Ph.D. Qualifying Examination

STATISTICAL INFERENCE

Tuesday, July 6, 1993
9:00 a.m. - 11:00 a.m.

Answer any four questions:

1. Let $X_1, \ldots, X_n$ be i.i.d. with common density $f_\lambda(x) = \lambda e^{-\lambda x}$, $x > 0$. Consider estimating $\theta = P_\lambda(X_1 \le 4) = 1 - e^{-4\lambda}$.

   (a) [15 points]
   Show that the estimator

$$\delta(X_1, \ldots, X_n) = \begin{cases} 1 - \left(1 - \frac{4}{\sum_1^n X_i}\right)^{n-1} & \text{if } \sum_{i=1}^n X_i \ge 4 \\ 1 & \text{otherwise} \end{cases}$$

   is uniformly minimum variance unbiased for estimating θ.

   (b) [10 points]
   Obtain the maximum likelihood estimator of θ. Prove that it is not an unbiased estimator of θ.

$\sum_i^n x_i$ is suff. Complete (exp. family) so, $E\{ I_{[X_1 \le 4]} \mid \sum x_i \}$ is UMVUE

$\sim U[0,u]$

$\min\{U_1 \cdots U_{n-1}\}$

when $\sum x_i < 4 \Rightarrow x_1 < 4 \Rightarrow I_{[\ ]} \equiv 1$

So, $E\{1\} = 1$ when $\sum x_i < 4$

$P(U_{(1)} \le 4)$
$= 1 - P(U_{(1)} > 4)$
$= 1 - P(U_1 \cdots U_{(n-1)} > 4)$
$= 1 - [1 - F(4)]^{n-1}$
$= 1 - [1 - \frac{4}{u}]^{n-1}$

$P((X_1 \le 4) \mid \sum X_i = u > 4) = \dfrac{P(X_1 \le 4, \sum X_i = u)}{P(\sum X_i = u)}$

=

2. Let $X_1, X_2, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$, with both μ and σ^2 unknown.

   (a) [13 points]
   Derive the level α likelihood ratio (LR) test for testing H_0: $\mu = 5$ vs. H_A: $\mu \neq 5$.

   (b) [12 points]
   Show that the LR test derived in (a) is UMP unbiased level α test.

3. Let $X_1, \ldots, X_n$ be i.i.d. $U[0, \theta]$. Let $X_{(n)}, X_{(n-1)}$ be the largest and second largest order statistics of $\{X_1, \ldots, X_n\}$.

   (a) [10 points]
   Find the asymptotic distribution of $n(X_{(n)} - \theta)$ and obtain a 95% asymptotic confidence interval for θ.

   (b) [9 points]
   Show that, under mean squared error criterion, $\hat{\theta} = X_{(n)} + [X_{(n)} - X_{(n-1)}]$ is a better estimator than $X_{(n)}$ for θ. need to compute $EX_{(n)}X_{(n-1)}$

   (c) [6 points]
   How does the estimator $\frac{n+1}{n} X_{(n)}$ compare with $\hat{\theta}$ and $X_{(n)}$ under the mean squared error for estimating θ?

the dist. of $X_{(n)}$ is $P(X_{(n)} < t) = \left(\frac{t}{\theta}\right)^n$, $0 < t \leq \theta$

the dist. of $n(X_{(n)} - \theta)$ is (for $t < 0$)

$$P(n(X_{(n)} - \theta) < t) = P\left(X_{(n)} - \theta < \frac{t}{n}\right)$$

$$= P\left(X_{(n)} < \theta + \frac{t}{n}\right)$$

$$= \left(\frac{\theta + \frac{t}{n}}{\theta}\right)^n = \left(1 + \frac{t}{n\theta}\right)^n$$

$$= \left(1 + \frac{t/\theta}{n}\right)^n$$

as $n \to \infty$ this $\to e^{t/\theta}$

4. Let $X_1, X_2, \ldots, X_n$ be i.i.d. with common unknown distribution function F. Consider estimating $\theta = F(2) = P(X_1 \le 2)$ with squared error loss. Let $\hat{F}_n$ be the empirical distribution of $X_1, \ldots, X_n$. $i=1,2,\cdots n$

   (a) [6 points] based on $I[X_i \le 2]$

   Derive the Bayes estimator of θ with respect to the prior $\pi(\theta) = \frac{\Gamma_{(\alpha+\beta)}}{\Gamma_{(\alpha)}\Gamma_{(\beta)}}\theta^{\alpha-1}(1-\theta)^{\beta-1}$ where α and β are known positive constants. $[X \le 2,$

   Ex

   (b) [13 points]

   Show that

   $I[X_1 \le 2] \sim B(\theta)$

$$\delta(X_1, \ldots, X_n) = \frac{1}{1+\sqrt{n}}\left(\sqrt{n}\,\hat{F}_n(2) + \frac{1}{2}\right)$$

   is minimax for θ.

   (c) [6 points]

   Is the minimax estimator [given in (b)] admissible? Justify your answer.

Hint: What is the distribution of $\hat{F}_n(2)$?

5. Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. with unknown continuous distribution function F.

   (a) [8 points]

   Derive a consistent estimator of the function $\Lambda(t_0) = -\log(1 - F(t_0))$ where t_0 is a fixed number. Prove your claim of consistency.

   $-\log[1-\hat{F}_n(t_0)]$ is a consistant est.

   need to redifine at 0, and 1

   (b) [12 points]

   Determine the asymptotic distribution of the estimator derived in (a). use delta method

   (c) [5 points]

   Hence or otherwise obtain a 99% asymptotic confidence interval for $\Lambda(t_0)$.