COMPREHENSIVE EXAMINATION

Statistical Inference

Wednesday, August 14, 1991
9:00 a.m. – 11:00 a.m.

This is a closed-book, closed-notes exam. Please start each problem on a new sheet of paper.

1. Student seeking a Master's Level Pass may attempt any five problems from problems 1-6.

2. Students seeking a Ph.D. Level Pass must attempt problems 6-10.

1. (20 points)

[7] (a) X and Y are independent $N(0,1)$ variables. Find the joint distribution

$$U = \frac{X}{Y}, \qquad V = X + Y.$$

[6] (b) Find also the marginal distribution of U.

[2] (c) Obtain the conditional distribution of V, given $U = u$

[3] (d) Identify the distribution of U in (b), and state, without derivation, the marginal distribution of V.

[2] (e) Are U and V independently distributed?

2. (20 points)

[2] (a) The cumulative distribution function ($c.d.f.$) of a random variable X is

$$F(x;\theta_1,\theta_2) = \begin{cases} 0 & \text{for} \quad x < \theta_2 \\ 1 - \exp\{-\frac{(x-\theta_2)}{\theta_1}\} & \text{for} \quad x \geq \theta_2 \end{cases}$$

where θ_1, θ_2 are unknown parameters, $\theta_1 > 0$. Obtain the probability density function ($p.d.f.$) of X.

[4] (b) If $(X_1, X_2 ..., X_n)$ is random sample from the population in (a) above, derive the maximum likelihood estimates of θ_1, θ_2.

[4] (c) For the sample in (b) above, find (jointly) minimal sufficient statistics.

[4] (d) Assume now $\theta_2 = 0$. Find the Cramer-Rao lower bound for the variance of an unbiased estimator of θ_1.

[6] (e) If $\theta_2 = 0$, check whether the MLE of θ_1 is $UMVUE$.

3. (20 points)
Let the distribution of X given θ be Binomial (n, θ) and let θ have the following beta distribution.

$$\pi(\theta) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a,b)} I(0 \leq \theta \leq 1),$$

where $a > 1$ and $b > 0$ are known constants. Using the loss function

$$l(\theta, a) = \frac{(\theta - a)^2}{\theta}$$

find the Bayes estimator of θ.

4. (20 points)
Let $X_1, ..., X_n$ be iid with density

$$f(x) = ax^{a-1} I(0 \le x \le 1),$$

where $a > 0$.

(a) Show that the test

$$\phi(X_1, ..., X_n) = \begin{cases} 1 & \sum \ln X_i > k; \\ 0 & \text{otherwise} \end{cases}$$

is UMP of its size for testing $H_0 : a \le 1$ versus $H_1 : a > 1$.

(b) Find k so that the test has size α $(0 < \alpha < 1)$ (Hint: consider the distribution of $-\ln X_i$).

5. (20 points)
Let $X_1, ..., X_n$ be iid with density

$$f(x) = \lambda \exp(-\lambda x) I(x \ge 0),$$

where $\lambda > 0$.

Derive the generalized likelihood ratio test of $H_0 : \lambda = 1$ versus $H_1 : \lambda \ne 1$ and show that it is equivalent to the test

$$\phi(X_1, ..., X_n) = \begin{cases} 1 & \text{if } \bar{X} < c_1 \text{ or } \bar{X} > c_2; \\ 0 & \text{otherwise} \end{cases}$$

where c_1 and c_2 satisfy

$$c_1 \exp(-c_1) = c_2 \exp(-c_2).$$

6. (20 points)
[3] (a) Define completeness and bounded completeness of a statistic for a real parameter.

[5] (b) Show, by means of an example, that bounded completeness does not imply completeness.

[6] (c) Let $X_1, \cdots, X_n$ be iid $N(\theta, 1)$, $\theta \in (-\infty, \infty)$. Obtain the uniformly minimum variance unbiased estimator of θ^2.

[6] (d) Let $X \sim f(x - \theta)$, $\theta \in (-\infty, \infty)$ with $Var(X) < \infty$. Consider estimating θ under squared error loss. Show that the best equivariant location estimator is also UMVUE if X is a complete sufficient statistic for θ.

