Univ. of Kentucky

COMPREHENSIVE EXAM

Probability & Stochastic Processes
August 24, 1987
2 Hours

To All Students;

This is a closed book, closed notes exam. *Please start each problem on a new sheet of paper.* Indicate clearly on each sheet which problem you are working. The problems are grouped into 2 parts as follows:

Part I. Problems 1—8: 524 and 624 Material

Part II. Problems 9—14: 703—704 Material.

Any standard results that you use should be specifically quoted. Each problem is worth 20 points.

Students Seeking a Master's Level Pass.

Work any *six* problems. Be advised that Part II deals with more advanced material of 703 and 704. Do not hand in more than six problems. (Possible total is 120 points.)

Students Seeking a Ph.D. Level Pass.

Work any *five* problems with the restriction that *at least 4 problems must be chosen from Part II: Problems 9—14.* (*either* choose 4 problems from Part II and 1 other problem, *or* choose 5 problems from Part II). Do not hand in more than five problems. (Possible total is 100 points.)

Part I.

1. Let $X_1, X_2, \ldots,$ be a sequence of i.i.d. continuous random variables. We say that a record occurs at time n if $X_n > \max\{X_1, \ldots, X_{n-1}\}$. Show

   (i) [5 Points]
   $P\{\text{a record occurs at time } n\} = \frac{1}{n}$

   (ii) [2 Points]
   $E\{\text{number of records by time } n\} = \sum_{i=1}^{n} \frac{1}{i}$

   (iii) [8 Points]
   $\text{Var}\{\text{number of records by time } n\} = \sum_{i=1}^{n} \frac{i-1}{i^2}$

   (iv) [5 Points]
   If $N = \min\{n: n > 1 \text{ and a record occurs at time } n\}$, then what is $E(N)$?

2. Let m be a fixed non-negative integer. Let $X_1, X_2, \ldots, X_n$ be random variables with $E(X_i^2) < \infty$ for each i. Assume $\mathrm{Cov}(X_i, X_j) = 0$ if $|j-i| > m$ and $\mathrm{Cov}(X_i, X_j) \leq k$ for all i and j. $k > 0$ and k does not depend on i or j.

   (a) [5 Points]
   Show that $\mathrm{Var}(\sum_{i=1}^{n} X_i) \leq kn(2m+1)$.

   (b) [5 Points]
   Using (a) show that for any $\epsilon > 0$,

$$P(|\bar{X} - n^{-1}\sum_{i=1}^{n} E(X_i)| > \epsilon) \to 0$$

   where $\bar{X} = \sum_{i=1}^{n} X_i / n$.

   (c) [10 Points]
   Let $Y_1, Y_2, \ldots,$ be independent identically distributed random variables with $E(Y_1) = 0$ and $E(Y_1^2) = 1$. Let $X_i = Y_i + Y_{i+1}$, $i = 1, 2, \cdots$. Show that for any $\epsilon > 0$,

$$P(|\bar{X}| > \epsilon) \to 0$$

   where $\bar{X} = \sum_{i=1}^{n} X_i / n$.

3. Let $X, X_1, X_2, \ldots,$ be independent identically distributed discrete random variables with

$$P(X=j) = \left.\begin{array}{ll} p_j & j=1,2,3,\ldots, \\ 0 & \text{elsewhere} \end{array}\right\}$$

and $E(X)<\infty$. Let $M_n = \min(X_1, X_2, \ldots, X_n)$ and let F denote the distribution function of X.

(a) [4 Points]
Give an expression for the distribution function of M_n in terms of F.

(b) [9 Points]
For arbitrary positive integers i and j, give an expression for the conditional probability $P(M_n = i \mid M_{n-1} = j)$ in terms of F and $\{p_r\}$.

(c) [7 Points]
Prove that $E(M_n \mid M_{n-1} = j) = \sum_{i=1}^{\infty} \mathrm{Min}(i,j) p_i$ for any integer $j \geq 1$.
Write down the limit of this expression as $j \rightarrow \infty$.

4. Let $\{X_i\}$ be i.i.d. and let M be a positive integer valued random variable which is independent of the X_i's. Consider the random sum of random variables defined by $Y = \sum_{i=1}^{M} X_i$.

   (a) [7 Points]
   Derive an expression for the moment generating function (m.g.f.) of Y in terms of the m.g.f. of the common distribution of the X_i's and the probability generating function of M.

   (b) [6 Points]
   Suppose that the p.d.f. of the X_i's is $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$ where $\lambda > 0$ and that $P(M=k) = p(1-p)^{k-1}$ for $k = 1, 2, \ldots,$ where $0 < p < 1$. Show that Y has an exponential distribution and identify its parameter.

   (c) [7 Points]
   Now suppose instead that M has a negative binomial distribution with probability mass function $P(M=k) = (k-1)p^2(1-p)^{k-2}$ for $k = 2, \cdots$. If the X_i's are still i.i.d. exponential as in (b), what is the distribution of Y?

