

Ph.D. Qualifying Examination

STATISTICAL INFERENCE

Tuesday, July 6, 1993
9:00 a.m. - 11:00 a.m.

Answer any four questions:

1. Let X_1, \dots, X_n be i.i.d. with common density $f_\lambda(x) = \lambda e^{-\lambda x}$, $x > 0$. Consider estimating $\theta = P_\lambda(X_1 \leq 4) = 1 - e^{-4\lambda}$.

(a) [15 points]

Show that the estimator

$$\delta(X_1, \dots, X_n) = \begin{cases} 1 - \left(1 - \frac{4}{\sum_{i=1}^n X_i}\right)^{n-1} & \text{if } \sum_{i=1}^n X_i \geq 4 \\ 1 & \text{otherwise} \end{cases}$$

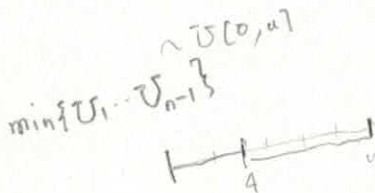
is uniformly minimum variance unbiased for estimating θ .

(b) [10 points]

Obtain the maximum likelihood estimator of θ . Prove that it is not an unbiased estimator of θ .

$\sum_{i=1}^n X_i$ is suff. so, $E\{I_{[X_1 \leq 4]} \mid \sum X_i\}$ is UMVUE

Complete (Exp. family)



when $\sum X_i < 4 \Rightarrow X_1 < 4 \Rightarrow I_{[]} \equiv 1$

So, $E\{I\} = 1$ when $\sum X_i < 4$

$$P(U_{(n)} \leq 4) = 1 - P(U_{(n)} > 4) = 1 - P(U_1, \dots, U_{(n-1)} > 4) = 1 - [1 - F(4)]^{n-1} = 1 - \left[1 - \frac{4}{u}\right]^{n-1}$$

$$P(X_1 \leq 4 \mid \sum X_i = u > 4) = \frac{P(X_1 \leq 4, \sum X_i = u)}{P(\sum X_i = u)}$$

2. Let X_1, X_2, \dots, X_n be i.i.d. $N(\mu, \sigma^2)$, with both μ and σ^2 unknown.

(a) [13 points]

Derive the level α likelihood ratio (LR) test for testing $H_0: \mu = 5$ vs. $H_A: \mu \neq 5$.

(b) [12 points]

Show that the LR test derived in (a) is UMP unbiased level α test.

3. Let X_1, \dots, X_n be i.i.d. $U[0, \theta]$. Let $X_{(n)}, X_{(n-1)}$ be the largest and second largest order statistics of $\{X_1, \dots, X_n\}$.

(a) [10 points]

Find the asymptotic distribution of $n(X_{(n)} - \theta)$ and obtain a 95% asymptotic confidence interval for θ .

(b) [9 points]

Show that, under mean squared error criterion, $\hat{\theta} = X_{(n)} + [X_{(n)} - X_{(n-1)}]$ is a better estimator than $X_{(n)}$ for θ . *need to compute $E X_{(n)} X_{(n-1)}$*

(c) [6 points]

How does the estimator $\frac{n+1}{n} X_{(n)}$ compare with $\hat{\theta}$ and $X_{(n)}$ under the mean squared error for estimating θ ?

the dist. of $X_{(n)}$ is $P(X_{(n)} < t) = \left(\frac{t}{\theta}\right)^n$, $0 < t \leq \theta$

the dist. of $n(X_{(n)} - \theta)$ is (for $t < 0$)

$$\begin{aligned} P(n(X_{(n)} - \theta) < t) &= P(X_{(n)} - \theta < \frac{t}{n}) \\ &= P(X_{(n)} < \theta + \frac{t}{n}) \\ &= \left(\frac{\theta + \frac{t}{n}}{\theta}\right)^n = \left(1 + \frac{t}{n\theta}\right)^n \\ &= \left(1 + \frac{t/\theta}{n}\right)^n \end{aligned}$$

as $n \rightarrow \infty$ this $\rightarrow e^{t/\theta}$

4. Let X_1, X_2, \dots, X_n be i.i.d. with common unknown distribution function F . Consider estimating $\theta = F(2) = P(X_1 \leq 2)$ with squared error loss. Let \hat{F}_n be the empirical distribution of X_1, \dots, X_n .

(a) [6 points]

Derive the Bayes estimator of θ with respect to the prior

$\pi(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}$ where α and β are known positive constants.

Ex

(b) [13 points]

Show that

$$I_{[X_1 \leq 2]} \sim B(\theta) \quad \delta(X_1, \dots, X_n) = \frac{1}{1 + \sqrt{n}} \left(\sqrt{n} \hat{F}_n(2) + \frac{1}{2} \right)$$

is minimax for θ .

(c) [6 points]

Is the minimax estimator [given in (b)] admissible? Justify your answer.

Hint: What is the distribution of $\hat{F}_n(2)$?

5. Suppose X_1, X_2, \dots, X_n are i.i.d. with unknown continuous distribution function F .

(a) [8 points]

Derive a consistent estimator of the function $\Lambda(t_0) = -\log(1 - F(t_0))$ where t_0 is a fixed number. Prove your claim of consistency.

(b) [12 points]

Determine the asymptotic distribution of the estimator derived in

(a). *use delta method.*

(c) [5 points]

Hence or otherwise obtain a 99% asymptotic confidence interval for $\Lambda(t_0)$.

*$-\log[1 - \hat{F}_n(t_0)]$
is a consistent est.
need to redefine
at 0, and 1*

*based on $I_{[X_1 \leq 2]}$
 $[X \leq 2]$*