Comprehensive Examination in

Probability and Stochastic Processes

May, 2007

You are to answer 6 problems; 3 problems must be chosen from each section.

This exam has two sections. Section 1 contains problems on Probability and Section 2 contains problems on Stochastic Processes.

Closed book, closed notes. No computer. Calculator, if needed, are allowed.

Section 1: Probability

Answer three questions from this section. Indicate which 3 you chose.

1. Suppose random variables (X, Y) are independent, both with exponential $(\lambda = 1)$ distribution.

   (a) find the joint distribution or pdf of (U, V) with

   $$U = X + Y, \quad V = X - Y.$$

   (b) find the marginal distribution or pdf of $V = X - Y$.

2. Suppose $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of exponential $(\lambda = 1)$ random variables.

   (a) Show that

   $$\frac{X_n}{n} \longrightarrow 0$$

   in probability as $n \to \infty$.

   (b) Show that

   $$\frac{X_n}{n} \longrightarrow 0$$

   almost surely as $n \to \infty$.

   (c) Show that

   $$\frac{\max_{1 \le i \le n} X_i}{n} \longrightarrow 0$$

   in probability as $n \to \infty$.

3. (a) For a negative random variable X with $E|X| < \infty$, show that

$$-\int_{-\infty}^{0} F(t)dt$$

is finite and is equal to $E(X)$ where $F(t)$ is the CDF of X.

(b) For a non-negative random variable Y, with $E(Y) < \infty$ show

$$\sum_{n=1}^{\infty}(1 - F(n)) \leq E(Y) \leq \sum_{n=0}^{\infty}(1 - F(n))$$

where F is the CDF of Y.

(c) What if the expectations $(E|X|, E(Y))$ are infinite in (a), (b) ? Does the equality [inequality] still hold?

4. Suppose X is a binomial (n, p) random variable.

(a) compute the moment generating function of X.

(b) If Y is a binomial (m, p) random variable and independent of X, show that $X + Y$ is binomial $(n + m, p)$.

(c) Suppose $n \to \infty$ and $p = p_n \to 0$ in such a way that $n \cdot p_n \to \lambda > 0$. Show that the binomial random variable X converges to a Poisson random variable, as $n \to \infty$.

(d) Does the "converge" above imply $P(X = 2)$ converge to a limit (as $n \to \infty$)? Justify your answer.

Section 2: Stochastic Processes

Answer three questions from this part. Indicate on your exam paper which 3 you chose.

Question 1: True or False Mark whether each of the following statement is true (T) or false (F). State a reason for each question. If the answer is TRUE, also specify the rate.

(a) Let $\{X_t,\, t \geq 0\}$ and $\{Y_t,\, t \geq 0\}$ be independent Poisson processes with parameter λ and β. Define $\{Z_t,\, t \geq 0\}$ such that $Z_t = X_t + Y_t$. $\{Z_t,\, t \geq 0\}$ is a Poisson process.

(b) Let $\{X_t,\, t \geq 0\}$ be a Poisson process with parameter λ. Define $\{Z_t,\, t \geq 0\}$ such that $Z_t = k \cdot X_t$ where $k > 1$ is a constant. $\{Z_t,\, t \geq 0\}$ is a Poisson process.

(c) Let $\{X_t,\, t \geq 0\}$ be a Poisson process with parameter λ. Define $\{Z_t,\, t \geq 0\}$ such that $Z_t = X_t + k$ where $k > 0$ is a constant. $\{Z_t,\, t \geq 0\}$ is a Poisson process.

(d) Let $\{X_t,\, t \geq 0\}$ and $\{Y_t,\, t \geq 0\}$ be independent Poisson processes with parameter $\lambda > 0$ and $\beta > 0$. Define $\{Z_t,\, t \geq 0\}$ such that $Z_t = X_t - Y_t$. $\{Z_t,\, t \geq 0\}$ is a Poisson process.

(e) Let $\{X_t,\, t \geq 0\}$ be a Poisson process with parameter λ. Define $\{Z_t,\, t \geq 0\}$ such that $Z_t = X_{k \cdot t}$, where $k > 0$ is a constant. $\{Z_t,\, t \geq 0\}$ is a Poisson process.

(f) Let $\{X_t,\, t \geq 0\}$ be a Poisson process with parameter λ. Define $\{Z_t,\, t \geq 0\}$ such that $Z_t = X_{t^2}$. $\{Z_t,\, t \geq 0\}$ is a Poisson process.

Question 2

(a) Given the following transition probability matrix

$$P = \begin{pmatrix} - & \frac{1}{5}\times\frac{1}{4} & \frac{1}{4} & \frac{1}{5}\times\frac{1}{4} \\ \frac{1}{5}\times\frac{1}{4} & - & \frac{1}{5}\times\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{5}\times\frac{1}{4} & - & \frac{1}{5}\times\frac{1}{4} \\ \frac{1}{5}\times\frac{1}{4} & \frac{1}{4} & \frac{1}{5}\times\frac{1}{4} & - \end{pmatrix},$$

(diagonal entries are set so that rows sum to 1), is this process time-reversible? What is the stationary distribution?

(b) A matrix $P = \{P_{i,j}\}$ is called *stochastic* if $P_{i,j} \geq 0$ for all i, j and $\sum_j P_{i,j} = 1$ for all i. A matrix P is called *doubly stochastic* if in addition to the above, $\sum_i P_{i,j} = 1$ for all j. Prove that if a finite irreducible Markov chain has doubly stochastic transition probability matrix, the stationary probabilities π_i for all i exist and are equal.

Question 3

Consider a Poisson process $\{X_t, t \geq 0\}$ with a parameter λ.

(a) Write down the rate matrix A for the process.

(b) Let $P_{i,j}(t)$ be the transition probability from a state i to state j. Write down Kolmogorov's forward and backward equations for this process, $\{X_t, t \geq 0\}$.

(c) What is the probability that k events occur in a time interval $(s, s+t]$ where $s, t > 0$ for the Poisson process, $\{X_t, t \geq 0\}$.

Question 4

(a) Consider a **discrete time** Markov chain. State the detailed balance equations and balance equations for a **discrete time** Markov chain. Prove that if the Markov chain satisfies the detailed balance equations, then it satisfies the balance equations.

(b) State the detailed balance equations and balance equations for a **continuous time** Markov chain. Prove that if the Markov chain satisfies the detailed balance equations, then it satisfies the balance equations.

Question 5

(a) There is a mission planned to Mars and you are responsible for equipping the exploration craft with enough battery power to last $t = 1$ year upon arriving on the red planet. Suppose cells in the battery fail according to a **single** Poisson process (regardless of how many cells are alive in the battery) with the fixed rate $\lambda = 2$ per year and the battery stops working when all cells have failed. If you equip the exploration craft with a 2-cell battery, what is the probability that the mission fails on account of your planning?

(b) What is this probability if the failure rate is doubled during the cold Mars nights which last 50% of everyday during the 1-year mission? (Hint: For this problem, the rate in the night and the rate in the day differ).