5. Let $\{N(t): t \geq 0\}$ represent a nonhomogeneous Poisson process having intensity function $\lambda(t) = \lambda t$.

   (a) [3 Points]
   The probability that n events occur between $t=4$ and $t=5$ is ________.

   (b) Let $T_1, T_2, \ldots,$ denote the interarrival times of events in the process.

      (i) [5 Points]
      Find the p.d.f. of T_1.

      (ii) [5 Points]
      By conditioning on the event $T_1 = s$ show that the p.d.f. of T_2 is $f_{T_2}(t) = \lambda^2 \int_0^\infty s(t+s)e^{-\lambda(t+s)^2/2}\,ds$.

      (iii) [2 Points]
      Are T_1 and T_2 independent or identically distributed?

   (c) [5 Points]
   Given that $N(T) = 1$ find the probability distribution of T_1, the time at which the lone event occurred.

6. Suppose that $\{X(t): t \geq 0\}$ is a pure death process in which $X(0) = N$ and in which for h small and $n = 1,2, \ldots, N$ we assume

$$P\{X(t+h) = n-1 | X(t) = n\} = \mu_n h + o(h).$$

(a) [4 Points]
Derive the set of Kolmogorov forward equations (KFEs) for this process.

(b) Solve the KFEs in the following 2 cases (use mathematical induction in each case).

(i) [5 Points]
$\mu_n = n\mu$, $1 \leq n \leq N$.

(ii) [4 Points]
$\mu_1, \mu_2, \ldots, \mu_N$ are all equal to the common positive number μ.

(c) [5 Points]
Derive an expression for $E\{X(t)\}$ and $\text{Var}\{X(t)\}$ in the case (b)—(i).

(d) [2 Points]
Let $\{x(t): t \geq 0\}$ represent the deterministic version of the process $\{X(t): t \geq 0\}$. State and solve the differential equation that $x(t)$ must satisfy when $\mu_n = n\mu$ and $x(0) = N$. Relate your result to part (c).

7. After treatment by irradiation a certain kind of insect will produce zero offspring with probability q or 2 offspring with probability $p = 1-q$. For $n = 1, 2, \ldots,$ let X_n = the number of insects in the n^{th} generation given that $X_0 = i$.

(a) [7 Points]
Find the probability distribution of X_2 when $i = 2$.

(b) [7 Points]
Find $E(X_n)$ and $\text{Var}(X_n)$ when $i = 1$. You may use the fact that for $n \geq 1$ we have

$$\text{Var}(X_n) = \mu^{n-1}\sigma^2 + \text{Var}(X_{n-1})\mu^2.$$

(c) [6 Points]
What is the probability that eventually there will be no descendants of 3 irradiated mice? Under what conditions is extinction a certainty?

8. All Markov chains in this question are time-homogeneous.

(a) [5 Points]

Suppose we are given an irreducible Markov chain with states 1,2,3, . . . , and initial state 1 and a stationary distribution π. Does it follow that π is also the limiting distribution? If so, prove it. If not, give a counterexample.

(b) [10 Points]

Prove that a Markov chain with more than one stationary distribution has infinitely many stationary distributions. Show that the Markov chain with transition matrix

$$P = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1/4 & 3/4 \end{pmatrix} \end{array}$$

is such a Markov chain. For this chain find the mean recurrence time for each state.

(c) [5 Points]

For the 2-state Markov chain with transition matrix

$$P = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} a & 1-a \\ 1-b & b \end{pmatrix} \end{array}$$

and $0 < a,\ b < 1$, find the recurrence time distribution for state A.

9. (a) [6 Points]

Using only the probability axioms prove that if $\{A_i\}$ is an increasing sequence of events with union A, then $P(A_n) \to P(A)$.

(b) [10 Points]

Let X, Y be random variables such that for all finite real x, y, $P(X \leq x, Y \leq y) = g(x)h(y)$ for some real valued functions g and h. Prove that $\lim_{x \to \infty} g(x) = g(\infty)$, $\lim_{y \to \infty} h(y) = h(\infty)$ exist and that X and Y are independent. Show carefully how the result of part (a) is used in your proof.

(c) [4 Points]

Suppose X, Y are continuous random variables such that for all finite x, y, some fixed set $A \subset \mathbb{R}^2$, and strictly positive real functions g and h,

$$P(X \leq x, Y \leq y) = \left. \begin{matrix} g(x)h(y) & (x,y) \in A \\ 0 & (x,y) \notin A \end{matrix} \right\}.$$

Prove that X and Y are independent if and only if A is of the form $\{(x,y): x > x_0, y > y_0\}$ for some $x_0 \geq -\infty$, $y_0 \geq -\infty$.

10. (a) [5 Points]
State the Lindeberg's Central Limit Theorem for an array of random variables.

