COMPREHENSIVE EXAM

Probability & Stochastic Processes

August 25, 1986

2 Hours

To All Students.

This is a closed book, closed notes exam. ***Please start each problem on a new sheet of paper.*** Indicate clearly on each sheet which problem you are working. The problems are grouped into 3 parts as follows:

Part I. Problems 1-5: 524 Material
Part II. Problems 6-8: 624 Material
Part III. Problems 9-14: 703-704 Material.

Each problem is worth 20 points.

Students Seeking a Master's Level Pass.

Work any *six* problems with the restriction that ***at least 2 problems must be chosen from Part II: Problems 6-8.*** (*either* choose 2 Problems from Part II and 4 other problems *or* choose 3 problems from Part II and 3 other problems). Be advised that Part III deals with the more advanced material of 703 and 704. Do not hand in more than six problems. (Possible total is 120 points.)

Students Seeking a Ph.D. Level Pass.

Work any *five* problems with the restriction that ***at least 4 problems must be chosen from Part III: Problems 9-14.*** (*either* choose 4 problems from Part III and 1 other problem, *or* choose 5 problems from Part III.) Do not hand in more than five problems. (Possible total is 100 points.)

Part I. 524 Material

1. (a) State the Central Limit Theorem for a sequence $\{X_i\}$ of i.i.d. random variables. (6)

   (b) Show that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}. \qquad (14)$$

2. An urn contains a white and b black balls. After a ball is drawn, it is returned to the urn if it is white; but if it is black, it is replaced by a white ball from another urn. Let M_n denote the expected number of white balls in the urn after the foregoing operation has been repeated n times.

   (a) Derive the recursive equation

$$M_{n+1} = \left(1 - \frac{1}{a+b}\right) M_n + 1 \qquad (10)$$

   (b) Use part (a) to prove that $M_n = a + b - b\left(1 - \frac{1}{a+b}\right)^n$. (10)

3. Let $\{X_i\}$ be a sequence of i.i.d. Bernoulli random variables with $P(X_1=0) = 1-p$ and $P(X_1=1) = p$ for some $p\,\{0<p<1\}$. Let N be an integer-valued random variable independent of the X_i's such that $P(N=k) = e^{-\lambda}\lambda^k/k!, \quad k=0,1,\ldots.$ Find the distribution of $S_N = X_1+\cdots+X_N$. (20)

4. Let $X_1,X_2,\ldots,X_n$ be independent identically distributed random variables with common distribution function F. Define the empirical distribution function

$$F_n(x) = \frac{1}{n} \times \text{number of observations} \leq x\ .$$

(a) If x and y are any two fixed real numbers with $x \leq y$, find expressions for $E(F_n(x))$, $\mathrm{Var}(F_n(x))$ and $\mathrm{Cov}(F_n(x),F_n(y))$ in terms of $F(x)$ and $F(y)$. (17)

(b) Describe the distribution of $nF_n(x)$ for fixed x and n, and give an expression for $P(nF_n(x)=r)$ in terms of n,r and $F(x)$. (3)

[Hint: Use the indicator variables $I_i(x)$ defined by $I_i(x)=1$ (or 0) if and only if $X_i \leq x$ (or $X_i > x$)].

5. (a) Consider the gamma distribution with parameters α and γ having probability density function

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x},\ x \geq 0\ .$$

Find its moment generating function. (5)

(b) Use the m.g.f. to find the mean and variance of this distribution. (9)

(c) If X_1 and X_2 are independent random variables such that X_1 has a gamma distribution with parameters α_1 and λ while X_2 has a gamma distribution with parameters α_2 and λ , prove that X_1+X_2 has a gamma distribution with parameters $\alpha_1+\alpha_2$ and λ. (6)

Part II. 624 Material

6. (a) Find the Forward Kolmogorov equations for a birth and death process with constant birth and death rates λ and μ respectively. Assume the process is initially at 0. (6)

(b) Define for this process $X(t)$:

$$P_n(t) = P(X(t)=n)$$
$$\phi_n(\theta) = \int_0^\infty e^{-\theta t} P_n(t)\mathrm{dt}, \quad \theta>0$$
$$G(\theta,z) = \sum_{n=0}^{\infty} \phi_n(\theta) z^n, \qquad |z|\leq 1 .$$

Show that for $n>0$

$$\int_0^\infty e^{-\theta t} P_n'(t)\mathrm{dt} = \theta\phi_n(\theta)$$

and obtain a similar formula for the case $n=0$. (5)

