QUALIFYING EXAMINATION — MATHEMATICAL STATISTICS

Part II, Afternoon — May 29, 1958

DO FIVE PROBLEMS

1. Let $E(Y_{ij}) = \alpha + \beta_i x_{ij}$, for $i = 1,2$, $j = 1,2$, with $(x_{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

   (a) State additional assumptions sufficient for a test of the hypothesis $\alpha = 0$.

   (b) Give a procedure for this test at the .10 significance level.

   (c) What is the power of this test?

   (d) Ignoring the problem of testing the hypothesis $\alpha = 0$, give a procedure for simultaneous confidence limits, at the .80 confidence level, on all quantities of the form $\alpha + \beta_i x$ for $i = 1,2$ and for all x.

2. (a) State the Gauss-Markov Theorem.

   (b) Prove the theorem.

3. (a) Let $Z = (X,Y)$ have a 2-dimensional normal distribution with mean 0 and unknown covariance matrix. On the basis of n observations on Z describe how you would decide whether or not X and Y are independent. Explain why your procedure seems reasonable.

   (b) For $0 \leq a \leq \frac{3}{20}$ let $Z = (X,Y)$ have a 2-dimensional distribution function $F_a(x,y)$ determined by

$$
\begin{aligned}
P_a\{X = x, Y = y\} &= a && \text{if } x = 1,\ y = 1 \\
&&& \phantom{\text{if }} = 0,\ \ = 1 \\
&&& \phantom{\text{if }} = -1,\ \ = 0 \\
&&& \phantom{\text{if }} = -1,\ \ = -1 \\
&= \frac{a}{3} && \text{if } x = 1,\ y = 0 \\
&&& \phantom{\text{if }} = 0,\ \ = -1 \\
&= \frac{2a}{3} && \text{if } x = -1,\ y = 1
\end{aligned}
$$

Problem 3 continued

$$= \frac{4a}{3} \quad \text{if } x = 1, y = -1$$

$$= 1 - \frac{20}{3} a \quad \text{if } x = 0, y = 0$$

$$= 0 \quad \text{otherwise.}$$

Assuming that a is unknown and given n observations on Z describe how you would decide whether or not X and Y are independent.

4. Let $X_1, \ldots, X_n$ be real random variables satisfying

   (a) X_1 is normal with mean 0 and variance 1,

   (b) $X_i - X_j$ is normal with mean 0 and variance $|i - j|$

   (c) for $i < j \leq k < \ell$, $X_\ell - X_k$ is independent of $X_j - X_i$.

   What is the joint distribution of $X_1, \ldots, X_n$?

5. Let X be a p-dimensional random vector with mean 0 and covariance matrix A. For $p = 1$ Chebyshev's inequality says $P\{|X| > t\} \leq \frac{\sigma^2}{t^2}$ where $\sigma^2 = \text{Var } X$.

   Give and prove a similar inequality for general p.

6. Let X_1, X_2 have the multinomial distribution i.e.,

   for $0 \leq \nu_1, \nu_2 \leq n$, $\nu_1 + \nu_2 \leq n$

$$P\{X_1 = \nu_1, X_2 = \nu_2\} = \frac{n!}{\nu_1! \nu_2! (n - \nu_1 - \nu_2)!} p_1^{\nu_1} p_2^{\nu_2} (1 - p_1 - p_2)^{n - \nu_1 - \nu_2}$$

   Here p_1, p_2 are non-negative with $p_1 + p_2 \leq 1$. Describe a way of utilizing the Hotelling T^2 statistic to test whether or not $p_1 = p_2 = 1/3$.