FREENESS AND DISCRETENESS OF ACTIONS ON R-TREES BY FINITELY GENERATED FREE GROUPS, I

By Li Lee

0. Introduction

In this paper, we investigate minimal actions of a finitely generated group G on an $\mathbf{R}$-tree T. We address the question of whether the action of G on T is free or discrete as suggested by the title.

We study this question inductively by decomposing G as a free product of smaller rank free groups G' and G'', i.e. $G = G' * G''$. Let T' and T'' be the minimal invariant subtrees for the groups G', G'' respectively.

The idea is to study the intersection T_0 of T' and T'' and the partial actions defined by G' and G'' on T_0. We prove that the action $T \times G \to T$ is free (discrete resp.) if and only if the partial action on T_0 by the set of alternating combinations of Σ' and Σ'' is free (dose not have an infinite orbit resp.), where Σ' and Σ'' are the sets of partial isometries on T_0 defined by the elements of G' and G'' respectively (see Proposition 4.1 and 4.2).

Let us now introduce the following property for an action on an $\mathbf{R}$-tree by a finitely generated group:

Property (P): The action is discrete provided that it is free.

There is an example (Bestvina-Handel) of a minimal action of F_3 on an $\mathbf{R}$-tree, which is free but not discrete, this implies that Property (P) is not true in general. In the following sections, we will investigate what actions satisfy Property (P), which is the basic theme that motivates this paper.

In order for the data to be sufficient for our study, we will introduce a hypothesis, Condition **A** (see page 8-9). We prove that if Condition **A** is true, and if the set of end points of domains of all the partial isometries on T_0 defined by elements of G is finite, then the action $T \times G \to T$ satisfies Property (P) (see Theorem 4.11). Another main result of this paper is that Condition **A** is a kind of freeness condition (in the sense of Theorem 7.1).

Section 1 contains some preliminary materials. In Section 2 and 3, we introduce the basic assumptions and conditions. The main results of this paper along with their proofs are contained in Section 4. In Section 5 and 6, we study the relations of Condition **A** to other conditions. Section 7 is devoted to the discussions on the 'freeness' (in the sense of Theorem 7.1) of Condition **A**, and Section 8 shows that this condition really makes things different. In Section 9 and 10, we provide examples which make it clear that when we adopt our assumptions, we do not loss the generality.

1. Preliminary

As good references [1], [6], [8] and [10] are highly recommended to the materials in this section.

Throughout this paper, G always represents a finitely generated free group and T always stands for an $\mathbf{R}$-tree. We use $T \times G \to T$ for the action of G on T, and $u \cdot g$ for the image of the pair (u, g) under the action, where $u \in T$ and $g \in G$.

By assumption, $G = G' * G''$. An **alternating word** (with respect to G' and G'') is an ordered family $\{a_1, a_2, \ldots a_n\}$ of elements of $G' \cup G'' - \{1\}$, such that $a_{2k} \in G'' - \{1\}$, $a_{2k+1} \in G' - \{1\}$ or $a_{2k} \in G' - \{1\}$, $a_{2k+1} \in G'' - \{1\}$ for all k. We allow the empty word to be an alternating word. For every element $g \in G$, there is a unique alternating word $\{a_1, a_2, \ldots, a_n\}$ such that g is the product of a_i's, i.e. $g = a_1 a_2 \cdots a_n$. ($g = 1$ if and only if the corresponding word is empty.) Call this word as the **alternating word of** g (in elements of G' and G''), call n as the (alternating) **word length** of g and denote it by $L(g)$. Set $g_b = a_1, g_e = a_n$ and

$$g_i = \begin{cases} 1, & \text{if } i = 0; \\ a_1 \cdots a_i, & \text{if } i \leq n \text{ and } i > 0\ ; \\ g, & \text{if } i > n. \end{cases}$$

Recall that an element $g \in G$ of positive translation length $\tau(g)$ is called a **hyperbolic element.** g acts on its axis A_g by a translation with length $\tau(g)$. If G acts on an $\mathbf{R}$-tree T and G contains a hyperbolic element, then T contains a unique minimal invariant subtree.

If S is a subset of T, H is a subset of G, set

$$S \cdot H = \{s \cdot h | s \in S, h \in H\}$$

The action $T \times G \to T$ is said to be **discrete** or **Properly discontinious** if for every $u \in T$, there is an open set U containing u such that the set $\{g \in G | U \cdot g \cap U \neq \emptyset\}$ is finite. When the action $T \times G \to T$ is free, it is discrete if and only if all the G-orbits are discrete. When G is torsion free, the action $T \times G \to T$ is discrete implies that it is free. In this paper, we deal only with finitely generated free groups, so the freeness of an action is always implied by its discreteness.

An $\mathbf{R}$-graph is called **finite** if it consists of finitely many edges and has a finite total measure; it is called **nondegenerate** if it has a positive total measure. The quotient T/G for the action $T \times G \to T$ is a finite $\mathbf{R}$-graph.

A closed subtree F of T is called a **fundamental domain** of T mod G if the following conditions are satisfied:

(a) $F \cdot G = T$

(b) If $g \in G - \{1\}$, then either $F \cap F \cdot g = \emptyset$, or $F \cap F \cdot g$ consists of a single point.

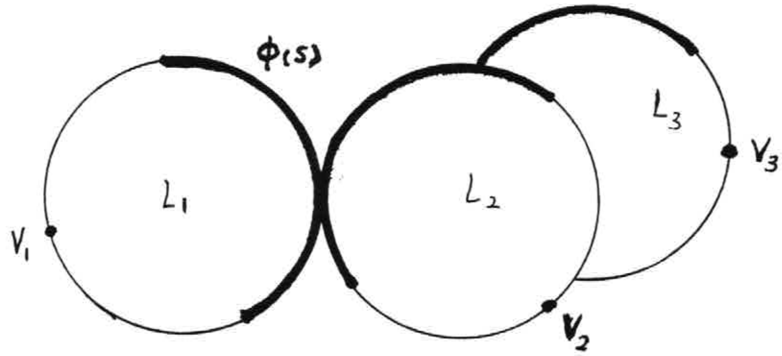
We use the letter d for the distance between points or sets as usual. Assume $p: X \to Y$ is a map, S is a subset of X, we use $p|_S$ for the map p restricted on S. and use $(S)^\circ$ for the interior of S with respect to X. When S is the union of a family of **R**-trees or **R**-graphs, we denote by $Y(S)$ ($E(S)$ resp.) the set of branch points (end points resp.) of connected components of S. If S is empty, so also are $Y(S)$ and $E(S)$.

Lemma 1.1: *Assume G is a finitely generated free group, G acts on T freely, then*

(a) If the action $T \times G \to T$ is discrete and minimal, then for every closed subtree s of T such that s is embedded into T/G under the quotient map $\phi: T \to T/G$, there is a nondegenerate finite fundamental domain F of T mod G which contains s in its interior.

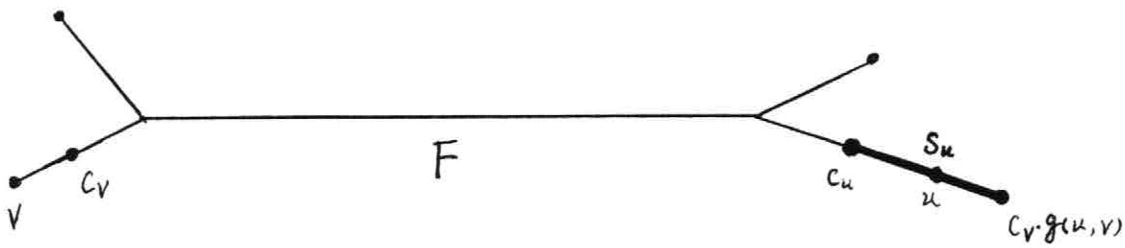
(b) If there is a nondegenerate finite fundamental domain F of T mod G, then the action $T \times G \to T$ is discrete.

Proof: (a) Since G is a finitely generated free group and the action $T \times G \to T$ is minimal, we have $E(Q) = \emptyset$ and Q consists of only finitely many subloops $l_1, l_2, \ldots, l_n$ (by a loop we mean an **R**-tree which is homeomorphic to a circle). For each subloop l_i of Q, we pick up a point $v_i \in l_i - \phi(s) - Y(Q)$ (this set is not empty because $\phi(s)$ is a closed subtree of Q). Set $\bar{L} = Q - \{v_i | i = 1, 2, \ldots, n\}$, then $\bar{L}$ is an open maximum subtree of Q containing $\phi(s)$ in its interior. There is a lift L of $\bar{L}$ in T containing s in its interior. The closure F of L is a fundamental domain of T mod G. F is finite and nondegenerate because Q is so.



