Inference Examination

June 3, 1998
9:00 to 11:00

1. Answer all problems.
2. Start each problem on a new sheet of paper with your name.
3. Number of points for each problem is shown in parenthesis. Maximum possible is 100.

1. (18 points) Suppose $X_i \sim f_i(x, \theta)$, $i = 1, 2, \cdots, n$ are independent random variables where

$$f_i(x, \theta) = a_i \theta (1 - x)^{a_i \theta - 1} \quad \text{for} \quad 0 < x < 1.$$

where $a_i > 0$ are known constants and $\theta > 0$ is the parameter.

(a) Find the MLE of $\eta = \log \theta$ and construct an asymptotic 95% confidence interval for η based on the central limit theorem for MLE.

(b) Obtain a 95% confidence interval for θ based on the result in (a).

(c) Assume that θ is also a random variable with exp(1) distribution. Find the Bayes estimator of θ based on the observations $X_1, \cdots, X_n$ if the loss function is squared error loss.

2. (14 points) In n independent Bernoulli trials, let $X_i = 0$ or 1 according as the i^{th} trial is a success or failure. Also let

$$T = \sum_1^n X_i \; .$$

Assume the probability of success at each trial is p.

Find the UMVU (uniformly minimum variance unbiased) estimator of p^2, using the fact that an unbiased estimator of p^2 is δ given by

$$\delta = \begin{cases} 1, & \text{if } X_1 = 1 \text{ and } X_2 = 1 \\ 0, & \text{otherwise} \end{cases}$$

3. (18 points) Suppose X_1, X_2, X_3 are independent normal random variables such that

$$X_1 \sim N(3\theta + 1, \sigma^2 = 2)$$

$$X_2 \sim N(-2\theta, \sigma^2 = 1)$$

$$X_3 \sim N(\theta - 4, \sigma^2 = 4)$$

(a) Find a most powerful size $\alpha = 0.05$ test for

$$H_0 : \theta = 2, \quad vs \quad H_A : \theta = 1$$

Justify your answer.

(b) Find a UMP test with $\alpha = 0.05$ for

$$H_0 : \theta \geq 2, \quad vs \quad H_A : \theta < 2$$

Justify your answer.

4. (a) (20 points) When do we say that a family of densities $f(x, \theta)$ or probability mass function $p(x, \theta)$ has Monotone Likelihood Ratio property? (Define it carefully)

(b) Let $X_1, \cdots, X_n$ be a random sample from the probability density given by

$$f(x, \theta) = \theta x e^{-x^2\theta}, \quad x > 0 \quad \theta > 0,$$

i. Construct a UMP test of

$$H_0 : \theta \leq \theta_0 \quad \text{against} \quad H_1 : \theta > \theta_0$$

~~4 MLR~~
4, $\phi(T)$
2, H_A
2, H_0
2 MLR

based on the observations $X_1, \cdots, X_n$.
Justify that the test you obtained is indeed UMP. Base your argument on the Neyman-Pearson lemma only.

ii. Obtain an explicit expression (in terms of a known distribution) for the critical point of the test when $n = 10$ and $\alpha = 0.05$.

5. (15 points) Let $X_1, X_2, \cdots X_n$ be i.i.d. random variables with density function

$$f(x) = (\theta+1)x^{\theta} \ ; \qquad 0 < x < 1, \ \ \theta > 0.$$

(a) Find the distribution of $X_{(n)}$, the maximum of $X_1, X_2, \cdots X_n$.

(b) Find a pivotal quantity based on $X_{(n)}$ and construct a $(1-\alpha)100\%$ confidence interval for θ based on the pivotal.

6. (15 points) Consider the **Option-Pricing Model** for stock option prices over a given period of time which is divided into equally spaced time points: $t_0 < t_1 < \cdots < t_n$. Let S_i denote the price of the stock at t_i $(i = 0, 1, \cdots n)$ with $S_0 = S$. From time t_i to time t_{i+1} the price of the stock will either increase by a factor of $(1+\rho)$ with probability θ or decrease by a factor $(1-\rho)$ with probability $1-\theta$, where $0 < \theta < 1$. For example

$$S_1 = \begin{cases} S(1+\rho) & \text{with probability } \theta \ ; \\ S(1-\rho) & \text{with probability } 1-\theta \ . \end{cases}$$

Assume that both S and ρ are known ($S > 0$ and $1 > \rho > 0$) and we want to estimate θ. Assume changes in price of S over successive time periods are mutually independent.

(a) Find a 1 dimensional sufficient statistic for estimating θ, justify your answer.

(b) What is the probability distribution of S_n?

(c) Compute the expected value of S_n.

(hint: let $X_1, X_2, \cdots X_n$ be the indicators of changes at each stage. That is $X_i = 1$ if the stock price increases from time t_{i-1} to t_i and 0 otherwise.)