Inference Examination

June 6, 2005

9:00 to 11:00

Full credit will be given only if your results are justified and details are shown.

1. Answer all problems. Write only on one side of the paper.

2. Start each problem on a new sheet of paper with your name.

3. Number of points for each problem is shown in parentheses. Maximum possible is 100.

1. (25 points)

Suppose $X_1, \cdots, X_n$ are iid with a logistic distribution with density $f(x|\theta_0)$.

(Please note the logistic density is defined as

$$f(x|\theta) = \frac{e^{-(x-\theta)}}{[1+e^{-(x-\theta)}]^2}$$

where $-\infty < \theta < \infty$ is the parameter.)

Let

$$M_n(\theta) = \sum_{i=1}^{n} e^{-(X_i-\theta)2} .$$

$e^{-(X_i-\theta)^2}$

Define the maximizer of M_n as our estimator $\hat{\theta}$:

$$\hat{\theta} = arg \max M_n(\theta) .$$

(a) show that the estimator is consistent: $\hat{\theta} \to \theta_0$ as $n \to \infty$. (either weak consistent or strong consistent is OK, but identify which one you have proved).

(b) Assume the estimator defined above is root-n consistent; (i.e. $\hat{\theta} = \theta_0 + O_p(1/\sqrt{n})$), sketch a proof of a central limit theorem for the estimator $\hat{\theta}$: i.e.

$$\sqrt{n}(\hat{\theta} - \theta_0) \to N(0, \sigma^2)$$

in distribution as sample size n increases to infinity.

Identify the $\sigma 2$ above.

2. (25 points)

Let $X_1, \ldots, X_n$ ($n \geq 2$) be an iid sample of Bernoulli random variables, $P(X_i = 1) = \theta,\ 0 < \theta < 1$. Define $\phi(\theta) = \theta(1-\theta)$.

(a) Show that $X_2(1 - X_1)$ is an unbiased estimator of $\phi(\theta)$.

(b) Find an improvement of the estimator in (a) based on a complete sufficient statistic.

Hint: Take the conditional expectation.

3. (25 points) The following table contains the risk $R(\theta_i, d_j)$ for different combinations of parameters θ_i, $i = 1, 2$, and decision rules d_j, $j = 1, \ldots, 4$.

	d_1	d_2	d_3	d_4
θ_1	3	2	1	4
θ_2	6	4	2	8

(a) Draw the risk set.

(b) Find the set of admissible rules.

(c) Find the Bayes rule(s) and the Bayes risk for the prior $\pi = (3/4, 1/4)$, where $P_\pi(\theta = \theta_1) = 3/4$ and $P_\pi(\theta = \theta_2) = 1/4$.

(d) Is this Bayes rule admissible? Why?

(e) Find the least favorable prior distribution. Verify that it is least favorable.

4. (25 points) Let the random variables $X_1, \cdots, X_n$ be iid having the negative binomial distribution with unknown parameter $p \in (0, 1)$:

$$P(X = x) = \frac{\Gamma(m + x)}{\Gamma(x + 1)\Gamma(m)} p^m (1 - p)^x, \; x = 0, 1, 2, \ldots$$

where m is a specified positive number.

4 (a) Let p have the beta prior $B(\alpha, \beta)$ where $\alpha > 1, \beta > 0$ are given. Find the posterior distribution.

7 (b) Find the Bayes estimator of $1/p$ using squared error loss. $= \dfrac{[nm + \alpha + \sum X_i + \beta - 1]}{nm + \alpha - 1}$

$\dfrac{\sum X_i + nm}{nm}$ = (c) Find the MLE of $1/p$ based on the n observations.

4

10 (d) Show that the MLE of $1/p$ is an extended Bayes estimator for squared error loss.

Hint: the mean and variance of a negative binormial r.v. are $m(1 - p)/p$ and $m(1 - p)/p2$ respectively.

Hint: If $X_1 \sim NB(m_1, p)$ and $X_2 \sim NB(m_2, p)$ and are independent of each other, then $X_1 + X_2 \sim NB(m_1 + m_2, p)$.