COMPREHENSIVE EXAM

Statistical Inference

August 23, 1985

This is a closed book, closed notes exam. Please start all problems on a new sheet of paper.

1. Student's seeking a Master's Level Pass may attempt any five problems.
2. Student's seeking a Ph.D. Level Pass should attempt problems 6 through 10.
3. Student's seeking a pass at both levels must clearly designate those problems to be considered for the Master's Pass.

Part I: Students seeking a M.S. Level Pass should first attempt Problems 1 - 5.

1. [20 Points]

Let $X_1, \ldots, X_n$ be a random sample from $U(0,\theta)$.

(i) [3 points] Find the method of moments estimator of θ.

(ii) [4 points] Find the maximum likelihood estimator of θ.

(iii) [7 points] Which of these estimators has the smallest mean square error? Defend your answer.

(iv) [6 points] Show that both of these estimators are inadmissible. {Hint: Consider an estimator of the form $c\hat{\theta}$, where $\hat{\theta}$ is the m.l.e. of θ.}

2. [20 Points]

(i) [7 points] State and prove the Rao-Blackwell Theorem.

(ii) [3 points] State the Lehman-Schéffe Theorem.

(iii) [10 points] Given $X_1, \ldots, X_n$ are Bernoulli variables with parameter $\theta = P\{X=1\}$. Use both theorems above to find an optimal estimator of $g(\theta) = Var(X)$.

3. [20 Points]

(i) [7 points] State and prove the Neyman-Pearson Lemma.

(ii) [3 points] State when this lemma can be extended to obtain a uniformly most powerful (UMP) test of H_0: $\theta = \theta_0$ versus H_1: $\theta > \theta_0$.

(iii) [5 points] Apply the results above to define a UMP test of H_0: $\lambda = 3$ versus H_1: $\lambda > 3$ when $X_1, \ldots, X_n$ are i.i.d. exponential (λ).

(iv) [5 points] Determine an expression for the power function of this test in terms of the c.d.f. of a well tabulated statistical function.

4. [20 Points]

Given $X_1, \ldots, X_n$ is a random sample from $N(\mu_1, \sigma_1^2)$ and $Y_1, \ldots, Y_m$ is a random sample from $N(\mu_2, \sigma_2^2)$ and the samples are independently chosen.

(i) [2 points each] Define an appropriate test statistic and critical region for each of the following hypotheses:

(a) H_0: $\mu_1 = \mu_2$ versus H_1: $\mu_1 \neq \mu_2$ when it is assumed that $\sigma_1^2 = \sigma_2^2$.

(b) H_0: $\mu_1 = \mu_2$ versus H_1: $\mu_1 \neq \mu_2$ when it cannot be assumed that $\sigma_1^2 = \sigma_2^2$.

(c) H_0: $\sigma_1^2 = \sigma_2^2$ versus H_1 $\sigma_1^2 \neq \sigma_2^2$.

(ii) [7 points] Derive the test in part c above via the likelihood ratio criteria and give an expression for the power function of the test.

(iii) [7 points] Using Satterwhaite's procedure derive the approximate degrees of freedom for the test statistic in part b when the sampling distribution of the test statistic under H_0 is approximated by a student's t distribution.

5. [20 Points]

Let X_1, X_2 and X_3 be i.i.d. $N(0,\theta)$. State the probability distributions of the following random variables:

(i) [2 points] $X_1 + X_2 - 2X_3$

(ii) [3 points] $(X_1 + X_2)^2/2\theta$

(iii) [3 points] $2X_1^2/(X_2^2 + X_3^2)$

(iv) [3 points] $X_1^2/(X_1^2 + X_2^2)$

(v) [3 points] $\sqrt{2}\,\bar{X}/S$,
where $\bar{X} = (X_1 + X_2 + X_3)/3$
and $S^2 = \Sigma(X_i - \bar{X})^2/3$

(vi) [3 points] Conditional distribution of X_1 given X_2

(vii) [3 points] Conditional distribution of $(X_1 + X_2)$ given S^2

COMPREHENSIVE EXAM
Part II
Statistical Inference

1. [20 Points]

Let $\Theta = \left\{\frac{1}{3}, \frac{2}{3}\right\}$, $A = (-\infty, \infty)$, and $L(\theta, a) = (\theta - a)^2$. A coin is tossed once, and the probability of heads is θ. A non-randomized decision rule is represented by a point (x,y) in $A \times A$ = Euclidean plane, where x is the action taken if heads occurs, and y is the action taken if tails occurs.

(i) [3 pts] Show that the rule $d = (x,y)$ has risk function

$$R(\tfrac{1}{3}, d) = \tfrac{1}{9} - \tfrac{2}{9}x + \tfrac{1}{3}x^2 - \tfrac{4}{9}y + \tfrac{2}{3}y^2 \;,$$

$$R(\tfrac{2}{3}, d) = \tfrac{4}{9} - \tfrac{8}{9}x + \tfrac{2}{3}x^2 - \tfrac{4}{9}y + \tfrac{1}{3}y^2 \;.$$

(ii) [3 pts] Show that the rule $d_0 = (\frac{5}{9}, \frac{4}{9})$ is Bayes w.r.t., the prior τ_0 placing probability $\frac{1}{2}$ at each point of Θ.

(iii) [3 pts] Show that d_0 is an equalizer rule.

(iv) [3 pts] Show that d_0 is admissible.

(v) [3 pts] Show that d_0 is a minimax rule.

(vi) [5 pts] Prove that τ_0 is least favorable and that the game has a value.

