COMPREHENSIVE EXAM
Statistical Inference
August 19, 1988

This is a closed book, closed notes exam. *Please start all problems on a new sheet of paper.*

1. Students seeking a Master's Level Pass may attempt any five problems.
2. Students seeking a Ph.D. Level Pass should attempt problems 6 through 10.
3. Students seeking a pass at both levels must clearly designate those problems to be considered for a Master's Pass.

Part I: Students weeking a M.S. level pass should first attempt Problems 1-5.

1. (20 Points)

Let X be a random variable with probability density function

$$f(x) = 2xe^{-x^2}, \quad x > 0.$$

(a) Prove that all positive moments of X exists.

(b) Derive the k^{th} moments of X.

3. (20 Points)

Let X be Poisson random variable with mean μ. Consider the problem of estimating μ^2.

(a) Show that $T = X^2 - X$ is the unique minimum variance unbiased estimator (UMVUE) of μ^2.

(b) What is the maximum likelihood estimator (m.l.e.) of μ^2?

(c) Is T efficient? Is the m.l.e. efficient? Why?

[*Hint:* $\mathrm{Var}(T) = 4\mu^3 + 2\mu^2$.]

2. (20 Points)

Let $X_1, \ldots, X_n$ be i.i.d. with cumulative distribution function

$$F(x;\theta_1,\theta_2) = 1 - \left(\frac{\theta_1}{x}\right)^{\theta_2}, \quad x \geq \theta_1,$$

where θ_1 and θ_2 are positive.

(a) Find a statistic, say

$$\mathrm{T}(X_1, \ldots, X_n) = \big(T_1(X_1, \ldots, X_n),\ T_2(X_1, \ldots, X_n)\big),$$

which is sufficient for (θ_1,θ_2).

(b) Derive the maximum likelihood estimator of (θ_1,θ_2).

(c) Prove that $\min(X_1, \ldots, X_n)$ is consistent in probability for estimating θ_1.

4. (20 Points)

A coin with unknown probability of success θ is tossed twice. Consider the statistical problem involving two decisions d_1 and d_2 with loss

$$W(\theta,d_1) = \begin{cases} 0 & \text{if } \theta \leq \frac{1}{2} \\ 1 & \text{if } \theta > \frac{1}{2}, \end{cases}$$

$$W(\theta,d_2) = \begin{cases} 2 & \text{if } \theta \leq \frac{1}{2} \\ 0 & \text{if } \theta > \frac{1}{2}, \end{cases}$$

i.e., d_1 corresponds to the statement that the coin is not biased towards heads and d_2 the opposite.

Derive the Bayes procedure with respect to the prior $\pi(\theta = \frac{1}{3}) = 0.2$, $\pi(\theta = \frac{3}{4}) = 0.8$.

5. (20 Points)

(a) Suppose $X_1, \ldots, X_n$ are uncorrelated random variables with $EX_i = a_i\mu$, $\text{Var}(X_i) = b_i\sigma^2$, $i = 1,2, \ldots, n$ where a_i and $b_i > 0$ are known. Show that the best linear unbiased estimator of μ (i.e. the minimum variance estimator in the class of all linear unbiased estimators) is

$$\sum_{i=1}^{n}(a_iX_i/b_i)/(\sum_{i=1}^{n}a_i^2/b_i).$$

(Clearly state any theorems used.)

(b) Two physics students, Nancy and John, measure the distance an object falls in a second (starting at rest). Nancy makes 5 measurements and John makes 10 measurements averaging 15.8 and 15.5, respectively. The variance of each measurement Nancy makes is ¼ that of John's. What is your estimate of the distance? (State clearly the assumptions and criteria imposed.)

(c) If $X_1, \ldots, X_n$ are independent normal random variables, what further properties does the estimator in (a) have?

6. (20 Points)

(a) State the Neyman-Pearson lemma.

(b) Prove that every Neyman-Pearson test is Bayes with respect to a prior under the 0-1 loss function.

(c) Is every Neyman-Pearson test admissible? Justify your answer.

If $X_1, X_2 \cdots X_n$ are iid $N(\theta, \sigma^2)$. θ σ both unknown prove that the t-test is UMPU for testing

H_0 $\theta = \theta_0$

H_A $\theta \neq \theta_0$

$X_1 X_2 \cdots X_n \sim$ iid $N(\theta_1, \sigma_1^2)$

$Y_1 Y_2 \cdots Y_m \sim$ iid $N(\theta_2, \sigma_2^2)$

when $\sigma_1^2 = \sigma_2^2$

H_0 $\theta_1 = \theta_2$

H_A $\theta_1 \neq \theta_2$

UMPU test?

7. (20 Points)

(a) If $T = T(X)$ is sufficient for the family $\mathbb{P}$ of distributions P_θ, $\theta \in \Theta$, of X, when is T said to be complete for $\mathbb{P}$?

