COMPREHENSIVE EXAM

Statistical Inference

August 21, 1987

This is a closed book, closed notes exam. *Please start all problems on a new sheet of paper.*

1. Students seeking a Master's Level Pass may attempt any five problems.
2. Students seeking a Ph.D. Level Pass should attempt problems 6 through 10.
3. Students seeking a pass at both levels must clearly designate those problems to be considered for a Master's Pass.

Part I: Students seeking a M.S. Level Pass should first attempt Problems 1—5.

1. [10 Points]

   Let Y_1, Y_2, and Y_3 be independent identically distributed exponential random variables with mean λ. State the distributions of the following random variables (specify all parameters):

   (a) $\dfrac{Y_1+Y_2+Y_3}{3}$

   (b) $\dfrac{2Y_1}{Y_2+Y_3}$

   (c) $\dfrac{Y_1}{Y_1+Y_2}$

   (d) $\min(Y_1, Y_2, Y_3)$

2. [15 Points]

   Let X_1 and X_2 be independent identically distributed random variables with probability density function $f(x)=1/x^2$, $1 \le x < \infty$.

   (a) Find the joint probability density function of $U=X_1X_2$ and $V=X_1$.

   (b) Find the marginal probability density function of U.

   (c) Are U and V independent? Why?

   (d) Does the moment generating function of X_1 exist? Why?

3. a. [10 Points]

State and prove the Neyman-Pearson Lemma (give the version including randomized tests).

b. [15 Points]

Consider the family of discrete distributions specified below:

$$\{f(x;\theta):\ \theta\in\{1,2,3,4\},\ x\in\{1,2,3\}\}.$$

		Value of x		
		1	2	3
	1	.5	.3	.2
Value of	2	.7	.2	.1
θ	3	.4	.3	.3
	4	.1	.8	.1

(i) Determine a level $\alpha=.10$ most powerful test of $H_0: \theta=1$ vs. $H_a: \theta=2$.

(ii) Determine a level $\alpha=.10$ uniformly most powerful test (if it exists) for the hypothesis specified below. If no uniformly most powerful test exists, explain why it does not exist.

$$H_0: \theta=3,\quad H_a: \theta<3$$

(iii) Obtain the level $\alpha=.10$ Likelihood Ratio Test of $H_0: \theta=2$ vs. $H_a: \theta\neq 2$.

4. [25 Points]

Let $Y_1, \ldots, Y_n$ be independent random variables such that the probability density function of Y_i $(1 \le i \le n)$ is

$$f_i(y;\beta) = (2\pi)^{-1/2} \exp\{-(y - x_i\beta)^2/2\}, \quad -\infty < y < \infty,$$

where the x_i are known constants such that $\sum_{i=1}^{n} x_i^2 > 0$ and $\beta \in (-\infty, \infty)$ is an unknown parameter.

(a) Show that the maximum likelihood estimator of β is

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}.$$

(b) Show that $\hat{\beta}$ is sufficient for β.

(c) Obtain the minimum variance unbiased estimator of β^2.

Hint: You may assume that $\hat{\beta}$ has a complete family of distributions.

(d) Show that the joint pdf of $Y_1, \ldots, Y_n$ has a monotone likelihood ratio with respect to $\hat{\beta}$.

(e) Obtain the level α $(0 < \alpha < 1)$ uniformly most powerful test of H_0: $\beta \le \beta_0$ vs. H_a: $\beta > \beta_0$.

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5. [25 Points]

Let $X_1, \ldots, X_n$ be independent random variables with probability density function

$$f(x;\theta) = \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta x}, \quad x>0$$

where $\alpha>0$ is a known constant and $\theta>0$ is unknown.

(a) Show that $\overline{X}/\alpha$ is the maximum likelihood estimator of θ^{-1}.

(b) Obtain the Fisher-Information of $X_1, \ldots, X_n$. (You may assume the regularity conditions hold.) ✓

(c) Use (b) to show that $\overline{X}/\alpha$ is the minimum variance unbiased estimator of θ^{-1}.

(d) Suppose that θ has the prior probability density function

$$g(\theta;\tau,\lambda) = \frac{\lambda^\tau}{\Gamma(\tau)} \theta^{\tau-1} e^{-\lambda\theta}, \quad \theta>0,$$

where $\tau>0$ and $\lambda>0$ are known constants. Show that the Bayes estimator of θ^{-1} under the loss function $L(\theta,a)=(a^{-1}-\theta)^2$ is

$$\hat{d}_b = \frac{\lambda+n\overline{X}}{\tau+n\alpha}.$$

(e) Discuss the relationship between the maximum likelihood estimator and $\hat{d}_b$ as the sample size (n) goes to infinity.

Part II: Students seeking a Ph.D. pass should attempt Problems 6--10 inclusive.

6. [20 Points]

Let $X_1, \ldots, X_n$ be a independent and identically distributed random variables with density

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1}, & 0<x<1,\ \theta>0, \\ 0, & \text{elsewhere}. \end{cases}$$

We wish to test H_0: $\theta=1$ against H_1: $\theta<1$.