(b) Suppose $X_1, X_2, \ldots, X_n$ is a sequence of independent random variables such that $P[X_n = \sqrt{n}] = \frac{1}{2} = P[X_n = -\sqrt{n}]$. Let $S_n = \sum_1^n X_i$ and N be the standard normal variable. Prove that

(i) [7 Points]

$$\frac{S_n}{n} \to \frac{1}{\sqrt{2}} N \text{ as } n \to \infty;$$

(ii) 8 Points]

$$E\frac{S_n^2}{n^2} \to \frac{1}{2} \text{ as } n \to \infty.$$

$ES_n^2 = \sum Var X_i$

$= \sum EX_i^2 = \sum_{i=1}^{n} i =$

$1+2+3+4+\cdots+n$

$\frac{n}{2} \cdot (n+1)$

11. Let $(\Omega, \mathcal{A}, P)$ be a probability space.

(i) [8 Points]

State and prove the Borel-Cantelli Lemmas for a sequence of independent events $\{A_n\}$.

(ii) [12 Points]

Let $\{X_n\}$ be a sequence of independent and identically distributed random variables with common distribution $N(0,1)$. Prove that

$$P\left[\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = 1\right] = 1.$$

$$\left\{\text{Hint: } \frac{y}{1+y^2} e^{-\frac{1}{2}y^2} \le \int_y^\infty e^{-\frac{1}{2}x^2}dx \le \frac{1}{y}e^{-\frac{1}{2}y^2}.\right\}$$

$$P\left(|X_n| > \alpha\sqrt{2\log n} \ \ i.o.\right) = \begin{cases} 0 & \alpha > 1 \\ 1 & \alpha \le 1 \end{cases}$$

$$P\left(X_n > \alpha\sqrt{2\log n} \ \ i.o.\right) = 0$$

$$\sum P(X_n > \alpha\sqrt{2\log n}) < \infty$$

$$\sum \int_{\alpha\sqrt{2\log n}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \sim \sum \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha\sqrt{2\log n}} e^{-\alpha^2\log n}$$

$$\sum \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha\sqrt{2\log n}} e^{\log n^{-\alpha^2}}$$

$$\sum \frac{1}{\sqrt{2\pi}\,\alpha\sqrt{2\log n}}\, n^{-\alpha^2}$$

$$\sum \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha\sqrt{2\log n}} \frac{1}{n^{\alpha^2}}$$

12. Let X and Y be two integrable non-negative random variables on $(\Omega,\mathbb{F},P)$. Suppose $\mathbb{F}_1 \subset \mathbb{F}_2 \subset \mathbb{F}$ are sub-σ-fields.

(i) [4 Points]
Prove $E\big(YE(X|\mathbb{F}_2)\big) = E\big(XE(Y|\mathbb{F}_2)\big)$;

(ii) [6 Points]
Prove $E\left\{\left(X-E(X|\mathbb{F}_2)\right)^2\right\} \le E\left\{\left(X-E(X|\mathbb{F}_1)\right)^2\right\}$ a.s.;

(iii) [10 Points]
Further suppose that Y is discrete and integer valued. Show that

$$E(Y|\mathbb{F}_1) = \sum_{n=0}^{\infty} P(Y > n|\mathbb{F}_1) \text{ a.s.}.$$

13. Let $Y_1, Y_2, \ldots,$ be a sequence of independent random variables. Set $\mathbb{F}_n = \sigma\{Y_1, \ldots, Y_n\}$ and $\tau_n = \sigma\{Y_n, Y_{n+1}, \cdots\}$. Let $A \in \mathbb{F}_\infty = \sigma(\bigcup_{n \geq 1} \mathbb{F}_n)$. Define $X_n = E(I_A | \mathbb{F}_n)$. Prove the following.

(i) [4 Points]
$\{X_n, \mathbb{F}_n\}$ is a Martingale;

(ii) [8 Points]
$X_n \to I_A$ a.e.;

(iii) [8 Points]
If $A \in \bigcap_{n \geq 1}^{\infty} \tau_n$ then $P(A) = 0$ or 1.

14. Let $(\Omega,\mathbb{F},P)$ be a probability space and $\mathbb{F}_n,\mathbb{G}_n$ be sequences of sub-σ-fields of $\mathbb{F}$. Assume $(X_n,\mathbb{F}_n)$ and $(Y_n,\mathbb{G}_n)$ are Martingales for some random variables X_n and Y_n. Let $\mathbb{H}_n = \sigma(X_1+Y_1, X_2+Y_2, \ldots, X_n+Y_n)$.

   (a) [6 Points]
   Prove that if $\mathbb{F}_n = \mathbb{G}_n$ then $(X_n+Y_n,\mathbb{F}_n)$ is a Martingale;

   (b) [14 Points]
   Prove that if $\{X_n\}$ and $\{Y_n\}$ are independent collections of random variables then $(X_n+Y_n,\mathbb{H}_n)$ is a Martingale (even if $\mathbb{F}_n \neq \mathbb{G}_n$).