(b) Suppose $X = (Y_1, \ldots, Y_m; Z_1, \ldots, Z_n)$, where Y_i's are i.i.d., $N(\mu_1, \sigma_1^2)$ variables independent of Z_j's which are i.i.d. $N(\mu_2, \sigma_2^2)$ variables.

(i) If all the parameters are unknown (with known sample sizes m and n), check whether the minimal sufficient statistic is complete and justify your answer.

(ii) If $\mu_1 = \mu_2 = \mu$ (say), which is unknown, find the minimal sufficient statistic and give your reasons for stating whether it is complete.

(c) State without proof a general version of Rao-Blackwell Theorem. (You are encouraged to state a more general version than the one usually used in the M.S. courses.)

(d) In part (b)(i), suggest some good estimator of σ_1^2/σ_2^2 and state briefly your reasons.

8. (20 Points)

(a) Define a uniformly most powerful unbiased (UMPU) test for $H\colon \theta \in \Omega_H$ against $K\colon \theta \in \Omega_K$.

If a UMP test exists, is it UMPU?

(b) Suppose T is a sufficient statistic for the family $\mathbb{P}_\omega = \{P_\theta, \theta \in \omega\}$, and the test ϕ has Neyman structure with respect to T for $H_\omega\colon \theta \in \omega$. Show that ϕ is a similar test of H_ω.

Does every similar test of H_ω have Neyman structure? Under what conditions is this true?

(c) X_i, $i = 1, \ldots, k$, are independent binomial variables with probability of success π_i in each of n trials. Assume the model

good question

$$\ln \frac{\pi_i}{1-\pi_i} = \alpha + \beta y_i, \quad i = 1, \ldots, k,$$

where y's are known and α, β are unknown parameters.

Find the UMPU test for $H\colon \beta = \beta_0$ against $\beta > \beta_0$.

$$\prod \binom{n}{x_i} \pi_i^{x_i} (1-\pi_i)^{n-x_i} = \prod \binom{n}{x_i} \left(\frac{\pi_i}{1-\pi_i}\right)^{x_i} (1-\pi_i)^n = \prod_i \binom{n}{x_i} (1-\pi_i)^n \, e^{x_i \log \frac{\pi_i}{1-\pi_i}} = (1-\pi_i)^n e^{x_i(\alpha+\beta y_i)}$$

$$= \prod_{i=1}^{k} \binom{n}{x_i} \cdot \prod_{i=1}^{k} (1-\pi_i)^n \, e^{\sum x_i(\alpha+\beta y_i)}$$

$$e^{\alpha \sum x_i + \beta \sum x_i y_i}$$

9. (20 Points)

(a) When is a distribution Λ over Ω_H said to be least favorable for testing H: $\theta \in \Omega_H$ against a simple hypothesis K_1: $\theta = \theta_1$?

(b) Suppose X has p.d.f. $p_\theta(x)$ with respect to measure μ, and

$$h_\lambda(x) = \int_{\Omega_H} p_\theta(x) d\Lambda(\theta),$$

where Λ is some probability distribution over Ω_H. ϕ_λ is a test which is most powerful at level α for H_λ: X has p.d.f. $h_\lambda(x)$ against K_1: $\theta = \theta_1$.

Show that if $E_\theta \phi_\lambda(X) \leq \alpha$ for all $\theta \in \Omega_H$, then (i) ϕ_λ is most powerful at level α for testing H: $\theta \in \Omega_H$ against K_1, and (ii) Λ is least favorable.

(c) Use the result in (b) to derive the UMP test for

$$H: \sum_{i=1}^{k} \mu_i \leq a,$$

against $\Sigma \mu_i > a$, given independent Poisson variables $X = (X_1, \ldots, X_k)$ with means μ_i, $i = 1, \ldots, k$, where a is a given number.

10. (20 Points)

$X = (X_1, \ldots, X_n)$ is a random sample from a distribution with probability density

$$f_\theta(y) = \frac{1}{\theta} f\left(\frac{y}{\theta}\right),$$

for some $\theta > 0$. Consider the class G of transformations g_b of X which tranform X_i to $X_i' = bX_i$ for $0 < b < \infty$.

(i) Suppose $h(\theta)$ is a real valued function to be estimated. When is the loss function said to be invariant under G?

Give one loss function which is invariant and another which is not invariant for estimating $h(\theta) = \theta^2$.

(ii) When is an estimator $\delta(x)$ said to be equivariant for $h(\theta)$?

Assuming any invariant loss function, give one estimator which is equivariant for $h(\theta) = \theta^2$, and another which is not equivariant.

(iii) Prove that $\delta(x)$ is equivariant for $h(\theta)$ if and only if $\delta(x) = \delta_0(x)/\nu(x)$, where $\delta_0(x)$ is a given equivariant estimator for $h(\theta)$ and $\nu(x)$ is invariant under G.

(iv) Describe briefly the technique of deriving the minimum risk equivariant (MRE) estimator of $h(\theta)$. (You may use a suitable example to illustrate the technique.)

Is the MRE minimax?