Qualifying Examination
Advanced Inference

June 3, 2002

Choose 5 out of the following 6 problems.

Full credit will be given only if your results are justified and details are shown.

1. The following table contains the risk $R(\theta_i, d_j)$ for different combinations of parameters θ_i, $i = 1, 2$, and decision rules d_j, $j = 1, \ldots, 3$.

	d_1	d_2	d_3
θ_1	1	2	2
θ_2	2	1	2

   (a) Draw the risk set.

   (b) Find the set of admissible rules.

   (c) Geometrically, find the Bayes rule(s) for the prior $\pi = (1/3, 2/3)$, where $P_\pi(\theta = \theta_1) = 1/3$ and $P_\pi(\theta = \theta_2) = 2/3$.

   (d) Numerically, find the Bayes rule(s) and the Bayes risk for the prior given in (c).

   (e) Is this Bayes rule admissible? Why?

   (f) Find the least favorable prior distribution geometrically *and* numerically.

2. Let D^* be the space of all decision rules, and let $\mathcal{B}$ denote the class of all Bayes rules.

   (a) Show that if $\mathcal{B}$ is essentially complete in D^* then $\mathcal{B}$ is complete in D^*.

   (b) Is this also true for the class $\mathcal{B}^E$ of extended Bayes rules? I.e., does the following hold: If $\mathcal{B}^E$ is essentially complete in D^* then $\mathcal{B}^E$ is complete in D^*.

20% 3. (a) State all the good and bad (if any) properties of the maximum likelihood 6% estimator of an unknown parameter (one dimensional case).

   (b) Let $\Theta = \mathbb{R}$, $L(\theta, a) = (\theta - a)^2$, $X \sim N(\theta, 1)$.

      5% i. Find the maximum likelihood estimator of θ, based on one observation, X.

      5% ii. Show that the maximum likelihood estimator obtained in (i) is not a Bayes rule with respect to any prior π that is a non-degenerate probability distribution on Θ.

      4% iii. Can a Bayes rule with respect to a prior π be unbiased in general? Why?

20% 4. Let $f(x, \theta)$ be a family of densities, with $\theta \in \Theta$ where Θ is an open subset of R^2.

Suppose $X_1, X_2, \cdots, X_n$ are iid random variables from the density $f(x, \theta_0)$ for a fixed θ_0 in the interior of Θ.

Show that (under regularity conditions*) for any given $\theta \in \Theta$ and $\theta \neq \theta_0$, we have

$$P(L_n(\theta) < L_n(\theta_0)) \to 1 \quad as\ n \to \infty \qquad 14\%$$

where $L_n(\theta)$ is the likelihood function $\prod_{i=1}^{n} f(X_i, \theta)$.

*Specify the regularity conditions needed and where they are needed.

6%

20% 5. Suppose $X_1, X_2, \cdots X_n$ are iid r.v.s from an exponential distribution with parameter $\lambda > 0$. I.e.,

$$f(\overset{t}{\cancel{x}}, \lambda) = \lambda \exp(-\lambda t) \quad \text{for} \quad t > 0.$$

4% (a) Find a complete, sufficient statistic for λ.

5% (b) Consider the nonrandomized decision rule $d(x_1, \ldots, x_n) = x_1$. Can this decision rule be improved by using the sufficient statistic? What assumptions have to be made?

11% (c) Show that

$$2 \log L_n(\hat{\lambda}) - 2 \log L_n(\lambda)$$

converges in distribution to a ... distribution (to which one?), where $\hat{\lambda}$ is the MLE of λ and $L_n(\lambda) = \prod_{i=1}^{n} f(X_i, \lambda)$.

20% 6. Let $\Theta = \mathcal{A} = \mathbb{R}$, and let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(\theta, 1)$. Let the loss function be $L(\theta, a) = (\theta - a)^2$ (squared error loss).

12% (a) Find a Bayes rule of estimating θ with respect to a $N(0, \sigma^2)$ prior distribution.

8% (b) Prove that the estimator $d(x) = \bar{x}$ is minimax.

extended Bayes + equalizer

$\bar{x} \, \frac{\sigma^2}{1+\sigma^2}$

$N(0,\sigma^2) \cdot N(\theta,1) =$ joint $= h(\theta,x) = \frac{1}{2\pi\sigma} e\left[-\frac{(x-\theta)^2}{2} - \frac{\theta^2}{2\sigma^2}\right.$

$f(x) = N(0, \underbrace{1+\sigma^2}_{sd})$.

$g(\theta|x) = N\left(\frac{x\sigma^2}{1+\sigma^2}, \sqrt{\frac{\sigma^2}{1+\sigma^2}} = sd\right)$