Ph.D. Qualifying Exam

Probability

May 29, 2002
9 a.m.-12 p.m.

1. Start each question on a clean sheet of paper.
2. Put your name (or at least initials) at the top of each page you turn in.
3. Point values: #1 is worth 10 points. All others are worth 15 points.

1. Let $X_1, X_2, \ldots$ be a sequence of independent Bernoulli variables such that

$$X_i = 1 \text{ with probability } p_i$$
$$= 0 \text{ with probability } 1 - p_i = q_i$$

for $i = 1, 2, \ldots$

if $S_n = X_1 + X_2 + \ldots + X_n$

Find sufficient conditions on the p_i such that

$$\frac{(S_n - \sum_1^n p_i)}{(\sum_1^n p_i q_i)^{1/2}}$$ is asymptotically standard normal as n becomes large.

[*Note: Don't just state conditions. Verify your claim that these conditions are sufficient.*]

2. (a) Let $X_1, X_2, \ldots, X_n$ be *iid* random variables having density function f or g defined on the real line where $f(x) > 0$ and $g(x) > 0$ for all x. Define the likelihood ratio by

$$L_n = \prod_{k=1}^{n} \frac{g(x_k)}{f(x_k)} \quad \text{where } L_o = 1.$$

(i) If f is the true density function of the X's, show that L_n is a martingale over $(\Omega, \mathcal{F}, P_f)$ where P_f denotes the probability under which the X_k have density f, for each n.

(ii) If the X_k have the density g under P_g, then show that L_n is a submartingale.

(b) Let $\{X_n\}_{n=1}^{\infty}$ be a martingale with respect to $\{\mathcal{F}_n\}_{n=1}^{\infty}$. Let

$Y_1 = X_1$ and $Y_n = X_n - X_{n-1}$ for $n \geq 2$. Suppose Z_{n+1} is $\mathcal{F}_n$ - measurable for $n = 1, 2, \ldots$ such that $E(Z_n Y_n)$ is finite for all $n \geq 2$.

Define $U_1 = Y_1$ and for $n \geq 2$, $U_n = Y_1 + \sum_{i=2}^{n} Z_i Y_i$.

Prove that $\{U_n\}_{n=1}^{\infty}$ is a martingale.

3. Prove or give counterexample: Let $S_n = \sum_{i=1}^{n} X_i$

(a) If X_n converges almost surely to X, then X_n converges in probability to X.

(b) If X_n converges in probability to 0, then $\frac{S_n}{n}$ converges in probability to 0.

(c) If $\frac{S_n}{b_n}$ converges in probability to 0, and $\frac{b_{n+1}}{b_n}$ converges to 1, then $\frac{X_n}{b_n}$ converges to 0 in probability.

4. Let $\{X_n\}_{n=1}^{\infty}$ and X be random variables. Suppose that $E(f(X_n)) \to E(f(X))$ for all bounded uniformly continuous functions *f*. Prove that $X_n \xrightarrow{d} X$.

5. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of d.f.'s with ch.f.'s $\{\phi_n\}_{n=1}^{\infty}$. Assume $\{F_n\}_{n=1}^{\infty}$ is a tight family. Suppose F is a d.f. with ch.f. ϕ. Prove that if $\lim \phi_n(t) = \phi(t)$ for all $t \in \mathbb{R}$, then $\lim F_n(x) = F(x)$ for every *x* which is a continuity point of *F*.

6. Prove the following:

(a) Suppose that X is a nonnegative random variable. Then, for any a > 0, $P(X \geq a) \leq \frac{E(X)}{a}$

(b) If X is a random variable with finite mean μ and variance σ^2, then for any a > 0 $P(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}$

(c) If X is a random variable with mean 0 and variance σ^2, then for any a > 0, $P(X \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$ [Hint: Observe that for $b > 0$, $X \geq a$ is equivalent to $X + b \geq a + b$, etc. Find a value of b which minimizes the probability.]

(d) If X is a random variable with mean μ and variance σ^2, then

$$P(X \geq \mu + a) \leq \frac{\sigma^2}{\sigma^2 + a^2} \quad \text{and}$$

$$P(X \leq \mu - a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

(e) Show that (d) implies that $P(|X-\mu| \geq a) \leq 2\frac{\sigma^2}{\sigma^2+a^2}$. How does this bound compare to the bound in (b)?

(f) Suppose that $\{X_n\}_{n=1}^{\infty}$ have common finite mean μ and finite positive variances $\{\sigma_n^2\}$ which are uniformly bounded (say by B). Assume further that $Cov(X_i, X_j) = 0$ if $|i-j| > 2$. Prove that $\frac{1}{n}\sum_{i=1}^{n} X_i$ converges in probability to μ.

7. (a) Suppose that for all $a, b \in \mathbb{R}$ with $a < b$, we have $P(X_n < a$ infinitely often, and $X_n > b$ infinitely often$) = 0$. Prove that $\lim X_n$ exists almost surely, although it may be infinite with positive probability.

(b) Suppose $\{X_n\}_{n=1}^{\infty}$ are independent random variables. Prove that $P(\lim X_n = 0)$ is either 0 or 1. [Hint: First, prove that $(\lim X_n = 0) = \bigcap_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (|X_n| \leq \frac{1}{m})$]