(b) We want to prove that for every point $u \in T$, there is an open subtree s_u of T containing u such that $s_u \cap \{u\} \cdot (G - \{1\}) = \emptyset$. Because if this is true, then the action $T \times G \to T$ is discrete.

We may assume that $u \in F$. Suppose $u \in E(F)$, we choose a point $c_u \in F$ such that $u \neq c_u$ and $[u, c_u] \cap Y(F) = \emptyset$. Also for each $u \in E(F)$, define $X_u = \{v \in E(F) | v \cdot g = u \text{ for some } g \in G\}$, and for $v \in X_u$, denote the unique element $g \in G$ satisfying that $v \cdot g = u$ by $g(u, v)$ if it exists. Because F is finite, we have $\#E(F) < \infty$ and $\#X_u < \infty$ for every $u \in E(F)$. Set $s_u = \bigcup\{[u, c_v \cdot g(u, v)) | v \in X_u\}$, then s_u is an open subtree of T containing u.



We claim that $s_u \cap \{u\}\cdot(G-\{1\}) = \emptyset$. Suppose there is an element $g \in G - \{1\}$ such that $u\cdot g \in s_u$, then $u \cdot g \in [u, c_v \cdot g(u,v))$ for some $v \in X_u$, we have $u \cdot g \in F \cdot g \cap F \cdot g(u,v) \subset E(F \cdot g(u,v))$, so $u \cdot g \in [v, c_v) \cdot g(u,v) \cap E(F \cdot g(u,v)) = \{v \cdot g(u,v)\} = \{u\}$, i.e. $u \cdot g = u$, this is impossible because the action $T \times G \to T$ is free.

Assume $u \in (F)^\circ$, take $s_u = (F)^\circ$, then $s_u \cap \{u\}\cdot(G-\{1\}) = \emptyset$ is obviously true also. This completes the proof of Lemma 1.1 ◇

Now let us look at some simple examples:

We use F_n for the free group of rank n, and denote the free group with a free basis $\{x_1, x_2, \ldots, x_n\}$ by $F(x_1, x_2, \ldots, x_n)$.

If $F_1 = F(x)$ acts on an $\mathbf{R}$-tree T minimally and freely, then T is isomorphic to $\mathbf{R}$, and F_1 acts on T by a translation. This action is discrete and the quotient $Q = T/F_1$ is a circle.

Assume F_2 acts on an $\mathbf{R}$-tree T minimally and freely, then there is a free basis $\{x, y\}$ of F_2 such that $|A_x \cap A_y| < \min\{\tau(x), \tau(y)\}$, (see [6] as a reference). We take closed subsegments F', F'' of A_x, A_y respectively, such that

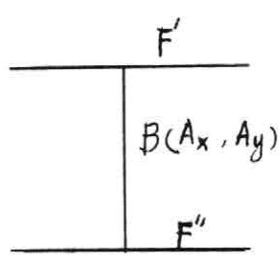
(a) $|F'| = \tau(x)$ and $|F''| = \tau(y)$.

(b) $A_x \cap B(A_x, A_y) \subset F'$ and $A_y \cap B(A_x, A_y) \subset F''$ if $A_x \cap A_y = \emptyset$, $A_x \cap A_y \subset F' \cap F''$ if $A_x \cap A_y \neq \emptyset$.

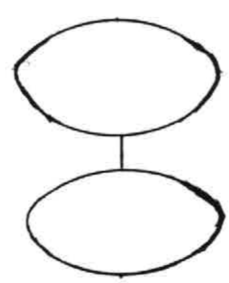
Set $F = F' \cup F''$, if $A_x \cap A_y \neq \emptyset$ and $F = F' \cup F'' \cup B(A_x, A_y)$, if $A_x \cap A_y = \emptyset$. Then F is a fundamental domain of T mod F_2, and the quotient $Q = T/F_2$ is an $\mathbf{R}$-graph with genus 2. By Lemma 1.1 (b), the action $T \times F_2 \to T$ is discrete.

Let us list the types of F and Q in all the possible cases as follows:

Case 1, $A_x \cap A_y = \emptyset$.

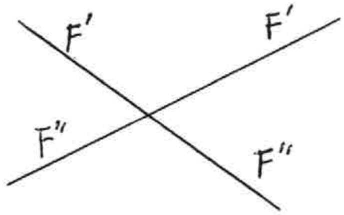


F

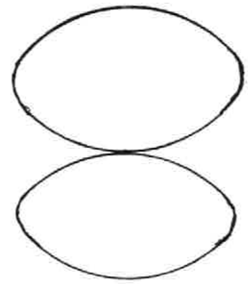


Q

Case 2, $A_x \cap A_y$ is a single point.

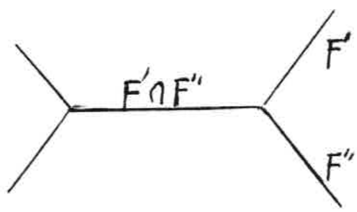


F

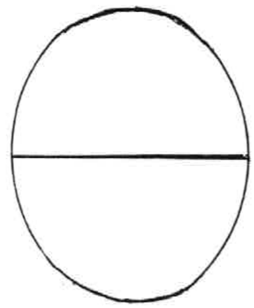


Q

Case 3, $A_x \cap A_y$ is a nondegenerate segment.



F



Q

In Case 1 and Case 3, there are two F_2-equivalence classes (or F_2-orbits) in $Y(T)$, and every point of $Y(T)$ has order 3. In Case 2, $Y(T)$ is a F_2-orbit and every point of $Y(T)$ has order 4.

Therefore actions of free group of rank 1 or 2 on **R**-trees always satisfy Property (P). But the example of Bestvina-Handel tells us that Property (P) is not true in general, if the rank of the group G is greater than 2.

2. General Assumptions

In this paper, we always make the following:

Assumption 1: The actions $T' \times G' \to T'$ and $T'' \times G'' \to T''$ are free and discrete.

This assumption is actually an inductive hypothesis for the proof of Property (P) under certain conditions. Notice that when $G = F_3$ acts freely on an **R**-tree T, if we take G' and G'' to be F_2 and F_1 respectively, then the action automatically satisfies Assumption 1.

Set

$$T_0 = T' \cap T''$$

Every element $g \in G$ induces an isometry from $T_0 \cdot g^{-1} \cap T_0$ to $T_0 \cap T_0 \cdot g$, we denote this partial isometry of T_0 by σ_g, denote its domain and range by D_g and R_g respectively, which are closed subtrees of T_0.

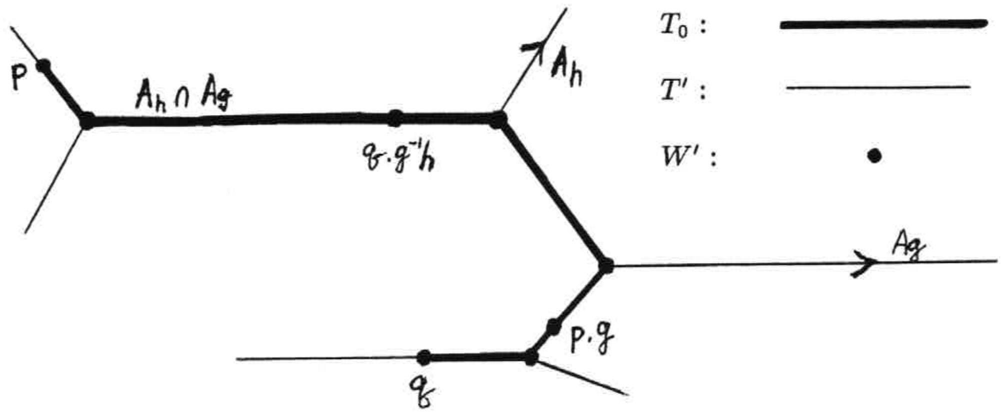
Let

$$\Sigma' = \{\sigma_g | g \in G', D_g \neq \emptyset\}$$

$$\Sigma'' = \{\sigma_g | g \in G'', D_g \neq \emptyset\}$$

$$\Sigma = \{\sigma_g | g \in G, D_g \neq \emptyset\}$$

Also let $W' = (Y(T') \cup E(T_0) \cdot G') \cap T_0$ and $W'' = (Y(T'') \cup E(T_0) \cdot G'') \cap T_0$.