(c) Obtain a set of recurrence relations for the sequence $\phi_n(\theta)$ defined above and hence show that

$$G(\theta,z) = \frac{z+\mu\phi_0(\theta)(z-1)}{z(\theta+\lambda+\mu)-\lambda z^2-\mu} . \qquad (9)$$

7. (a) Define a *stationary distribution* for a time-homogeneous Markov Chain. (3)

A store sells widgets. At the close of business on Friday the store counts the number of widgets on hand. If there are 0 or 1, the store orders an additional 2. These two arrive prior to the start of business on Monday morning (the next business day). Unfulfilled demand is assumed to be lost with no back-orders taken. Let X_n denote the number of widgets on hand at the end of the n'th week. Assume that the weekly demands are i.i.d. random variables, with $P(D=0) = 0.3$, $P(D=1) = 0.3$, $P(D=2) = 0.2$ and $P(D \geq 3) = 0.2$. Assume that initially there are 3 items on hand. Observe that X_n is a (time-homogeneous) Markov Chain.

(b) Check that a stationary distribution for X_n is given by $(\pi_0, \pi_1, \pi_2, \pi_3) = \left(\frac{26}{80}, \frac{21}{80}, \frac{24}{80}, \frac{9}{80}\right)$. Is this the only stationary distribution? (12)

(c) Does the limiting distribution $\lim_{n \to \infty} P(X_n = k)$, $k = 0,1,2,3$, exist? Explain your answer. If it does exist, find it. (5)

[Hint: How many equivalence classes are there?]

8. (a) Derive the backward Kolmogorov equations for a pure birth process $X(t)$ with $\lambda_n = n\lambda$ and 1 ancestor at $t=0$. (6)

(b) Let $\phi(s,t)$ denote the probability generating function of $X(t)$ (written as a function of s) for the given process. Show that ϕ satisfies the partial differential equation

$$\frac{\partial\phi}{\partial t} = \lambda\phi(\phi-1)\,. \qquad (4)$$

(c) Solve the above equation to obtain $\phi(s,t)$ and hence obtain $P(X(t) = n \mid X(0) = 1)$. (10)

Part III. 703-704 Material

9. Let a_n and b_n be sequences of real numbers such that $b_n > 1$ for all n, $\sum_{i=1}^{\infty} a_i^2 < \infty$ and, for some k with $0 < k \leq \infty$, $n^{-1} \sum_{j=1}^{n} a_j^2 b_j^2 \to k$ as $n \to \infty$. Let $\{X_n\}$ be a sequence of independent random variables with $P(X_n = 0) = (0.5)(1 - a_n^2)$, $P(X_n = 1) = P(X_n = -1) = (0.25)$, $P(X_n = b_n) = P(X_n = -b_n) = (0.25)a_n^2$. Define $S_n = \sum_{i=1}^{n} X_i$, $X_i' = X_i I_{[|X_i| \leq 1]}$ and $T_n = \sum_{i=1}^{n} X_i'$.

(a) By considering the event $\{X_i \neq X_i' \text{ infinitely often}\}$, show that

$$\frac{S_n - T_n}{\sqrt{n/2}} \xrightarrow{a.s.} 0. \qquad (7)$$

(b) Find the limiting distribution of $S_n / \sqrt{\mathrm{Var}(S_n)}$. (10)

(c) Hence show that the Lindeberg condition for the sequence $\{X_i\}$ does not hold. (3)

$$P\left(\left|\frac{S_n - T_n}{\sqrt{n/2}}\right| > \varepsilon \text{ i.o.}\right) = 0$$

$$\sum P(S_n \neq T_n) = \sum_{k=1}^{\infty} \sum_{n=1}^{k} a_n^2$$

10. (a) Let $\{X_n, \mathcal{a}_n\}$ be a martingale. Suppose for all positive integers k, ϵ_k is a random variable defined as $\epsilon_k = 1$ if and only if $(X_1, X_2, \ldots, X_k) \in B_k$, $\epsilon_k = 0$ otherwise, where B_k is a Borel set in E^k. Define $Y_1 = X_1$, $Y_2 = X_1 + \epsilon_1(X_2 - X_1), \ldots,$ $Y_n = X_1 + \epsilon_1(X_2 - X_1) + \cdots + \epsilon_{n-1}(X_n - X_{n-1})$. Prove that $\{Y_n, \mathcal{a}_n\}$ is a martingale and that $E(Y_n) = E(X_n)$ for all n. (10)

