Comprehensive Exam
Statistical Inference

June 4, 1997
10:00 am - 12:00 pm

Instructions

1. **Answer all the questions.**

2. **Number of points each problem carries is indicated in parentheses. Maximum possible score is 100.**

3. **Start each question on a new sheet of paper with your name on it.**

1. (16 points) Let $(X_1, ..., X_m)$ be a random sample from normal (μ, σ^2) and $(Y_1, ..., Y_n)$ be a random sample from normal (μ, τ^2). Assume that the X's and Y's are mutually independent.

   3-dimensional

   where.
   $-\infty < \mu < +\infty$
   $0 < \sigma^2 < +\infty$
   $0 < \tau^2 < +\infty$

   (a) Obtain a sufficient statistic for (μ, σ^2, τ^2).

   (b) Is the sufficient statistic obtained in (a) complete (justify your answer)?

2. (18 points)

   (a) State all the properties of the maximum likelihood estimates of an unknown parameter (one dimensional case).

   (b) Find the maximum likelihood estimates of θ and σ based on a random sample $(X_1, ..., X_n)$ from the density

$$f(x; \theta, \sigma) = \frac{1}{\sigma} e^{-(x-\theta)/\sigma}, \qquad x \geq \theta, \quad \sigma > 0$$
$$= 0 \qquad , \text{ elsewhere.}$$

   (c) Assuming that θ is known to be equal to zero, obtain the Cramér-Rao lower bound for the variance of any unbiased estimator of σ.

3. (16 points) Let $(X_1, ..., X_n)$ be a random sample from the Bernoulli population with unknown parameter p. Assume that p has a prior distribution that is uniform on $(0, 1)$.

   (a) Obtain the Bayes estimate of p assuming a quadratic loss function.

   (b) What is the limit of the mean square error of this estimator as n becomes large?

4. (16 points) Suppose that $X_1, X_2 ..., X_n$ are *iid* with the density

$$f(t) = \begin{cases} \beta\,(1-t)^{\beta-1}\,, & \text{for } 0 \le t \le 1 \\ 0 \quad , & \text{otherwise} \end{cases}$$

where $\beta > 0$ is a parameter.

(a) Based on the random sample $X_1, X_2 ..., X_n$ obtain a most powerful test of

$$H_0 : \beta = 1$$

versus

$$H_A : \beta = 2.$$

(b) What is the distribution of your test statistic under H_0?

(c) Find a UMP test of

$$H_0 : \beta \le 1$$

versus

$$H_A : \beta > 1.$$

(justify your answer)

5. (16 points) Suppose that X_1, X_2, X_3 are *iid* Poisson (λ). Independent of that we also have Y_1, Y_2 which are *iid* Poisson (μ). Find a UMPU test of

$$H_0 : \lambda \le \mu$$

versus

$$H_A : \lambda > \mu$$

based on X_1, X_2, X_3, Y_1, Y_2.

Indicate how you obtain the rejection region for a specified α (probability of type I error).

6. (18 points) Suppose X_i, $i = 1, 2, \ldots n$ are *iid* non negative observations such that $\sqrt{X_i}$ has the density

$$f_\theta(u) = \begin{cases} \theta^2 u e^{-\theta u}; & u > 0 \\ 0 & u \leq 0 \end{cases}$$

where $\theta > 0$ is an unknown parameter.

(a) Find the MLE of θ and its asymptotic distribution.

(b) Find a transformation $g(\theta)$ such that its MLE has an asymptotically constant variance (not depending on θ).

(c) If $n = 80$ and $\sum_{i=1}^{n} \sqrt{X_i} = 66$, use $\alpha = 0.05$ and obtain a generalized likelihood ratio test for

$$H_0 : \theta = 1$$

versus

$$H_A : \theta \neq 1.$$