$(g, h \in G')$

Lemma 2.1: *(a) $E(D_g) \subset W'$, for every $g \in G'$ and $E(D_g) \subset W''$ for every $g \in G''$.*

(b) If $|T_0| < \infty$, then $\#\Sigma' < \infty$ and $\#\Sigma'' < \infty$.

Proof: (a) Suppose $g \in G'$, we may assume that $D_g \neq \emptyset$. Take $u \in D_g$, if $u \notin W'$, then $u \cdot g \notin W'$, there is a small open segment $s \subset T_0$ containing u such that $u \cdot g \in s \cdot g \subset T_0$. Then $s \subset D_g$, so $u \notin E(D_g)$. Therefore, $E(D_g) \subset W'$.

(b) If $\text{rank}(G') = 1$, $G' = F(x)$, then T_0 is a subsegment of the axis A_x. There are only finitely many integers m satisfying $T_0 \cdot x^m \cap T_0 \neq \emptyset$, or equivalently, $D_{x^m} \neq \emptyset$, therefore $\#\Sigma' < \infty$. Assume that $\text{rank}(G') > 1$. Because $|T_0| < \infty$, $\#Y(T') \cap T_0 < \infty$. Since for every $u \in Y(T')$, Order(u) in $T' < \infty$ and for every edge e of T', $e \cap T_0 \cap Y(T') \neq \emptyset$ if $T_0 \not\subset e$, we see that T_0 is contained in finitely many edges of T', therefore, $\#E(T_0) < \infty$. As a consequence, W' is a finite set. By (a), for every $\sigma_g \in \Sigma'$, $E(D_g) \cup E(R_g) \subset W'$, since there are only finitely many partial isometries of T_0 satisfying this property, $\#\Sigma' < \infty$. Similarly, $\#\Sigma'' < \infty$. ◊

Remark: Lemma 2.1 (a) is equivalent to the following statment: For every $g \in G'$, we have $E(D_g) \cup E(R_g) \subset W'$, for every $g \in G''$, we have $E(D_g) \cup E(R_g) \subset W''$.

If $\sigma, \tau \in \Sigma$, the composition $\sigma\tau$ is defined at a point $u \in T_0$ if and only if u belongs to the domain of σ and $(u)\sigma$ belongs to the domain of τ. The product of partial isometries $\sigma_1, \sigma_2, \ldots, \sigma_n$ is the composition of them if exists, which is denoted by $\sigma_1\sigma_2 \cdots \sigma_n$. Σ acts from the right on T_0. Notice that the identity map of T_0 is included in Σ.

Because the examples in Section 9 and 10, **there is no loss of generality in making the following assumptions, as we shall do from now on:**

Assumption 2: $T_0 \neq \emptyset$.

Assumption 3: $|T_0| < \infty$.

3. Condition A and A′

Set

$$Y_0 = (Y(T') \cup Y(T'') \cup E(T_0)\cdot(G' \cup G'')) \cap T_0$$

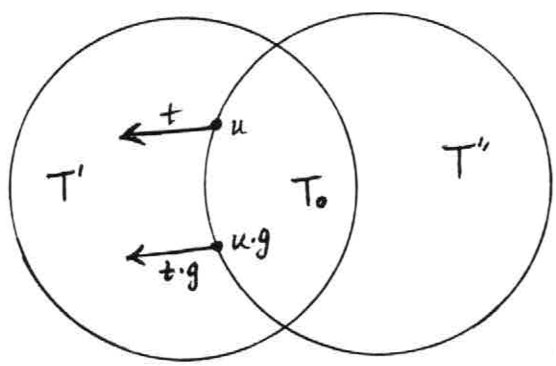
It is clear that Y_0 is a finite set.

$$S = \{(u,g) | u \in Y_0, g \in G, u \cdot g_i \in T_0 \ \forall i \geq 0\}$$

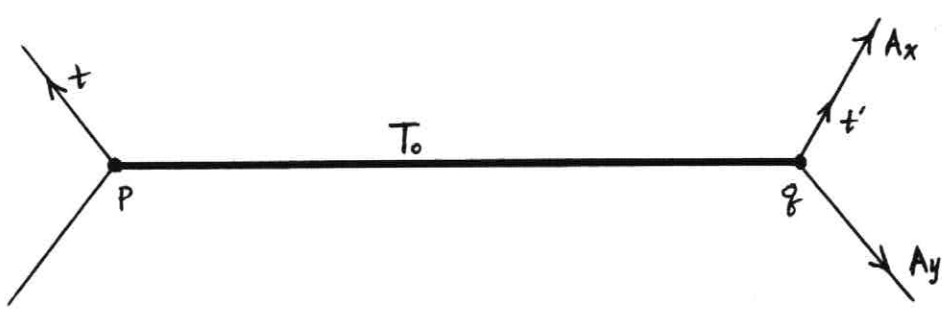
This is the set of pairs, a point u in Y_0 and an alternating word in elements of $G' - \{1\}$ and $G'' - \{1\}$ whose inductive images keep u belong to T_0.

Now we are in the position to introduce the following two equivalent conditions, which are essential in this paper. First, let us give some explaination:

Suppose $u \in Y_0$, $t \in C(u, T')$, $g \in G''$ and $u \cdot g \in Y_0 \subset T_0$, then $t \cdot g$ is a direction normal to (i.e. not belong to) T_0.



We do not want $t \cdot g$ to belong to T'. For example, $G = F(x,y)$, $G' = F(x)$ and $G'' = F(y)$, then $T' = A_x$ and $T'' = A_y$.



$T_0 = [p, q]$ for some $p, q \in A_x \cup A_y$. Denote the only direction in $C(p, T')$ ($C(q, T')$ resp.) by t (t' resp.). We want to avoid the kind of collapsings that $t \cdot y = t'$, Condition **A** and **A'** are designed for this perpose.

Set

$$G_n = \begin{cases} G', & \text{if } n \text{ is odd;} \\ G'', & \text{if } n \text{ is even.} \end{cases}$$

$$T_0 = T' \cap T''$$

$$T_1 = T'$$

$$T_2 = T_1 \cdot G_2 \cup T''$$

Define T_n inductively, for every positive integer n,

$$T_n = T_{n-1} \cdot G_n$$

$T_0 = T' \cap T''$ and $T_1 = T''$ are connected. By induction on n, we can prove that T_n is connected for every $n \geq 2$, since T_{n-2} is contained in $T_{n-1} \cdot g$ for every $g \in G_n$. Therefore T_n is a subtree of T for every $n \geq 0$. Because T is a minimal tree and $\bigcup_{n=0}^{\infty} T_n$ is invariant under G, we have $T = \bigcup_{n=0}^{\infty} T_n$.

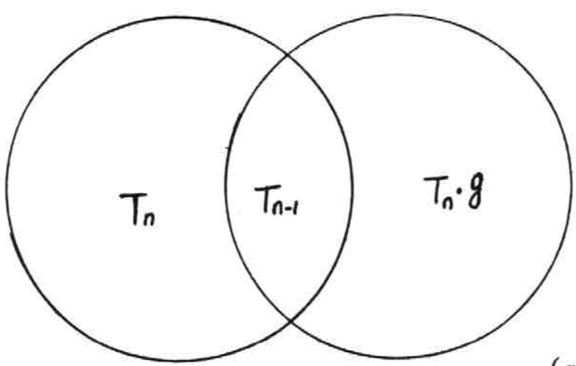
Condition A': *For every integer $n \geq 1$,*

$$T_n \cdot (G_{n+1} - \{1\}) \cap T_n \subset T_{n-1}$$

Remark: This is equivalent to the following formula:

$$T_n \cdot (G_{n+1} - \{1\}) \cap T_n = T_{n-1}$$

Because by definition, T_{n-1} is contained in T_n and is invariant under $G_{n+1} = G_{n-1}$, it is always contained in $T_n \cdot (G_{n+1} - \{1\}) \cap T_n$.



($g \in G_{n+1}$)

We also have need of local condition which is equivalent to Condition **A′**. To formulate this, we introduce the following notation:

Assume $\bar{T}$ is a subtree of T, u is a point of T_0, define $D(u,\bar{T})$ to be the set of directions in $\bar{T}$ starting from the point u, and $C(u,\bar{T})$ the subset of directions in $D(u,\bar{T})$ which are normal to T_0, in other words, $C(u,\bar{T}) = D(u,\bar{T}) - D(u,T_0)$.