7. (20 points)

Let X be distributed according to $Bin(9, \theta)$ with $\theta \in \Theta = (0, 1)$. Consider the problem of estimating θ with a loss of the form $L(\theta, a) = \omega(\theta)(\theta - a)^2$ where $\theta \in \Theta = (0, 1)$ and $\omega(\theta) > 0$.

[7] (a) Show that the natural estimator $\delta_0(X) = \frac{X}{9}$ is not Bayes with respect to any prior probability distribution if $\omega(\theta) \equiv 1$.

[7] (b) Assume that $\omega(\theta) \equiv 4.5$. Derive the Bayes estimator with respect to the prior distribution of Beta($\frac{3}{2}$, $\frac{3}{2}$). Is this estimator minimax? (Justify your answer)

[6] (c) Prove that the estimator $\delta_1(X) \equiv 1$ is minimax if $\omega(\theta) = \frac{1}{(1-\theta)^2}$.

8. (20 points)

The cumulative distribution function of a random variable X is given by

$$G_\beta(x) = 1 - [1 - F(x)]^\beta, \quad \beta > 0$$

where β is an unknown parameter and F is an absolutely continuous distribution function.

[10] (a) Assuming $F(\cdot)$ is known, find the UMP level α $(0 < \alpha < 1)$ test for $H_0 : \beta = 1 \;\; vs \;\; H_A : \beta > 1$.

[10] (b) Derive the locally most powerful level α $(0 < \alpha < 1)$ rank test for $H_0 : \beta = 1 \;\; vs \;\; H_A : \beta > 1$.

9. (20 points)

Let $(X_1, Y_1) \ldots (X_n, Y_n)$ be iid random vectors with $EX_1 = EY_1 = 0$, $\text{Var}(X_1) = \text{Var}(Y_1) = 1$. In addition, suppose that X_i is uncorrelated with Y_i for $i = 1, 2 \cdots, n$.

Define the sample correlation-coefficient r to be

$$r = \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{S_x^2}\sqrt{S_y^2}} \qquad \text{where}$$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i, \quad \bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i \;\; and \;\; S_x^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$$

$$S_y^2 = \frac{1}{n-1}\sum_{i=1}^n (Y_i - \bar{Y})^2$$

[10] (a) Assuming that $E|X|^4 < \infty$ show that

$$\sqrt{n}(S_x - 1) \xrightarrow{D} N(0, \tau^2)$$

for some suitable $\tau^2 > 0$. Also determine the value of τ^2.

[10] (b) Prove that $\sqrt{n}\, r$ is also asymptotically normal and obtain the variance of the limiting distribution.

10. (20 points)

Let $X_1, ..., X_n$ be iid random variables with a common discrete distribution. Let $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ be the order statistics.

[7] (a) Find the conditional distribution of $(X_1, ..., X_n)$ given $X_{(1)}, ..., X_{(n)}$.

[8] (b) Let $\mathcal{C}$ denote the class of all unbiased estimators with finite variance of a real parameter $\theta \in \Theta$,

i.e

$$\mathcal{C} = \left\{ S(X_1, ..., X_n) \,\middle|\, \begin{array}{l} ES(X_1, ..., X_n) = \theta \text{ and} \\ V(S) < \infty \text{ for all } \theta \in \Theta \end{array} \right\}$$

Now for a given $S \in \mathcal{C}$, let

$$\mathsf{E}S = \frac{1}{n!} \sum S(X_{i_1}, ..., X_{i_n})$$

where the summation is over all permutations $(i_1, \cdots, i_n)$ of $(1, 2, \cdots, n)$, and set $\mathcal{C}^* = \{S^* | S^* = \mathsf{E}S \text{ for some } S \in \mathcal{C}\}$. Note $\mathcal{C}^* \subset \mathcal{C}$. Show that, for the decision problem with $\Theta = \mathcal{A} = \mathbb{R}^1$ and squared error loss, $\mathcal{C}^*$ is essentially complete *relative to* $\mathcal{C}$.

[5] (c) Is $\mathcal{C}^*$ complete? Justify your answer.