(b) An urn contains m balls. Some are white while the remainder are blue. One ball is drawn and then replaced by two of the same color. This process is repeated. (For example, whenever the urn contains c white and $r-c$ blue balls and a white ball is drawn, then the fraction of white balls in the urn before the next drawing is $(c+1)/(r+1)$.) If X_n denotes the fraction of white balls in the urn after the $(n-1)$'st draw and just before the n'th draw show that $\{X_n\}$ is a martingale, that it converges to a limit X_∞, and that $E(X_\infty) = E(X_1)$. (10)

11. Let $(\Omega,\mathcal{A},P)$ be a probability space. Let $A_1,A_2,\ldots,A_n$ be any events. For each $w \in \Omega$ define the event

$$S_w = \left(\bigcap_{i:w\in A_i} A_i\right) \cap \left(\bigcap_{i:w\notin A_i} A_i^c\right).$$

Define the real valued function P from $\Omega \times \mathcal{A}$ into R by

$$P(w,A) = \begin{cases} P(A\cap S_w)/P(S_w) & \text{if } P(S_w)>0 \\ P(A) & \text{if } P(S_w)=0 \end{cases}$$

Show that $P(w,A)$ is a regular conditional probability on $\mathcal{A}$ given $\sigma(A_1,A_2,\ldots,A_n)$.

[Hint: It may be helpful to consider the events

$$G_T = \left(\bigcap_{i\in T} A_i\right) \cap \left(\bigcap_{i\notin T} A_i^c\right)$$

where T is a subset of $\{1,2,3,\ldots,n\}$.] (20)

12. Let $\{X_{in}:1\leq i\leq n\}$ be a triangular array such that for each w, $X_{1n},X_{2n},\ldots,X_{nn}$ are independent normal random variables and suppose that $E(X_{in})=0$, $\mathrm{Var}(X_{in})=\sigma_{in}^2$. Also, assume that $\sum_{i=1}^{n}\sigma_{in}^2\leq n^{-2}$ for each n. Show that $S_n=X_{1n}+X_{2n}+\cdots+X_{nn}$ converges to zero almost surely.

[Hint: Recall that $S_n \xrightarrow{a.s.} 0$ if $\sum_{n=1}^{\infty}P(|S_n|>\epsilon)<\infty$ for all $\epsilon>0$.] (20)

13. (a) Let X be a random variable with distribution function F. Assuming that F is continuous, find the distribution of $F(X)$. (11)

(b) Let U be a random variable which is uniformly distributed on (0,1). Let F be any distribution function. Define G by $G(u)=\sup\{x: F(x)<u\}$ for $u\in[0,1)$.. Prove that $G(U)$ has distribution function F. (9)

14. (a) On a probability space $(\Omega, \mathcal{A}, P)$ is defined an i.i.d. sequence of random variables $V_1, V_2, \ldots,$ with a common uniform distribution on $[0,1]$. $X_n = \max(V_1, V_2, \ldots, V_n)$. On another probability space $(\Omega', \mathcal{A}', P')$ is defined a sequence of random variables $\{X_n'\}$ such that X_n and X_n' are identically distributed for each n. Show that $X_n \xrightarrow{d} 1$, $X_n \xrightarrow{a.s.} 1$, $X_n' \xrightarrow{d} 1$, $X_n' \xrightarrow{a.s.} 1$. $\xrightarrow{d}$ and $\xrightarrow{a.s.}$ denote convergence in distribution and almost surely, respectively. (10)

(b) Let $W, W_1, W_2, \ldots,$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{A}, P)$ and let $W', W_1', W_2', \ldots,$ be a sequence of random variables defined on another probability space $(\Omega', \mathcal{A}', P')$. Suppose that for each n, W_n and W_n' are identically distributed and that W and W' are also identically distributed. For each of the following three statements, either prove it if it is true, or give a counter example if it is false:

(i) $W_n \xrightarrow{d} W$ implies $W_n' \xrightarrow{d} W'$

(ii) $W_n \xrightarrow{a.s.} W$ implies $W_n' \xrightarrow{d} W'$

(iii) $W_n \xrightarrow{a.s.} W$ implies $W_n' \xrightarrow{a.s.} W'$

(10)

$\forall \varepsilon$. $P\left(|X_n - 1| > \varepsilon \text{ i.o.}\right) = 0$

$P(|X_n - 1| > \varepsilon) = P(X_n < 1-\varepsilon)$

$= P(X_1 < 1-\varepsilon \ldots X_n < 1-\varepsilon)$

$= (1-\varepsilon)^n$