Suppose $u \in Y_0$, $g \in G$ and $(u,g) \in S$, define

$$H_b(u,g) = \begin{cases} C(u,T')\cdot g, & \text{if } g_b \in G'' \\ C(u,T'')\cdot g, & \text{if } g_b \in G' \end{cases}$$

$$H_e(u,g) = \begin{cases} C(u\cdot g,T'), & \text{if } g_e \in G'' \\ C(u\cdot g,T''), & \text{if } g_e \in G' \end{cases}$$

Set

$$H(u,g) = H_b(u,g) \cap H_e(u,g)$$

Condition A: *For every $(u,g) \in S$ such that $g \neq 1$ and $u\cdot g \in Y_0$, we have*

$$H(u,g) = \emptyset$$

This is to say that g does not carry a direction starting from u which is in T' or T'' (depending on $g_b \in G''$ or G') and is outside T_0 to a direction starting from $u\cdot g$ which is in T' or T'' (depending on $g_e \in G''$ or G') and is normal to T_0, if $(u,g) \in S$, $g \neq 1$ and $u\cdot g \in Y_0$. For example, $g = ab$ with $a \in G'$, $b \in G''$, $u \in Y_0$ and $u\cdot a \in T_0$, $u\cdot g = u\cdot ab \in Y_0$, then g dose not carry a direction from u in T'', which is normal to T_0 to a direction from $u\cdot g$ in T', which is outside T_0.

Condition **A** and **A′** are equivalent to each other, the proof can be fined in Section 5.

4. Main Theorems

Let us first introduce the following notation:

Assume $u \in T_0$, set

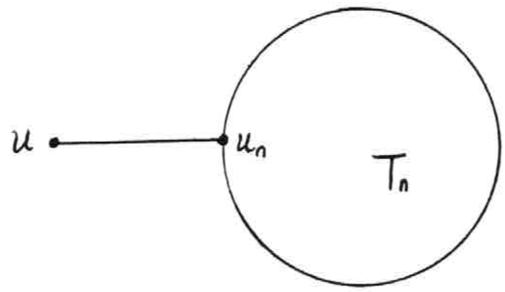
$$F(u) = \{u\cdot g | g \in G, u\cdot g_i \in T_0 \ \forall\ i \geq 0\}$$

Also, let

$$\Omega = \bigcup_{g\in G} E(D_g)$$

$$B_0 = \{u\cdot g_i | u \in Y_0, g \in G, u\cdot g_i \in T_0 \ \forall i \geq 0\}$$

Assume u is a point of T, n is a nonnegative integer, we define a point u_n as follows: if $u \in T_n$, then $u_n = u$. If $u \notin T_n$, there is a unique point $v \in T_n$, such that the bridge between u and T_n is the segment between u and v, i.e. $B(u, T_n) = [u, v]$, then we take v to be our u_n.



In this section, Condition **A** and **A′** are assumed without mention everywhere. We work out some results using Condition **A′**, among which the most impotant are the following:

Proposition 4.1: *(a) If the action $T \times G \to T$ is not free, then Σ has fixed points both in $\Omega \subset B_0$ and in $Y_0 \subset B_0$.*

(b) If the action $T \times G \to T$ is free, then Σ dose not have any fixed point in T_0.

Remark: When we say Σ has a fixed point, we mean that there is a point $u \in T_0$, an element $\sigma \in \Sigma - \{id\}$ such that σ is defined at u and $(u)\sigma = u$.

Proposition 4.2: *Assume the action $T \times G \to T$ is free, then*

(a) If the action $T \times G \to T$ is not discrete, then $\#\Omega = \infty$ and there is a point $u \in Y_0 \subset B_0$, such that $\#F(u) = \infty$.

(b) If $\#B_0 = \infty$, then the action $T \times G \to T$ is not discrete.

The proofs of the above two propositions can be found later in this section.

According to Proposition 4.1 and Proposition 4.2, to investigate whether the action $T \times G \to T$ is free or discrete, we need only look at how Σ acts on the points of B_0. We can find examples using this idea in Part 2.

Before proving Proposition 4.1 and Proposition 4.2, we give the following lemmas, which are direct consequences of Condition **A′**:

Lemma 4.3: *For $n \geq 2$, we have $(T_n - T_{n-1})\cdot g \subset T_{n+L(g)} - T_{n+L(g)-1}$ if $L(g) > 0$ and $g_1 \in G_{n+1}$.*

Proof: First we assume that $L(g) = 1$, so $g = g_1 \in G_{n+1}$. If $n > 1$, because T_{n-1} is invariant under $G_{n+1} = G_{n-1}$, this formula is a direct consequence of Condition **A′**. For $n = 1$, $(T_1 - T_0)\cdot(G_2 - \{1\}) = (T_1 - T'')\cdot(G'' - \{1\})$, since T'' is invariant under G'', this set does not intersect T''. On the other hand, by Condition **A′**, the intersection of this set and T_1 is contained in $T_0 \subset T''$, so this intersection must be empty. Therefore, $(T_1 - T_0)\cdot(G_2 - \{1\}) \subset T_2 - T_1$.

Assume $L(g) > 1$, $h = gg_e^{-1}$, then $L(h) = L(g) - 1$. By induction, $(T_n - T_{n-1})\cdot h \subset T_{n+L(h)} - T_{n+L(h)-1}$ and $(T_{n+L(h)} - T_{n+L(h)-1})\cdot g_e \subset T_{n+L(h)+1} - T_{n+L(h)} = T_{n+L(g)} - T_{n+L(g)-1}$. So, $(T_n - T_{n-1})\cdot g \subset (T_{n+L(h)} - T_{n+L(h)-1})\cdot g_e \subset T_{n+L(g)} - T_{n+L(G)-1}$. ◊

Lemma 4.4: *Suppose g is an element of G, $k \geq 0$, u is a point of T_k. Suppose there is an integer $i \leq n = L(g)$ such that $u\cdot g_i \notin T_k$, then $u\cdot g \notin T_k$.*

Proof: Assume $a_1 a_2 \cdots a_n$ is the alternating word of g. Suppose that i is the smallest number such that $u\cdot g_i \notin T_k$. If $k \geq 1$, then $a_i \notin G_k$, so $a_i \in G_{k+1}$, and therefore, $u\cdot g_i = u\cdot g_{i-1}a_i \in T_{k+1} - T_k$. Assume $k = 0$, then $u\cdot g_i \in T_1 - T_0$ if $a_i \in G'$, and $u\cdot g_i \in T_2 - T_1$ if $a_i \in G''$. In every case, there is an integer $l \geq k$ such that $u\cdot g_i \in T_{l+1} - T_l$ and $a_i \in G_{l+1}$, by Lemma 4.3, $u\cdot g \in T_{n-i+l+1} - T_{n-i+l}$, so $u\cdot g \notin T_k$. ◊

We denote the domain and the range of a partial isometry $\sigma \in \Sigma$ by $D(\sigma)$ and $R(\sigma)$ respectively, thus if σ is induced by an element $g \in G$ (i.e. if $\sigma = \sigma_g$), then $D(\sigma) = D_g$ and $R(\sigma) = R_g$.

Lemma 4.5: *For any element $g \in G$, if $D_g \neq \emptyset$ and $g = g_1 g_2 \cdots g_n$ is the alternating word of g, then we have $\sigma_{g_1}\sigma_{g_2}\cdots\sigma_{g_n} = \sigma_g$.*

Proof: We proceed by induction on $L(g)$. Set $h = gg_e^{-1} = g_1 g_2 \cdots g_{n-1}$, then $L(h) = L(g) - 1$. By inductive hypotheses, $\sigma_{g_1}\sigma_{g_2}\cdots\sigma_{g_{n-1}} = \sigma_h$, it is enough to prove that $\sigma_h\sigma_{g_n} = \sigma_g$. Since obviously $\sigma_h\sigma_{g_n}$ is σ_g limited on a subset of its domain, we only have to prove $D_g \subset D(\sigma_h\sigma_{g_n})$. Suppose there is a point $u \in D_g - D(\sigma_h\sigma_{g_n})$, if $u \in D_h$, then $(u)\sigma_h \notin D_{g_n}$, therefore $u\cdot g = u\cdot hg_n = ((u)\sigma_h)\cdot g_n \notin T_0$, this implies that $u \notin D_g$, contradicting the assumption. Assume that $u \notin D_h$, then $u\cdot g_{L(g)-1} = u\cdot h \notin T_0$, by Lemma 4.4, $u\cdot g \notin T_0$, this is impossible. ◊

Proposition 4.6: *$\Sigma - \{id\}$ consists of alternating combinations of elements of $\Sigma' - \{id\}$ and $\Sigma'' - \{id\}$.*

Proof: This is implied by Lemma 4.5. ◊

Corollary 4.7: *(a) For every point $u \in T_0$, we have $F(u) = (u)\Sigma$.*

(b) We have $B_0 = (Y_0)\Sigma$.

