COMPREHENSIVE EXAMINATION

STATISTICAL INFERENCE

Monday, August 14, 1989
3 Hours

This is a closed-book, closed-notes exam. *Please start all problems on a new sheet of paper.*

1. Students seeking a Master's Level Pass may attempt any six problems, but should attempt 1-6.
2. Students seeking a Ph.D. Level Pass must attempt problems 7 through 11. You may choose one *and only one* from the two problems marked #7 and one *and only one* from the two problems marked #8.

1. (15 Points)
Suppose that S and θ are independent random variables, uniformly distributed on $(0,1)$ and $(0,2\pi)$, respectively. Let

$$X_1 = (-2 \ln S)^{\frac{1}{2}} \cos\theta$$
$$X_2 = (-2 \ln S)^{\frac{1}{2}} \sin\theta\,.$$

Show that X_1 and X_2 are independent standard normal variables.

HINT: You may want to obtain first the distribution of $R^2 = -2 \ln S$.

2. (20 Points)
   Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a $N(\mu, 1)$ distribution.

   ✓ (a) Find the MLE of $\theta = P(X_1 \le 1)$.

   (b) Consider the following estimator U of $P(X_1 \le 1)$:

$$U = \begin{cases} 1, & \text{if } X_1 \le 1, \\ 0, & \text{if } X_1 > 1. \end{cases}$$

   Show that U is unbiased estimator of $P(X_1 \le 1)$

   (c) Use the sufficient statistic $\bar{X}$ to obtain an unbiased estimator T of $P(X_1 \le 1)$ with smaller variance than U.

   HINT: The distribution of X_1 given $\bar{X} = y$ is normal with mean y and variance $1 - 1/n$.

$$\theta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1} e^{-\frac{(x-\mu)^2}{2}} dx$$

a). $\hat{\theta}$

a). $W = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1} e^{-\frac{(x-\bar{x})^2}{2}} dx$

$W = W(X_1, X_2 \ldots X_n)$.

$\widehat{g(\theta)} = g(\hat{\theta})$

c). $T = W - E(W) + P(X_1 \le 1)$.

$\bar{X} \to$ unbiased est?

3. (20 Points)
Consider a random variable X with a Laplace distribution with density

$$f(x;\beta) = \frac{1}{2\beta}e^{-|x|/\beta}, \quad -\infty < x < \infty, \ \beta > 0.$$

Let $X_1, \ldots, X_n$ be a random sample from the population.

(a) Show that the moment generating function of X is $(1 - \beta^2 t^2)^{-1}$, $-\frac{1}{\beta} < t < \frac{1}{\beta}$, and derive from it the variance of X.

(b) Use the method of moments to derive an estimator T_1 for β based on $X_1, \ldots, X_n$, where $X_1, \ldots, X_n$ are i.i.d. with density of $(x;\beta)$.

(c) Derive the MLE of β. Denote it by T_2.

(d) Show that T_2 is UMVUE of β.

(e) Is T_2 efficient?

4. (20 Points)
Let $X_1, \ldots, X_n$ be independent and identically distributed with probability density function

$$f(x) = \beta^{-2} x e^{-x/\beta}, \; x > 0.$$

We want to test H_0: $\beta \leq 1$ vs. H_A: $\beta > 1$ and let

$$\phi(X_1, \ldots, X_n) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} X_i > k \\ 0, & \text{otherwise,} \end{cases}$$

where k is a constant.

(a) Find an expression for k such that the size of the test is 0.05.

(b) Argue that ϕ is UMP of its size.

5. (10 Points)
Let $\Theta = \{\theta_1, \theta_2, \theta_3\}$. We want to test H_0: $\theta = \theta_1$ vs. H_A: $\theta = \theta_2$ or $\theta = \theta_3$. Our test will be based on the discrete random variable Z which can assume one of three values, z_1, z_2, z_3. The distribution of Z for $\theta = \theta_1$, θ_2 or θ_3 is shown below.

$P_\theta[Z = z_i]$

z_i	θ_1	θ_2	θ_3
z_1	.2	.3	.5
z_2	.2	.4	.3
z_3	.6	.3	.2

(a) Construct a likelihood ratio test of size 0.2.

(b) Is the test given in 5(a) UMP? Why?

6. (15 Points)
Let X have probability mass function

$$f(x|\theta) = \frac{e^{-\theta}\theta^x}{x!}, \; x = 0, 1, \ldots .$$

Let θ have prior distribution

$$g(\theta) = \frac{\alpha^m}{(m-1)!}\theta^{m-1}e^{-\alpha\theta}, \; \theta > 0,$$

where m is a positive integer and $\alpha > 0$. Compute the Bayes estimator with respect to the loss function

$$L(\theta, a) = (\theta - a)^2 .$$

7. (20 Points)

(a) State Jensen's inequality. Use it to establish essential completeness of the class of estimators based on sufficient statistics for convex loss functions.

