STATISTICS QUALIFYING EXAMINATION JANUARY 23, 1981

<u>INSTRUCTIONS</u>: Answer <u>ANY FOUR</u> problems, each in a separate blue book.

1. Let $(X_1,\ldots,X_n)$ be a random sample from the uniform distribution on $(\theta,\theta+1)$, where θ is unknown. To test

$$H_0:\ \theta=0 \quad \text{v.}$$
$$H_1:\ \theta>0$$

the following procedure is used: Reject H_0 iff $\max(X_1,\ldots,X_n)>1$ or $\min(X_1,\ldots,X_n)\geq C$ and $\max(X_1 \cdots X_n) \leq 1$

(A) Determine C so that the test will have size α. $[1-C]^n = \alpha$ $c = 1-\sqrt[n]{\alpha}$

(B) Find the power function of the test.

(C) Prove or disprove: If C is chosen so that the test has size α, then it is U.M.P. among all tests of level α.

2. Suppose $(X_1,\ldots,X_n)$ is a random sample from a $N(\mu,1)$ population, with μ unknown. It is desired to estimate the quantity

$$P_\mu\{X>\alpha\} = \frac{1}{\sqrt{2\pi}}\int_\alpha^\infty e^{-(x-\mu)^2/2}\,dx$$
$$= 1-\Phi(\alpha-\mu)$$

on the basis of the observed values $X_1,\ldots,X_n$. Two estimators are proposed:

(1) $\hat\theta = T_n/n$ where $T_n = \sum_{j=1}^{n} 1\{X_j>\alpha\}$, and

(2) $\tilde\theta = 1-\Phi(\alpha-S_n/n)$. ↳ is max likelihood estimate. MLE

(A) Discuss the relative advantages and disadvantages of these two estimators. Consider such properties as

(i) bias ✓ ✗
(ii) asymptotic normality and efficiency
(iii) consistency ✓✓

(B) Find a M.V.U.E. of $P_\mu\{X>\alpha\}$, based on $X_1,\ldots,X_n$.

3. Let $X_1, \ldots, X_n$ be independent Bernoulli random variables with success parameter p.

(A) Show that no unbiased estimator of $1/p$ exists.

(B) There are, however, unbiased <u>sequential</u> estimation procedures. Find one.
HINT: Consider the stopping rule

$$T = \min\{n : X_n = 1\}.$$

$ET = \frac{1}{p}$ $ET = \sum n P(T=n) = \sum n p q^{n-1} = p \sum_1^\infty n q^{n-1} = p \frac{1}{(1-q)^2} = \frac{p}{p^2}$

(C) Once again consider estimators of $1/p$ based on a fixed number n of observations $X_1, \ldots, X_n$. Find a sequence of estimators

$$\theta_n(X_1, \ldots, X_n)$$

which are consistent and asymptotically normal. Is your sequence asymptotically efficient (i.e., <u>best</u> asymptotically normal)?

4. A response variable y is observed at each of n values of a prediction variable x, and it is desired to fit the line $y_i = bx_i$.

(A) Derive the estimate $\hat{b}$ for which

$$\sum_1^n (y_i - bx_i)^2$$

is a minimum.

Let the observations be

x	1	2	3	4	5	6	$\Sigma x_i^2 = 91$
y	1	1	4	3	6	10	$\Sigma x_i y_i = 117$

(C) Find $\hat{b}$ and $\bar{b}$; compare the results. What would happen to each if for $x = 5$, $y = 20$?

$\hat{b} = \frac{\sum_1^n b_i}{n} = \frac{\sum_1^n \frac{y_i}{x_i}}{n}$

5. A machine has two parts A and B that can fail. It is known that their probabilities of failure are

$$p_A = .4$$

in one test

test of independence

$$p_B = .3$$

The parts can be replaced if they fail. In 200 tests, both parts fail 30 times and one fails 92 times.

(A) Test the hypothesis that A and B fail independently of each other.

(B) How would the result change if all you were told was that one part failed 92 times?

6. Suppose X has probability density

$$f_1(x|\theta) = \frac{1}{\sqrt{2\pi}} \cdot e^{-(x-\theta)2/2}$$

$$\text{or} \quad f_2(x|\theta) = \tfrac{1}{2} \cdot e^{-|x-\theta|}$$

where the parameter θ is unknown.

(A) Given n observations $X_1, \ldots, X_n$ on X, test

$$H_0 \div f(x) = f_1(x|\theta) \quad \text{for some } \theta$$

$$\text{vs.} \quad H_1 \div f(x) = f_2(x|\theta) \quad \text{for some } \theta$$

Does your test procedure have any optimality properties?

(B) Show that if the significance level of your test is .05, then the power converges to 1 as $n \to \infty$, uniformly in θ.