(c) B_0 and $F(u)$ for all $u \in T_0$ are Σ invariant.

Proof: (a), (b) are deduced from the definitions of B_0 and $F(u)$ for $u \in T_0$ and from Proposition 4.6.

(c) If $\sigma_g, \sigma_h \in \Sigma$ with $g, h \in G$, then $\sigma_g\sigma_h$ is $\sigma_g h$ restricted to a subset of its domain. Therefore for every point $u \in T_0$, we have $\{u\}\cdot\Sigma\Sigma = \{u\}\cdot\Sigma$. Then (c) comes from (a) and (b) easily. ◊

Lemma 4.8: $\Omega \subset B_0$.

Proof: Assume that $g \in G$ and $D_g \neq \emptyset$, we want to prove that $E(D_g) \subset B_0$.

By Lemma 2.4, if $g \in G' \cup G''$, then $E(R_g) = E(D_{g^{-1}}) \subset Y_0 \subset B_0$. For the general cases, we prove the lemma by induction on $L(g)$. Assume $a_1 a_2 \cdots a_n$ is the alternating word of g, set $h = a_1^{-1} g$, then $L(h) = L(g) - 1$. By inductive hypotheses, $E(D_h), E(R_h), E(D_{a_1})$ and $E(R_{a_1})$ are all contained in B_0. According to Lemma 4.5, $\sigma_g = \sigma_{a_1}\sigma_h$, so $D_g = (R_{a_1} \cap D_h)\sigma_{a_1}^{-1}$, and then $E(D_g) = (E(R_{a_1} \cap D_h))\sigma_{a_1}^{-1} \subset (B_0)\sigma_{a_1}^{-1} \subset (B_0)\Sigma \subset B_0$. This proves the lemma. ◊

Lemma 4.9: *If the action $T \times G \to T$ has a fixed point $v \in T$, then there is a point $u \in T_0$, an element $\sigma_g \in \Sigma$ for some $g \in G$ such that $u \in D_g$ and $(u)\sigma_g = u$. Furthermore, u is conjugate to v, i.e. they are in the same G-orbit.*

Proof: Assume $v \in T_n$ for some $n \geq 0$, by the definition of T_i for $0 \leq i \leq n$, there is a point $u \in T' \cup T''$, an element $h \in G$ such that $v = u \cdot h$. Since u is conjugate to v, u is fixed by a nontrivial element $g \in G$.

Assume that $u \in T'' - T_0 \cdot G'' \subset T_2 - T_1$. If $g_1 \in G'$, by Lemma 4.8, $u \cdot g \in T_{L(g)+2} - T_{L(g)+1}$, since $L(g) \geq 1$, $T_2 \subset T_{L(g)+1}$, so $u = u \cdot g \notin T_2$, this is a contradiction. Assume $g_1 \in G''$, if $L(g) = 1$, then $g \in G''$, but we assumed that the action $T'' \times G'' \to T''$ is free, u can not be fixed by g, so $L(g) > 1$. Because $u \notin T_0 \cdot G''$, $u \cdot g_1 \notin T_0$, therefore $u \cdot g_1 \in T_2 - T_1$, then by Lemma 4.8, $u \cdot g \in T_{L(g)+1} - T_{L(g)}$, so $u = u \cdot g \notin T_2$, impossible. Similarly we can prove $u \notin T' - T_0 \cdot G'$, hence $u \in T_0 \cdot (G' \cup G'')$, or equivalently, there is an element $s \in G' \cup G''$ such that $u \cdot s \in T_0$. By taking $u \cdot s$ for u, we may assume that $u \in T_0$.

Since u is fixed by g and $u \in T_0$, we have $u \in D_g$ and $(u)\sigma_g = u$. ◊

Lemma 4.10: *If $v \in T$ is such that the orbit $\{v\} \cdot G$ is indiscrete, then there is a point $u \in T_0$, such that $d(u, F(u) - \{u\}) = 0$ and u is conjugate to v.*

Proof: As in the proof of Lemma 4.9, there is a point $w \in T' \cup T''$, an element $g \in G$ such that $v = w \cdot g$.

Suppose that $w \in T'' - T_0 \cdot G''$, Since the action $T'' \times G'' \to T''$ is discrete and T_0 is compact, the set $T_0 \cdot G''$ is closed in T'', we have $d(w, T_0 \cdot G'') > 0$ or equivalently, $d(\{w\} \cdot G'', T_0) > 0$.

Claim 1: $\mathrm{dis}(w, \{w\} \cdot (G - G'')) \geq \rho = d(\{w\} \cdot G'', T_0)$.

Proof of Claim 1: Take any element $g \in G - G''$, assume that $g_1 \in G' = G_3$, since $[w, w_0) \subset T'' - T_0 \subset T_2 - T_1$, by Lemma 4.8, $[w, w_0) g \subset T_{2+L(g)} - T_{1+L(g)}$, but $w_0 \cdot g \in T_{1+L(g)}$, so $[w, w_0] g \cap T_{1+L(g)} = \{w_0 \cdot g\}$, i.e. $w_0 \cdot g = (w \cdot g)_{1+L(g)}$. Then $w_0 \cdot g \in B(w \cdot g, T'')$. We have $d(w \cdot g, T'') \geq \mathrm{dis}(w \cdot g, w_0 \cdot g) = \mathrm{dis}(w, w_0) \geq \rho$. Assume that $g_1 \in G''$, since $g \notin G''$, $g_2 \in G' - \{1\}$, take $w' = w \cdot g_1$, $h = g_1^{-1} g$, then by the above argument, $d(w \cdot g, T'') = d(w' \cdot h, T'') \geq \rho$. As a consequence, $\mathrm{dis}(w \cdot g, w) \geq \rho$, the claim is true.

Because w is conjugate to v, $\{w\}\cdot G$ is indiscrete, so $d(w,\{w\}\cdot(G-\{1\}))=0$. From claim 1 we deduce that $d(w,\{w\}\cdot(G''-\{1\}))=0$, but we assumed that the action $T''\times G''\to T''$ is discrete, this is impossible.

Similarly, w can not belong to $T'-T_0\cdot G'$, hence $w\in T_0\cdot(G'\cup G'')$. As in the proof of Lemma 4.9, we may assume that $w\in T_0$.

Assume that $\#F(w)<\infty$, then $F(w)\cdot(G'\cup G'')-T_0$ is a discrete set, because T_0 is compact, $\lambda=d(F(w)\cdot(G'\cup G'')-T_0,T_0)>0$.

Claim 2: $d(\{w\}\cdot G-T_0,w)\geq\lambda$.

Proof of Claim 2: Suppose $g\in G$ is such that $w\cdot g\notin T_0$, assume i is the smallest integer such that $w\cdot g_i\notin T_0$, then $w\cdot g_{i-1}\in F(w)$, so $w\cdot g_i\in F(w)\cdot(G'\cup G'')-T_0$, therefore, $d(w\cdot g_i,(w\cdot g_i)_0)\geq\lambda$. As in the proof of Claim 1 (take $w\cdot g_i$ for the w there), by applying Lemma 4.8, we can find a nonnegative integer m such that $(w\cdot g_i)_0\cdot g_i^{-1}g=((w\cdot g_i)\cdot g_i^{-1}g)_m=(w\cdot g)_m$, then $(w\cdot g_i)_0\cdot g_i^{-1}g\in B(w\cdot g,T_0)$, so $d(w\cdot g,w)\geq d(w\cdot g,T_0)\geq d(w\cdot g,(w\cdot g_i)_0\cdot g_i^{-1}g)=d(w\cdot g_i,(w\cdot g_i)_0)\geq\lambda$. This proves the Claim 2.

Because $d(w,\{w\}\cdot(G-\{1\}))=0$, by Claim 2, $d(w,\{w\}\cdot(G-\{1\})\cap T_0)=0$. By Lemma 4.4, $\{w\}\cdot(G-\{1\})\cap T_0=\{w\cdot g|g\in G-\{1\},w\cdot g_i\in T_0,\forall i\geq 0\}=\{(w)\sigma_g|g\in G-\{1\},w\in D_g\}=F(w)-\{w\}$, so $d(w,F(w)-\{w\})=0$, contradicting our assumption that $\#F(w)<\infty$, therefore, $\#F(w)=\infty$.

Because $|T_0|<\infty$, there is a point $u\in F(w)$ such that $d(u,F(w)-\{u\})=0$, i.e. $d(u,F(u)-\{u\})=0$. The lemma is thus proved. ◊

Now, let us proceed the proofs of Proposition 4.1 and Proposition 4.2.

