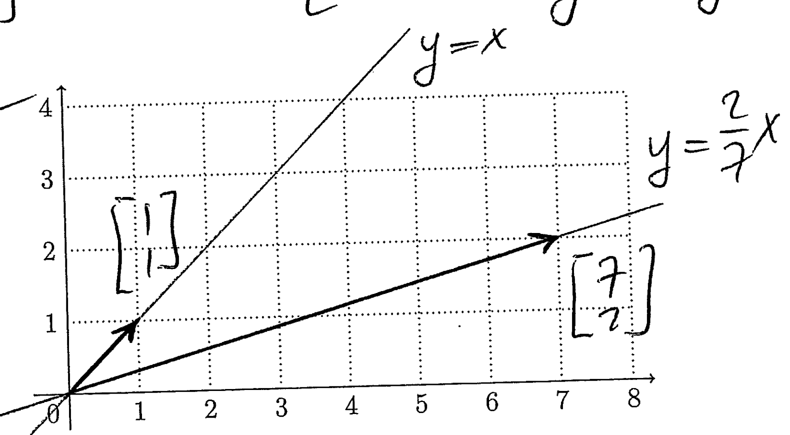
eigenvector corresponding to $\underline{\lambda_2 = 2}$. We need to solve

$\begin{bmatrix} 4 & -7 \\ 2 & -5 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{v_2} = 2 \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{v_2} \iff \begin{cases} 4x - 7y = 2x \\ 2x - 5y = 2y \end{cases}$

$\iff 2x - 7y = 0$

OR $y = \frac{2}{7}x$

So we can choose



pts: /10

14. Consider the vector-valued function $\mathbf{f}(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$ where

$$f_1(x,y) = \frac{x}{y} \quad \text{and} \quad f_2(x,y) = \frac{y}{x}.$$

(*a*) Find the linearization of $\mathbf{f}(x,y)$ at the point $(2,-3)$.

$$L(x,y) = f(2,-3) + \begin{bmatrix} \frac{\partial f_1}{\partial x}(2,-3) & \frac{\partial f_1}{\partial y}(2,-3) \\ \frac{\partial f_2}{\partial x}(2,-3) & \frac{\partial f_2}{\partial y}(2,-3) \end{bmatrix} \begin{bmatrix} x-2 \\ y+3 \end{bmatrix}$$

$$\frac{\partial f_1}{\partial x} = \frac{1}{y}; \quad \frac{\partial f_1}{\partial y} = -\frac{x}{y^2}; \quad \frac{\partial f_2}{\partial x} = -\frac{y}{x^2}; \quad \frac{\partial f_2}{\partial y} = \frac{1}{x}$$

$$\therefore L(x,y) = \begin{bmatrix} 2/-3 \\ -3/2 \end{bmatrix} + \begin{bmatrix} -1/3 & -2/9 \\ 3/4 & 1/2 \end{bmatrix} \begin{bmatrix} x-2 \\ y+3 \end{bmatrix}$$

$$= \begin{bmatrix} -2/3 - 1/3(x-2) - 2/9(y+3) \\ -3/2 + 3/4(x-2) + \frac{1}{2}(y+3) \end{bmatrix}$$

(*b*) Use your answer to part (*a*) to approximate $\mathbf{f}(1.9,-2.9)$.

$$f(1.9,-2.9) \approx L(1.9,-2.9) = \begin{bmatrix} -2/3 - 1/3(-0.1) - 2/9(0.1) \\ -3/2 + \frac{3}{4}(-0.1) + \frac{1}{2}(0.1) \end{bmatrix}$$

$$= \begin{bmatrix} -0.6556 \\ -1.525 \end{bmatrix}$$

note that the exact value is

pts: /10

15. Conside the system of linear autonomous differential equations below:

$$\frac{dx_1}{dt} = 4x_1 - 7x_2 \qquad \frac{d}{dt}\underline{x} = \begin{bmatrix} 4 & -7 \\ 2 & -5 \end{bmatrix}\underline{x}$$
$$\frac{dx_2}{dt} = 2x_1 - 5x_2.$$

(a) Find the general solution to the above system

Go and use the results from Problem # 13

$\lambda_1 = -3$, $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $\lambda_2 = 2$, $\underline{v}_2 = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$

Thus the general solution to the linear autonomous system of D.E.s is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underline{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 7 \\ 2 \end{bmatrix} e^{2t}$$

OR
$$\begin{cases} x_1(t) = c_1 e^{-3t} + 7c_2 e^{2t} \\ x_2(t) = c_1 e^{-3t} + 2c_2 e^{2t} \end{cases}$$

(b) Find the particular solution to this system such that $x_1(0) = 13$ and $x_2(0) = 3$.

$$\begin{bmatrix} 13 \\ 3 \end{bmatrix} = \underline{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3\cdot 0} + c_2 \begin{bmatrix} 7 \\ 2 \end{bmatrix} e^{2\cdot 0} \Longleftrightarrow$$

$$\begin{bmatrix} 1 & 7 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 13 \\ 3 \end{bmatrix} \cdots \quad c_1 = -1, \quad c_2 = 2$$

So
$$\begin{cases} x_1(t) = -e^{-3t} + 14e^{2t} \\ x_2(t) = -e^{-3t} + 4e^{2t} \end{cases}$$

(c) Classify the stability of the equilibrium point (0,0).

Since the eigenvalues are real and have opposite sign (0,0) is

pts: /10

Bonus. Consider the system of nonlinear autonomous differential equations below:

$$\frac{dx}{dt} = 2xy - x^2 + 6x \qquad = f_1(x,y)$$
$$\frac{dy}{dt} = (x-2)(y+9). \qquad = f_2(x,y)$$

(*a*) Find all three equilibrium points for the above system.

$$\begin{cases} 2xy - x^2 + 6x = 0 \\ (x-2)(y+9) = 0 \end{cases}$$

From the second equation $\hat{x} = 2$ OR $\hat{y} = -9$.

When $\hat{x} = 2$ the first equation becomes $4y + 8 = 0$ so that $\hat{y} = -2$ $\boxed{(2,-2)}$

when $\hat{y} = -9$ the first equation becomes $-18x - x^2 + 6x = 0 \iff x^2 + 12x = 0$ so $\hat{x} = 0$, $\hat{x} = -12$

$\boxed{(0,-9)}$ OR $\boxed{(-12,-9)}$

(*b*) For the only equilibrium point $(\hat{x}, \hat{y})$ found ~~in part (*a*) with $\hat{x}$~~ > 0, use the analytic approach (stated in the Hartman-Grobman Theorem) to classify its stability.

The equilibrium point we are interested in is $\boxed{(2,-2)}$. We need the Jacobi matrix evaluated at $(2,-2)$

$$Df(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2y - 2x + 6 & 2x \\ y + 9 & x - 2 \end{bmatrix}$$

$$Df(2,-2) = \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix}$$

the determinant is negative so $(2,-2)$ is an unstable equilibrium

pts: /10