(a) Obtain a uniformly most powerful test of H_0 against H_1 at level of significance α, $0<\alpha<1$.

(b) Obtain a uniformly most accurate $1-\alpha$ lower confidence bound for θ.

7. [20 Points]

Let X and Y be independently distributed as Poisson random variables with parameters λ and μ, respectively.

(a) Find the UMP unbiased test of H_0: $\mu=\lambda$ against H_1: $\mu\neq\lambda$. Also give an analytical expression for the power of the above test.

(b) Obtain the uniformly most accurate unbiased confidence interval for μ/λ.

8. [20 Points]

(a) Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with density functions $f_i(x-\theta)$, $i=1, \ldots, n$, where θ denotes the unknown location parameter. We wish to test

$X_n - X_1, \cdots X_n - X_{n-1}$

$$H_0: f(x) = \begin{cases} 1, & \theta - ½ \le x \le \theta + ½, \\ 0, & \text{elsewhere} \end{cases}$$

versus

$$H_1: f(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2), \quad -\infty < x < \infty.$$

Construct a most powerful univariant test of H_0 against H_1.

(b) Let $X_1, \ldots, X_n$ be independent and identically distributed $N(\mu, \sigma^2)$ random variables. Find the lower bounds for variances of unbiased estimators of $\theta_1 = \mu$, $\theta_2 = \sigma^2$. Do the minimum variance unbiased estimators attain these lower bounds? Justify your answer.

$X_1 \cdots X_n \rightarrow X_n, \; X_n - X_1 \cdots$

$X_1, \; X_2 - X_1, \; \cdots \; X_n - X_1$

$f(x_1 \cdots x_n)$ $\quad \dfrac{\partial(x_1 \cdots x_n)}{\partial(x_1 \; x_2 - x_1 \cdots x_n - x_1)}$ $\quad dx_1 \, d(x_2 - x_1) \cdots d(x_n - x_1)$

$\dfrac{\partial x_2}{\partial(x_2 - x_1)} \begin{pmatrix} 1 & — \\ & \end{pmatrix}$ Jacobi = 1

$X \,.\, X+Y$
$Y \,.\, X-Y$

$\dfrac{\partial(x_1 \, x_2 - x_1 \cdots)}{\partial(x_1 \cdots x_n)} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & & 1 \end{pmatrix} = 1$

9. [20 Points]

(a) Suppose X has density $f(x,\theta)$ and the parameter Θ has prior density $\lambda(\theta)$. Show that for the loss function

$$L(\theta,d) = \omega(\theta)[d-g(\theta)]^2, \quad \omega(\theta)>0,$$

the Bayes estimator of a real-valued function $g(\theta)$ is given by

$$\delta_\lambda(x) = \frac{E[\omega(\Theta)g(\Theta)|x]}{E[\omega(\Theta)|x]}.$$

(b) Let X have binomial distribution with n (known) trials and probability of 'success' θ. Obtain the Bayes estimator of θ for the loss function $L(\theta,d) = \dfrac{(d-\theta)^2}{\theta(1-\theta)}$, $0<\theta<1$, for the prior density

$$\lambda(\theta) = \frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}, \quad 0<\theta<1,$$

where $\alpha>0$, $\beta>0$.

(c) Hence show that the MLE $\delta(X) = \dfrac{X}{n}$ is a minimax estimator of θ.

10. [20 Points]

Let $X=(Y,Z)=(Y_1,\ldots,Y_m;Z_1,\ldots,Z_n)$, where Y_i's and Z_j's are independently distributed as $N(\theta_1,1)$ and $N(\theta_2,1)$, respectively, for $i=1,\ldots,m$ and $j=1,\ldots,n$. Consider the transformations

$$X'=g_{a,b}(X),\quad -\infty<a,b<\infty,$$

where

$$Y_i'=Y_i+a\,,\quad Z_j'=Z_j+b\,,\quad i=1,\ldots,m\,,\quad j=1,\ldots,n\,.$$

Let $\theta=(\theta_1,\theta_2)$ and it is desired to estimate $h(\theta)=\theta_1-\theta_2$.

(i) State the induced transformations $\bar{g}_{a,b}$ on the parametric space Ω of θ and $g^*_{a,b}$ on the space of values of estimators $\delta(X)$.

(ii) Show that the loss function $L(\theta;d)$ to estimate $h(\theta)$ is invariant if and only if $L(\theta;d)=P(d-\theta_1+\theta_2)$, where P is some function.

(iii) When is the estimator $\delta(X)$ of $h(\theta)$ said to be equivariant?

(iv) Prove that the risk function of any equivariant estimator is independent of θ.

(v) Give any equivariant estimator of $L(\theta)$ based on the complete sufficient statistic. What is your guess for the MRE (minimum risk equivariant) estimator for $h(\theta)$?

(Hint: Proofs are not required for part (v).)