Proof of Proposition 4.1:

(a) Suppose the action $T\times G\to T$ is not free then according to Lemma 4.9, there is a point $v\in T_0$, an element $\sigma_h\in\Sigma$ for some $h\in G$, such that $(v)\sigma_h=v$. We may assume that $v\notin\Omega$, then $v\notin R=E(D_h)\cup E(R_h)$. Since $\#R<\infty$, we have $d(v,R)>0$.

Set $s=\{u\in T_0|d(v,u)\leq d(R,v)\}$, then $s\subset D_h\cap R_h$ and $E(s)=\{u\in T_0|d(u,v)=d(R,v)\}$. It is clear that $E(s)\cap R\neq\emptyset$. Because $(v)\sigma_h=v$, we have $(s)\sigma_h=s$ and $(E(s))\sigma_h=E(s)$. Since $|s|\leq|T_0|<\infty$, $\#E(s)<\infty$, $\sigma_h|_{E(s)}$ is a permutation. Then for every $u\in E(s)\cap R\subset\Omega$, there is a positive integer k such that $(u)(\sigma_h)^k=u$, we take $g=h^k$, then $u\in D_g$ and $(u)\sigma_g=(u)(\sigma_h)^k=u$. Therefore, Σ has a fixed point in $\Omega\subset B_0$.

Assume $u=(v)\sigma_h\in(Y_0)\Sigma=B_0$ is fixed by $\sigma_g\in\Sigma-\{id\}$, where $v\in Y_0$, $\sigma_h\in\Sigma$, then v is fixed by $\sigma_h\sigma_g\sigma_{h^{-1}}$, according to the proof of Corollary 4.7 (c), v is fixed by $\sigma_{hgh^{-1}}\in\Sigma-\{id\}$. Hence Σ has a fixed point in $Y_0\subset B_0$.

(b) If $u\in T_0$ if fixed by $\sigma_g\in\Sigma-\{id\}$, then u is fixed by $g\in G-\{1\}$, so the action is not free. ◊

Proof of Proposition 4.2:

(a) Assume $\#\Omega < \infty$, then $\#\Sigma < \infty$, for any point $v \in T_0$, $\#F(v) \leq \#\Sigma < \infty$. But since the action $T \times G \to T$ is free and not discrete, there are indiscrete G-orbits, according to Lemma 4.10, there is a point $v \in T_0$ such that $\#F(v) = \infty$, this is a contradiction.

Therefore, $\#\Omega = \infty$. According to Lemma 4.4, $\Omega \subset B_0$, so, $\#B_0 = \infty$. By definition, $B_0 = \bigcup_{v \in Y_0} F(v)$, since $\#Y_0 < \infty$, there is a point $v \in Y_0 \subset B_0$, such that $\#F(v) = \infty$.

This also proves (b). ◊

Now, we introduce the following

Condition B: $\#\Omega < \infty$.

Theorem 4.11: *Assume Condition* **A** *(or equivalently,* **A**$'$*), and the action $T \times G \to T$ is free, then Condition* **B** *is true if and only if the action is discrete.*

Proof: It follows directly from Proposition 4.2 and its proof. ◊

To conclude this section, we give the following remark: Property (P) is implied by Condition **A** and **B**. We think that Condition **A** is an essential one. The example of a free indiscrete action by the free group of rank 3 on an **R**-tree given by Bestvina-Handel violates Condition **A**. While we feel Condition **B** could be true for all actions which are free and satisfy Condition **A**. This needs further study.

Examples using the theorems in this section can be find in Part 2 of this paper.

5. The equivalence of A and A$'$

Before proving the equivalence of Condition **A** and **A**$'$, let us see a couple of lemmas as follows.

Lemma 5.1: *If $u \in T' \cup T'' - T_0$, then $u_0 \in Y_0$.*

Proof: We may assume that $u \in T' - T_0$. Suppose $u_0 \notin Y(T')$, then $\#D(u_0, T') = 2$ (T' has no end point). Because $t_{u_0}(u) \in D(u_0, T') - D(u_0, T_0) = C(u_0, T')$, $\#D(u_0, T_0) \leq 1$. Since $u_0 \in T_0$ and T_0 is nondegenerate, $D(u_0, T_0)$ can not be empty. So $\#D(u_0, T_0) = 1$, hence $u_0 \in E(T_0) \subset Y_0$. ◊

Lemma 5.2: *Assume $T_{n-1} \cap T_{n-1} \cdot (G_n - \{1\}) \subset T_{n-2}$ for an integer $n \geq 2$, suppose $u \in T_n - T_{n-1}$, $u = v \cdot g$ for some $v \in T_{n-1}$, $g \in G_n - \{1\}$, then $u_{n-1} = u_{n-2} = v_{n-2} \cdot g \in T_{n-2}$.*

Proof: Because T_{n-2} is invariant under $G_n = G_{n-2}$, we have $(T_{n-1} - T_{n-2}) \cdot (G_n - \{1\}) \cap T_{n-1} = \emptyset$. Assume $n = 2$ and $u \in T''$, then $u_0 = u_1 \in T_0$. Now suppose $u = v \cdot g$ for some $v \in T_{n-1} - T_{n-2}$, $g \in G_n - \{1\}$ with $n > 2$. Since $[v, v_{n-2}) \subset T_{n-1} - T_{n-2}$, we have $[u, v_{n-2} \cdot g] \cap T_{n-1} = [v, v_{n-2}] \cdot g \cap T_{n-1} = \{v_{n-2} \cdot g\}$, so $u_{n-1} = v_{n-2} \cdot g$. Because $v_{n-2} \cdot g \in T_{n-2}$, $u_{n-1} = u_{n-2} \in T_{n-2}$. ◊

From now on until Proposition 5.7, Condition **A** is assumed, we intend to show that $\mathbf{A} \Longrightarrow \mathbf{A}'$.

Lemma 5.3 *We have* $T_1 \cap T_1 \cdot (G'' - \{1\}) \subset T_0$.

Proof: Suppose this is not true, then there are points $u \in T_1 - T_0$, $v \in T_1$ and an element $g \in G'' - \{1\}$ such that $u = v \cdot g$.

If $v \in T_0$, then $u = v \cdot g \in T'' \cap T' = T_0$, which contradicts the assumption. So, $v \in T_1 - T_0$. Because $[u, v_0 \cdot g] \cap T_0 = [v, v_0] \cdot g \cap T_0 = \{v_0 \cdot g\}$, $B(u, T_0) = [u, v_0 \cdot g]$, so, $u_0 = v_0 \cdot g$. Then, $t_{u_0}(u) = t_{v_0}(v) \cdot g$. By Lemma 5.1, $u_0, v_0 \in Y_0$, so $(v, g) \in S$. Since $t_{u_0}(u) \in C(u_0, T') \cap C(v_0, T') \cdot g = H(v, g)$, this contradicts Condition **A**. ◊

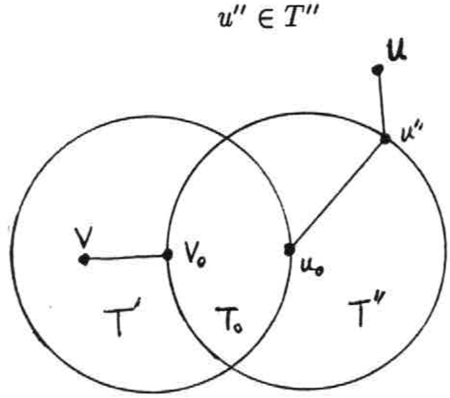
Suppose u is any point in the tree T, then $u'' \in T''$ is such a point that $B(u, T'') = [u, u'']$, or equivalently, $[u, u''] \cap T'' = \{u''\}$.

Lemma 5.4: *Assume* $u \in T_2 - T_1$. *If* $u'' \notin T_0$, *then* $u_0 \in Y_0$, $t_{u_0}(u) \in C(u_0, T'')$; *if* $u'' \in T_0$, *then there is a point* $u' \in Y_0$ *and an element* $g \in G''$ *such that* $u_0 = u'' = u' \cdot g$ *and* $t_{u_0}(u) \in C(u', T') \cdot g = H_b(u', g)$.