(b) Assume the usual regularity conditions for the distribution of X and suppose that the information matrix, $I(\theta)$ is positive definite, θ being the $p \times 1$ vector of parameters. If $\delta(X)$ is any unbiased estimator (with finite variance) of real-valued function $g(\theta)$, show that

$$\mathrm{Var}_\theta \delta(X) \geq \alpha' I^{-1}(\theta)\alpha,$$

where $\alpha_i = \frac{\partial g}{\partial \theta_i}$, $\alpha' = (\alpha_1, \ldots, \alpha_p)$.

OR

7. (20 Points)

(a) State the supporting hyperplane theorem.

(b) Use it to sketch a proof of Jensen's inequality.

(c) Using part 7(b), discuss the completeness of nonrandomized decision rules.

8. (20 Points)
Suppose $X = (Y_1, \ldots, Y_m; Z_1, \ldots, Z_n)$ has joint pdf

$$p_\theta(x) = f(y_1 - \xi, \ldots, y_m - \xi; z_1 - \eta, \ldots, z_n - \eta),$$

$-\infty < y_i < \infty$, $-\infty < z_j < \infty$, $-\infty < \xi < \infty$, $-\infty < \eta < \infty$, with $\theta = (\xi, \eta)$. Consider transformations

$$\begin{aligned} Y_i' &= Y_i + a,\ i = 1, \ldots, m \\ Z_j' &= Z_j + b,\ j = 1, \ldots, n \end{aligned}$$

and the problem of estimation of $h(\theta) = \eta - \xi$.

(a) Show that the loss function $L(\theta; d)$ is invariant if $L(\xi, \eta; d) = L(\xi + a, \eta + b; d + (b - a))$ for all a, b.

(b) When is the estimator $\delta(x)$ of $h(\theta)$ said to be equivariant? Give one estimator which is equivariant, and another which is not equivariant.

(c) If Y_i's are i.i.d. $N(\xi, 1)$, and Z_j's are i.i.d. $N(\eta, 1)$, independent of Y_i's and $L(\theta; d) = [d - (\eta - \xi)]^2$, sketch the argument to show that $\bar{Z} - \bar{Y}$ is MRE estimator of $h(\theta)$.

OR

8. (20 Points)
$X_1, \ldots, X_n$ i.i.d. F_θ with density

$$f(x|\theta) = \frac{\theta}{x^2} I_{(\theta, \infty)}(x),\ \theta > 0.$$

Find the best invariant estimate of θ if the loss function is $L(\theta, a) = |\log a - \log \theta|$.

9. (20 Points)

   (a) Prove that a sufficient condition for the admissibility of a Bayes rule with respect to any given prior is its uniqueness up to equivalence.

   (b) State and prove Basu's theorem concerning independence of ancillary statistic $V = V(X)$ and sufficient statistic $T = T(X)$, stating clearly any additional conditions. Illustrate briefly its use in any inference problem.

10. (20 Points)

(a) Define a uniformly most powerful unbiased (UMPU) test for $H\colon \theta \in \Omega_H$ against $K\colon \theta \in \Omega_K$. Give without proof a situation where a UMP test does not exist, but a UMPU test exists for testing H against K.

(b) $X_i = (Y_i, Z_i)$, $i = 1, \ldots, n$ are independent bivariate observations with independent components Y_i, Z_i, each normally distributed with variance σ^2, and means $E(Y_i) = \xi_i$, $E(Z_i) = \xi_i + \eta$, $i = 1, \ldots, n$. Derive the UMPU test for $H\colon \eta \leq 0$ against $K\colon \eta > 0$.

(c) Show that this UMPU test can be displayed as an unconditional t-test.

11. (20 Points)

   (a) Define a UMP invariant (UMPI) test for H against K. Give without proof an example where a UMP test does not exist, but a UMPI test exists.

   (b) Define a maximal invariant function. Prove that the power of an invariant test depends only on the maximal invariant function on the parametric space.

   (c) $X_i = (Y_i, Z_i)$ are i.i.d. bivariate normal variables with means μ_Y, μ_Z, variances σ_Y^2, σ_Z^2 and correlation coefficient ρ. Give a sketch of the proof to show that the one-sided test based on sample correlation coefficient r is UMPI for $H: \rho \leq 0$ against $K: \rho > 0$ under the class of transformations involving change of locations and scales.