COMPREHENSIVE EXAMINATION

Statistical Inference

Wednesday, August 20, 1992

9:00 a.m. – 11:00 a.m.

This is a closed-book, closed-notes exam. Please start each problem on a new sheet of paper.

1. Students seeking a Master's Level Pass must attempt problems 1 - 5.
2. Students seeking a Ph.D. Level Pass must attempt problems 6 - 10.

1. A physician wishes to determine the blood type of patients in rural Kentucky. She has a device which can supposedly do this, but it has not arrived at her clinic. Indeed, the only information she has is the following table describing the losses for misclassifying the blood.

		Classified As			
		AB	A	B	O
True	AB	0	1	1	2
Blood	A	1	0	2	2
Type	B	1	2	0	2
	O	3	3	3	0

(a) With only this information what would be the nonrandomized minimax classification rule?

(b) Still no device has arrived. However, suppose she discovers in one of her reference books that "blood type" corresponds physiologically to some parameter θ, according to the following relationship. If $0 < \theta < 1$, the blood is of type AB; if $1 < \theta < 2$, the blood is of type A; if $2 < \theta < 3$, the blood is of type B; otherwise the blood is of type O. Finally, in the population as a whole, θ is distributed according to the density $\pi(\theta) = e^{-\theta} I_{(0,\infty)}(\theta)$.

i. What are the prior probabilities of each of the four blood types? (Hint: $e^{-1} = .368$; $e^{-2} = .135$; and $e^{-3} = .050$.)

ii. What is the Bayes rule for classifying blood type?

(c) Finally, her device arrives. It measures a quantity X, which has density $f(x|\theta) = e^{-(x-\theta)} I_{(\theta,\infty)}(x)$, where θ is described in part (b).

i. If $x = 4$ is observed, show that the posterior distribution for θ is uniform on $(0, 4)$.

ii. If $x = 4$ is observed, what is the Bayes action for estimating blood type?

2. Consider a random sample of size n from a gamma distribution with pdf

$$f(x;\theta) = \frac{1}{\theta^{\kappa}\Gamma(\kappa)} x^{\kappa-1} e^{-x/\theta}, \quad \kappa > 0 \text{ known, } \theta > 0,\ x > 0 .$$

(a) Find a complete sufficient statistic for θ.

(b) Find the Cramer-Rao lower bound for all unbiased estimators of θ.

(c) Verify that $(1/\kappa)\bar{X}$ is an efficient and consistent estimator of θ.

(d) Find the UMVUE of θ^2.

3. Consider a random sample of size 2 from an exponential distribution, $f(x;\theta) = (1/\theta)e^{-x/\theta}$, $x > 0$, $\theta > 0$; say X_1 and X_2.

(a) Derive the distribution of $X_1 + X_2$.

(b) Give the cumulative distribution function of $X_1 + X_2$.

(c) An estimator $\hat{\theta}$ is said to be **median unbiased** if $P[\hat{\theta} < \theta] = P[\hat{\theta} > \theta]$.

i. Show that $\hat{\theta} = 1.19\bar{X}$ is a median unbiased estimator of θ.

ii. Find the relative efficiency of $\bar{X}$ with respect to $\hat{\theta}$.

4. Assume that X is distributed as $\mathrm{BIN}(n = 4, p)$, $p \in (0, 1)$, where BIN denotes the binomial distribution. Consider a test of the hypothesis:

$$\begin{aligned} H_0 &: \quad p = 0.25 \\ H_1 &: \quad p = 0.75 \end{aligned}$$

(a) Compute $\Lambda = f_0(x)/f_1(x)$ for each x and order according to Neyman-Pearson theory.

(b) Determine all nonrandomized most powerful tests of H_0 vs. H_1.

(c) Find the most powerful test of H_0 corresponding to an α level of 0.10.

(d) Prove that the test in part (c) is uniformly most powerful test for the hypothesis:

$$\begin{aligned} H_0 &: \quad p \leq 0.25 \\ H_1 &: \quad p \geq 0.75 \end{aligned}$$

5. Suppose $Y_1, Y_2, \ldots, Y_n$ are independent, $Y_i \sim N(\beta_1 u_i + \beta_2 v_i, \sigma^2)$ with β_1 and β_2 unknown; σ^2 known. For simplicity, assume $\sum_{j=1}^{n} u_j v_j = 0$ and $\sum_{j=1}^{n} u_j^2 = \sum_{j=1}^{n} v_j^2 = 1$.

(a) Show that the (joint) maximum likelihood estimates for β_1 and β_2 are $\hat{\beta}_1 = \sum_{j=1}^{n} u_j y_j$ and $\hat{\beta}_2 = \sum_{j=1}^{n} v_j y_j$, respectively.