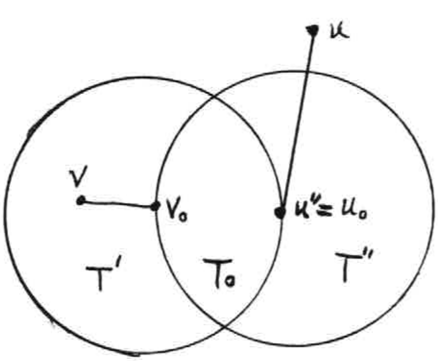
Proof: First assume that $u \in T'' - T_0$, then by Lemma 5.1 and Lemma 5.2, $u_1 = u_0 \in Y_0$ and obviously, $t_{u_0}(u) \in C(u_0, T'')$.

Now assume that $u \notin T''$, then $u = v \cdot g$ for some $v \in T_1 - T_0$, and $g \in G''$. Because T'' is invariant under g, $[u, v_0 \cdot g] \cap T'' = [v, v_0] \cdot g \cap T'' = \{v_0 \cdot g\}$, so $u'' = v_0 \cdot g$. Because $T_0 \subset T''$, $u_0 = (u'')_0$. By Lemma 5.3, $[u, v_0 \cdot g) \cap T_1 = [v, v_0) \cdot g \cap T_1 = \emptyset$, it is clear that $u_1 = (v_0 \cdot g)_1 = (u'')_1 = (u'')_0 = u_0$. Assume $u'' \notin T_0$, by Lemma 5.1, $(u'')_0 \in Y_0$, $t_{u_0}(u) = t_{u_0}(u'') \in C(u_0, T'')$. Suppose $u'' \in T_0$, take $u' = v_0$, then $u_1 = u_0 = u'' = u' \cdot g$, and $t_{u_0}(u) = t_{v_0}(v) \cdot g \in C(v_0, T') \cdot g = H_b(u', g)$. ◊

$u'' \in T''$



$u'' \notin T''$



Lemma 5.5: $T_2 \cdot (G' - \{1\}) \cap T_2 \subset T_1$.

Proof: Suppose this is not true, there are points $u, v \in T_2 - T_1$, element $g \in G' - \{1\}$, such that $u = v \cdot g$. Since T' is invariant under g, it is easy to see that $u_1 = v_1 \cdot g$ and $t_{u_1}(u) = t_{v_1}(v) \cdot g$. According to Lemma 5.3, $u_0 = u_1 \in Y_0$ and $v_0 = v_1 \in Y_0$.

Case 1: $u'', v'' \notin T_0$. Then $u_0, v_0 \in Y_0$, so $(v_0, g) \in S$. $t_{u_0}(u) = t_{v_0}(v) \cdot g \in C(u_0, T'') \cap C(v_0, T'') \cdot g = H(v_0, g)$, contradicting Condition **A**.

Case 2: $u'' \in T_0$, $v'' \notin T_0$. By Lemma 5.3, there is a point $u' \in Y_0$, an element $h \in G''$ such that $u_0 = u' \cdot h$ and $t_{u_0}(u) \in C(u', T') \cdot h$. Take $g' = h \cdot g^{-1}$, then $(u', g') \in S$, and $t_{v_0}(v) \in C(v_0, T'') \cap C(u', T') \cdot g' = H(u', g')$, which violates Condition **A**.

Case 3: $v'' \in T_0$, $u'' \notin T_0$. Similar to Case 2, this is impossible.

Case 4: $u'', v'' \in T_0$. There are $u', v' \in Y_0$, $h_1, h_2 \in G''$ such that $u_0 = u' \cdot h_1$, $v_0 = v' \cdot h_2$ and $t_{u_0}(u) \in C(u', T')$, $t_{v_0}(v) \in C(v', T')$. Take $h = h_2 \cdot g \cdot h_1^{-1}$, then $(v', h) \in S$. $t_{u_0}(u) \cdot h_1^{-1} = t_{v_0}(v) \cdot g \cdot h_1^{-1} \in C(u', T') \cap C(v', T') \cdot h = H(v', h)$, this contradicts Condition **A**.

Lemma 5.5 is thus proved. ◊

Lemma 5.6: *If $u \in T_3 - T_2$ and $u_2 \in T_0$, then there is a point $u' \in Y_0$, an element $h \in G$ such that $(u', h) \in S$, $u_2 = u' \cdot h$, $h_e \in G'$ and $t_{u_2}(u) \in H_b(u', h)$.*

Proof: There is a point $v \in T_2 - T_1$, $g \in G'$ such that $u = v \cdot g$. By Lemma 5.2, we have $u_2 = u_1 = v_1 \cdot g$ and $v_0 = v_1 \in T_0$. Assume that $v'' \notin T_0$, then $v_0 \in Y_0$ and $t_{v_0}(v) \in C(v_0, T'')$. Take $u' = v_0$, $h = g$, then $t_{u_2}(u) = t_{v_0}(v) \cdot g \in C(u', T'') \cdot h = H_b(u', h)$.

Suppose $v'' \in T_0$, by Lemma 5.4, there are $v' \in Y_0$, $g' \in G''$, such that $v_0 = v' \cdot g'$ and $t_{v_0}(v) \in C(v', T') \cdot g'$. Take $u' = v'$, $h = g' \cdot g$, then $u' \cdot h = v_0 \cdot g = u_2$, $t_{u_2}(u) = t_{v_0}(v) \cdot g \in C(v', T') \cdot h = H_b(u', h)$. Then u' and h satisfy all the desired properties. ◊

Proposition 5.7: *For $n \geq 1$, we have:*

(a) $T_n \cap T_n \cdot (G_{n+1} - \{1\}) \subset T_{n-1}$.

(b) If $u \in T_{n+1} - T_n$, then either $u_n \in T_0$, or there is an integer k such that $0 \leq k < n$ and $k = n$ mod 2 such that $u_n \in T_{k+1} - T_k$.

(c) If $u \in T_{n+1} - T_n$ and $u_n \in T_0$, then there are $u' \in Y_0$, $h \in G$, such that $(u', h) \in S$, $u_n = u' \cdot h$, $h_e \in G_{n+1}$ and $t_{u_n}(u) \in H_b(u', h)$.

Proof: We assume that $n = 1$, then Lemma 5.3 $\Longrightarrow$ (a), Lemma 5.4 $\Longrightarrow$(c) and (b) is trivial.

If $n = 2$, then Lemma 5.5 $\Longrightarrow$(a), Lemma 5.6 $\Longrightarrow$ (c) and (b) is trivial.

Assume $n \geq 3$, let us prove (a), (b) and (c) by induction on n.

(a) Suppose this is not true, then there are $u, v \in T_n - T_{n-1}$, $g \in G_{n+1} - \{1\}$ such that $u = v \cdot g$. Because T_{n-1} is invariant under g, we have $u_{n-1} = v_{n-1} \cdot g$, and $t_{u_{n-1}}(u) = t_{v_{n-1}}(v) \cdot g$.

By induction hypotheses, either

A: there is an nonnegative integer $k < n-1$, $k = n-1 \bmod 2$ such that $v_{n-1} \in T_k - T_{k-1}$.

or B: $v_{n-1} \in T_0$.

and either C: there is an nonnegative integer $k' < n-1$, $k' = n-1 \bmod 2$ such that $u_{n-1} \in T_{k'} - T_{k'-1}$.

or D: $u_{n-1} \in T_0$.

Assume A is true, by induction , $T_k \cdot g \cap T_k \subset T_{k-1}$, therefore $u_{n-1} = v_{n-1} \cdot g \in T_{k+1} - T_k$, so C is true and $k' = k+1$, but $k = k' \bmod 2$, this can not happen. Hence A is impossible, and similarly, C is also impossible. Therefore, $u_{n-1} \in T_0$ and $v_{n-1} \in T_0$.

By induction, there are $u'v' \in Y_0$, $h, f \in G$ such that $u_{n-1} = u' \cdot h$, $v_{n-1} = v' \cdot f$, $h_e, f_e \in G_n$, $(u', h), (v', f) \in S$ and $t_{u_{n-1}}(u) \in H_b(u', h)$, $t_{v_{n-1}}(v) \in H_b(v', f)$. Take $s = f \cdot g \cdot h^{-1}$, since $h_e, f_e \in G_n$, $g \in G_{n+1}$, there is no cancellation when we multiply f, g and h together, so $s_b = f_b$ and $s_e = h_b^{-1}$. Then $H_b(v', s) \cdot s^{-1} = H_b(v', f) \cdot f^{-1}$ and $H_e(v', s) = H_b(u', h) \cdot h^{-1}$.