2. [20 Points]

(i) [10 Points]

Let $X_1, \ldots, X_n$ be i.i.d. $N(\theta, M)$, where M is a fixed constant $0 < M < \infty$. Under squared error loss, derive a minimax estimator of θ, where $\theta \in \Theta$, $\Theta = (-\infty, \infty)$. Is it admissible? Is it unique?

(ii) [10 Points]

Let $X_1, \ldots, X_n$ be i.i.d. F, where $F \in \mathcal{F}_M = \{F: \; V_F \leq M\}$, where V_F is the variance of the distribution F and M is a fixed constant $0 < M < \infty$. Consider the problem of estimating the mean, θ_F, of F under squared error loss.

Show that $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$ is the unique minimax estimator.

3. [20 Points]

Let $\underset{\sim}{X} = (X_1,\ldots,X_n)'$ have i.i.d. elements with p.d.f.

$$f(x;\theta,\beta) = \beta^{-1} I_{(\alpha,\infty)}(x)\exp\left[\frac{-(x-\theta)}{\beta}\right],\ \theta,\ \beta > 0 .$$

Let G be the group of transformations defined by:

$$G = \{g_c:\ c \in \mathbb{R}\},\ \text{where}$$

$$g_c(\underset{\sim}{x}) = \underset{\sim}{x} + c\underset{\sim}{1},\quad \text{where } \underset{\sim}{1} = (1,1,\ldots,1)' \text{ and } c \in \mathbb{R}.$$

(i) [5 pts] Show that a maximal invariant with respect to G is

$$X_{(2)} - X_{(1)}, X_{(3)} - X_{(2)}, \ldots, X_{(n)} - X_{(n-1)}$$

where $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ are the order statistics.

(ii) [5 pts] Show that $D_i = (n-i+1)(X_{(i)} - X_{(i-1)})$, $i = 2,\ldots,n$, are i.i.d. with p.d.f. $\frac{1}{\beta}\exp(\frac{x}{\beta})I_{(0,\infty)}(x)$.

(iii) [10 pts] Derive the UMP invariant test of size α_0, $0 < \alpha_0 < 1$, of

$$H_0:\ \beta \leq 1 \quad \text{vs} \quad H_1:\ \beta > 1 .$$

Be sure to include a justification of the UMP invariant property of your test.

4. [20 Points]

Let (Y_i, X_i), $i=1,\ldots,n$, be i.i.d. random vectors with mean $(\mu\delta,\delta)$, $0 < \delta < 1$, and covariance matrix

$$\Sigma = \begin{pmatrix} \mu^2 & \mu\delta^2 \\ \mu\delta^2 & \delta^2 \end{pmatrix} .$$

(i) [12 pts] Discuss the asymptotic properties (consistency and asymptotic distribution, suitability normalized) of

$$T_n = \frac{\sum_{i=1}^{n} Y_i}{\sum_{i=1}^{n} X_i}$$

(ii) [8 pts] Derive a "large-sample" test of

$$H_0:\ \mu = 1$$

$$H_1:\ \mu \neq 1$$

for $0 < \alpha_0 < 1$. (Here α_0 is the level of significance.)

5. Choose either (a) or (b).

(a) [20 Points]

Let $\underset{\sim}{X} = (X_1, X_2, X_3)'$ be a trivariate normal with

$$EX_1 = 3\theta, \quad \theta > 0,$$
$$EX_i = 0, \quad \text{if } i = 2, 3$$

and dispersion matrix

$$\Sigma = \begin{pmatrix} 2+\theta^2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

The joint density of $X = (X_1, X_2, X_3)'$ is

$$\frac{1}{\theta(2\pi)^{3/2}} \exp\left(-\frac{1}{2}\left[\frac{3^2}{2}(\bar{X}-\theta)^2 + \sum_{i=2}^{3} X_i^2\right]\right), \quad -\infty < X_1, X_2, X_3 < \infty .$$

Let θ_0 and $\theta_1 (\theta_1 > \theta_0)$ be two distinct parameter values. Show that a U.M.P. test of size $\alpha_0, 0 < \alpha_0 < 1$, for $H_0: \theta = \theta_0$ vs $H_1: \theta \geq \theta_1$ does not exist.

(b) Let $X_1, X_2, \ldots, X_n$ be i.i.d. with common density

$$\frac{1}{\beta} \exp\left(\frac{-(x-\theta)}{\beta}\right) \text{ for } x \geq \theta, \ \theta, \ \beta > 0.$$

(i) [15 pts] Show that the U.MP.U. test of size α_0, $0 < \alpha_0 < 1$ for $H_0: \beta = 1$ vs $H_1: \beta \neq 1$, has the acceptance region

$$C_{1\alpha_0} \leq 2 \sum_{i=1}^{n} [X_i - \min(X_1, \ldots, X_n)] \leq C_{2\alpha_0} .$$

(Be sure to indicate how $C_{1\alpha_0}$ and $C_{2\alpha_0}$ could be determined.)

(ii) [5 pts] What is the distribution of the test statistic under H_0?

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6. [20 Points]

Let $\Theta = \left\{\frac{1}{3},\frac{2}{3}\right\}$, $A = (-\infty,\infty)$, and $L(\theta,a) = (\theta-a)^2$. A coin is tossed once, and the probability of heads is θ. A non-randomized decision rule is represented by a point (x,y) in $A \times A$ = Euclidean plane, where x is the action taken if heads occurs, and y is the action taken if tails occurs.

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