(b) Show that a (generalized) likelihood ratio test of H_0: $\beta_1 = \beta_2$ against H_a: $\beta_1 \neq \beta_2$ suggests a rejection region of the form $|\hat{\beta}_1 - \hat{\beta}_2| > C$, where C is a constant.

6. (a) Let $X_1, \ldots, X_n$ be a random sample from the uniform distribution on $(0, \theta)$. Find the uniformly minimum variance unbiased estimator (UMVUE) of θ^k for any integer $k > -n$.

   (b) Let $X_1, \ldots, X_n$ be *i.i.d.* according to the Poisson distribution with parameter λ. Using the fact that an unbiased estimator of $e^{-\lambda}$ is

$$\begin{aligned} \delta(x_1) &= 1, \quad \text{if } x_1 = 0, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

   Find the UMVUE of $e^{-\lambda}$.

7. Let $X_1, \ldots, X_n$ be a random sample from the Poisson distribution with parameter λ. Let λ have a gamma density given by

$$g(\lambda; a, b) = \frac{1}{\Gamma(a) b^a} \lambda^{a-1} e^{-\lambda/b}, 0 < \lambda < \infty.$$

   (a) For squared error loss, determine the Bayes estimator $\delta_{a,b}$ of λ.

   (b) What happens to $\delta_{a,b}$ as $a \to 0$ and $b \to \infty$?

   (c) If λ has the improper prior $d\lambda/\lambda$ (corresponding to $a = 0, b = \infty$) under what circumstances is the posterior distribution proper?

8. Let X be a random variable with distribution P_θ for $\theta \epsilon \Omega$. Let $G = \{g\}$ be a group of transformations of X inducing a group $G' = \{g'\}$ on Ω. Let $\delta(X)$ be an estimator of $h(\theta)$. Let $L(\theta, d)$ be the loss incurred in using d as an estimate of $h(\theta)$.

(a) When is $L(\theta, d)$ said to be invariant?

(b) When is the estimator $\delta(X)$ said to be equivariant?

(c) Let $X_1, X_2, \ldots, X_n$ be a random sample from

$$p(x, \theta) = \begin{cases} e^{-(x-\theta)}, & x \geq \theta, \\ 0, & \text{otherwise.} \end{cases}$$

Evaluate the Pitman estimator of θ given by

$$\delta^*(\underline{x}) = \int_{-\infty}^{\infty} u f(x_1 - u, \ldots, x_n - u) du \Big/ \int_{-\infty}^{\infty} f(x_1 - u, \ldots, x_n - u) du$$

when f is the joint density of the $X_1, \ldots, X_n$.

(d) Mention the optimal properties of δ^* in the context of (a) and (b).

9. Let $X_1, \ldots, X_n$ be a random sample from the gamma distribution $\Gamma(a, b)$ the density of which is given by

$$g(x; a, b) = \begin{cases} \frac{1}{\Gamma(a) b^a} \; x^{a-1} \; e^{-x/b}, & 0 < x < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Obtain a UMP unbiased test of $H_0 : a \leq a_0$ versus $H_1 : a > a_0$ as a conditional test.

(b) Use Basu's theorem to show the independence of

$$W = \Pi_{i=1}^{n}(X_i/\bar{X}) \text{ and } \bar{X} \text{ when } a = a_0.$$

(c) Hence sketch the proof for representing the UMP unbiased test in (a) as an unconditional test based on W.

10. Let $X = (Y_1, Y_2, \ldots, Y_m; Z_1, Z_2, \ldots, Z_n)$ where $Y_i's$ are *i.i.d.* $N(\mu_1, \sigma^2)$ variables and $Z_j's$ are *i.i.d.* $N(\mu_2, \sigma^2)$ variables, independent of the $Y_i's$.

Consider the group G of transformations $\{g_{a,b}\}$ where

$$X' = (Y_1', \ldots, Y_m'; Z_1', \ldots, Z_n')$$

is defined by

$$\begin{array}{ll} Y_i' = \ a + bY_i & , \ i = 1, \ldots, m \\ Z_j' = \ a + bZ_j & , \ j = 1, \ldots, n \end{array},$$

for $-\infty < a < \infty, \quad b > 0.$

(a) Show that the problem of testing $H_0 : \mu_1 \leq \mu_2$ against $K : \mu_1 > \mu_2$ remains invariant under the induced group G' of transformations $\{g'_{a,b}\}$ of the parameter $\theta = (\mu_1, \mu_2, \sigma^2)$.

(b) Identify any maximal invariant function of the minimal sufficient statistics.

(c) Hence argue that a well-known classical test is UMP Invariant for this testing problem.