Inference Examination

May 27, 2008
9:00 to 11:00

1. Answer all problems. Write only on one side of the paper.
2. Start each problem on a new sheet of paper with your name.
3. Number of points for each problem is shown in parentheses. Maximum possible is 100.

1. (20 points)

   (a) Define M-Estimator and Z-Estimator, and write the following three as M- and Z-Estimators: Least squares estimator, maximum likelihood estimator, sample median. (assume a random sample of n iid random variable is available)

   (b) Assume the consistency of the Z-estimator, state a theorem on asymptotic normality of Z-Estimators and outline a proof.

   Mention the necessary assumptions where they are needed in the proof.

2. (30 points)

The following table contains the risk $R(\theta_i, d_j)$ for different combinations of parameters $\theta_i,\ i = 1, 2$, and decision rules $d_j,\ j = 1, \ldots, 4$.

	d_1	d_2	d_3	d_4
θ_1	5	1	7	3
θ_2	6	3	7	2

   (a) Draw the risk set, and mark the set of admissible rules as well as the set of lower boundary points.

   (b) Find the Bayes rule(s) and the (minimal) Bayes risk for the prior $\pi = (4/5, 1/5)$, where $P_\pi(\theta = \theta_1) = 4/5$ and $P_\pi(\theta = \theta_2) = 1/5$.

   (c) Is this Bayes rule admissible? Why?

   (d) Find the least favorable prior distribution. Verify that it is least favorable.

   (e) Assume that there is another rule d_5 with $R(\theta_1, d_5) = 0,\ R(\theta_2, d_5) = 5$. Find the Bayes rule(s) and Bayes risk for the prior in (c).

   (f) Consider the randomized rule δ that chooses each of d_2 and d_5 with probability 1/2. Under which conditions can we find a nonrandomized rule that is at least as good as δ?

3. (25 points)

Let (X_n) be a sequence of *i.i.d.* random variables with exponential(λ) distribution.

(a) Derive the limit distribution of the minimum $X_{(1),n}$, after proper standardization.

(b) Derive the limit distribution of the maximum $X_{(n),n}$, after proper standardization.

Hint: Proper standardizations are of the form $(X_{(k),n} - a_n)/b_n$ for $k = 1, n$, and sequences a_n, b_n, so that the limiting distribution is non-degenerate.

Repeat the above questions with (X_n) a sequence of *i.i.d.* random variables with Uniform[0,1] distribution.

4. (25 points)

Suppose $X_1, \cdots, X_n$ are iid random variables with a CDF $F(t)$. Assume $F(t)$ is continuous. Denote by $\hat{F}_n(t)$ the empirical distribution.

(a) Show that

$$\sup_t |\hat{F}_n(t) - F(t)|$$

converge to zero in probability as $n \to \infty$. i.e. for any $\epsilon > 0$,

$$P(\sup_t |\hat{F}_n(t) - F(t)| > \epsilon)$$

goes to zero as $n \longrightarrow \infty$.

(b) For a fixed a, we want to estimate the conditional distribution

$$P(X_1 \leq t | X_1 > a) = \frac{F(t) - F(a)}{1 - F(a)}, \qquad t > a.$$

Consider the estimator

$$\frac{\hat{F}_n(t) - \hat{F}_n(a)}{1 - \hat{F}_n(a)} = \hat{G}_n(t)\,.$$

Assume $F(a) < 1$, show that

$$\sup_{t>a} |\hat{G}_n(t) - P(X_1 \leq t | X_1 > a)|$$

converge to zero almost surely as $n \to \infty$.

(c) For a fixed $t > a$, find the asymptotic distribution of

$$\sqrt{n}\left[\hat{G}_n(t) - P(X_1 \leq t | X_1 > a)\right]$$

as $n \longrightarrow \infty$.