We can check easily that $u' = v' \cdot s$, $(v', s) \in S$, $t_{u_{n-1}}(u) \cdot h^{-1} \in H_b(u', h)h^{-1} = H_e(v', s)$ and $t_{u_{n-1}}(u) \cdot h^{-1} = t_{v_{n-1}}(v) \cdot g \cdot h^{-1} \in H_b(v', f) \cdot f^{-1} \cdot s = H_b(v', s)$, so $t_{u_{n-1}}(u) \cdot h^{-1} \in H_b(v', s) \cap H_e(v', s) = H(v', s)$, this contradicts Condition **A**.

(b) Assume that $u \in T_{n+1} - T_n$, there is a point $v \in T_n$, an element $g \in G_{n+1}$ such that $u = v \cdot g$. By (a) $T_n \cdot (G_{n+1} - \{1\}) \cap T_n \subset T_{n-1}$, then by Lemma 5.2 $u_n = u_{n-1} = v_{n-1} \cdot g$. By induction, either $v_{n-1} \in T_0$ or there is a nonnegative integer $k < n$, $k = n \bmod 2$, such that $v_{n-1} \in T_k - T_{k-1}$. Assume the later case is true, then $u_n = v_{n-1} \cdot g \in T_{k+1} - T_k$ and (b) is true. Suppose that $v_{n-1} \in T_0$, if $u_n = v_{n-1} \cdot g \notin T_0$, when n is even, $g \in G'$, so $u_n \in T_1 - T_0$, we take $k = 0$; when n is odd, $g \in G''$, so $u_n \in T_2 - T_1$, take $k = 1$. Then $k = n \bmod 2$ and $u_n \in T_{k+1} - T_k$.

(c) Assume $u \in T_{n+1} - T_n$ and $u_n \in T_0$. Suppose $v \in T_n - T_{n-1}$ and $g \in G_{n+1}$ are as in (b). We have $v_{n-1} \in T_0$, otherwise by the proof of (b), $u_n = v_{n-1} \cdot g \notin T_0$, this is impossible. By induction, there is a point $v' \in Y_0$, an element $f \in G$ such that $v_{n-1} = v' \cdot f$, $f_e \in G_n$ and $t_{v_{n-1}}(v) \in H_b(v', f)$.

Take $u' = v'$, $h = f \cdot g$, then $u_n = u' \cdot h$, $(u', h) \in S$, $h_e = g \in G_{n+1}$ and $t_{u_n}(u) = t_{v_{n-1}}(v) \cdot g \in H_b(v', f) \cdot g = H_b(u', h)$, violating Condition **A**.

This complete the proof of Proposition 5.7. $\diamondsuit$

Theorem 5.8: *Condition* **A** *and Condition* **A'** *are equivalent to each other.*

Proof: Proposition 5.7 implies that Condition **A** $\Longrightarrow$ Condition **A'**. Let us prove the converse now.

Assume Condition **A'** is true and **A** is not true. There is a point $u \in Y_0$, an element $g \in G - \{1\}$, such that $u \cdot g \in Y_0$, $u \cdot g_i \in T_0$ for every $i \geq 0$ and $H(u, g) = H_b(u, g) \cap H_e(u, g) \neq \emptyset$. Assume v is such a point that $t_{u \cdot g}(v) \in H(u, g)$, then $t_{u \cdot g}(v) \in C(u \cdot g, T' \cup T'')$ and $t_u(v \cdot g^{-1}) = (t_{u \cdot g}(v)) \cdot g^{-1} \in H_b(u, g) \cdot g^{-1} \subset$

$C(u, T' \cup T'')$. We may choose v so close to $u \cdot g$ that v and $v \cdot g^{-1}$ both belong to $T' \cup T'' - T_0$.

If $v \in T' \subset T_1$, then $H_e(u,g) = C(u \cdot g, T')$, so $g_e \in G''$, $(g^{-1})_1 = (g^{-1})_b = (g_e)^{-1} \in G'' = G_2$; similarly, if $v \in T'' \subset T_2$, $(g^{-1})_1 \in G' = G_3$.

We may apply Lemma 4.3 to see that for $k = 1$ or 2, $v \in T_k - T_{k-1}$ while $v \cdot g^{-1} \in T_{k+L(g)} - T_{k+L(g)-1}$, (notice that the proof of Lemma 4.3 of only needs Condition **A'**). If $L(g) \geq 2$, or $k = 2$, then $k + L(g) - 1 \geq 2$, so $v \notin T_2$ and therefore, $v \notin T' \cup T''$, contradicting our assumption. Assume that $L(g) = 1$ and $k = 1$, then $v \in T'$ and $g^{-1} = (g^{-1})_e = (g^{-1})_b \in G''$, so $v \cdot g^{-1} \in T' \subset T_1$, but $v \in T_{k+l(g)} - T_{k+L(g)-1} = T_2 - T_1$, this is a contradiction.

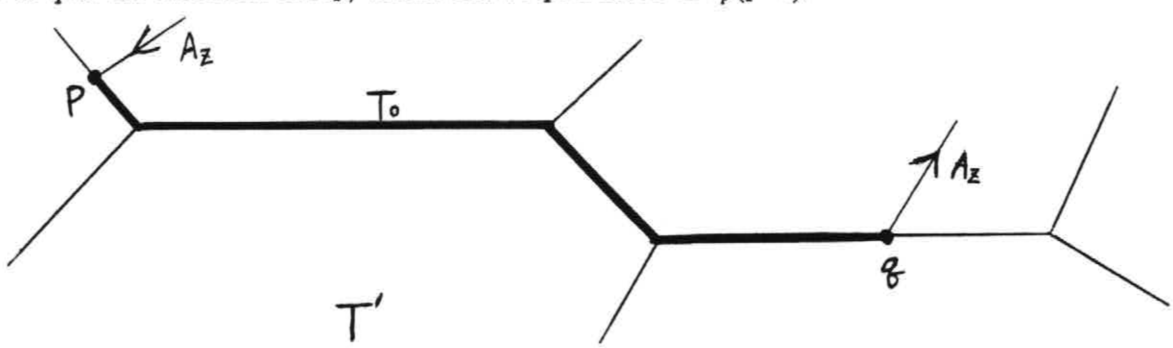
The equivalence of the two basic conditions is thus proved. ◊

6. A simpler condition for actions of the free group of rank 3

When we focus on minimal actions of $G = F_3$ (the free group of rank 3) on an **R**-tree T, Condition **A** is implied by the following Condition **D**, which is simpler.

Assume that $G = F(x,y,z)$. Take $G' = F(x,y)$, $G'' = F(z)$. We may assume $|A_x \cap A_y| < \min\{\tau(x), \tau(y)\}$, (cf. Section 1, page 4).

Since $T_0 \subset A_z$, we have $T_0 = [p,q]$ for some $p.q \in A_z$. We assume further that the direction $t_p(q)$ from p to q is the direction of A_z, which can be presented as $t_p(p \cdot z)$.



Assume K is a fundamental domain of T' mod G'. Then there are points $p', q' \in K$, elements $g_p, g_q \in G'$ such that $p \cdot g_p = p'$ and $q \cdot g_q = q'$. Set

$$X = Y(T') \cap K \cup \{p', q'\}$$

Then $\#X \leq 4$. We can choose K so that

$$X \cap E(K) = \emptyset$$

Since T'' is an axis, $C(p, T'')$ ($C(q, T'')$ resp.) consists of one direction, which is presented as $t_p(p \cdot z^{-1})$ ($t_q(q \cdot z)$ resp.). Set $t_p = t_p(p \cdot z^{-1}) \cdot g_p$, $t_q = t_p(q \cdot z) \cdot g_q$. Then $t_p \in D(p', T) - D(p', K)$ and $t_q \in D(q', T) - D(q', K)$.

Denote the the union of the set of directions in K starting from points of X with the two directions $\{t_p, t_q\}$ by Φ. Then $\#\Phi \leq 12$.

Assume $u \in X$, $g \in G - \{1\}$, then $R(u,g)$ is the following fact: $u \cdot g \in X$, $g_b \in G'$ if $u \notin T_0$, $g_e \in G'$ if $u \cdot g \notin T_0$ and $u \cdot g_i \in T_0$ if $0 < i < L(g)$.

Condition D: *(a) In case $p' = q'$, we have $t_p \neq t_q$.*

(b) If $u \in X$, $g \in G - \{1\}$ are such that g carries a direction $t \in \Phi$ starting from u to a direction in Φ, then $R(u,g)$ is not true.

Proposition 6.1: *When $G = F_3$, Condition* **D** $\Longrightarrow$ *Condition* **A**.

Proof: Assume that Condition **D** is satisfied.

Suppose Condition **A** is not true, then there is a pair $(u,g) \in S$, $g \neq 1$ such that $u \cdot g \in Y_0$ and $H(u,g) = H_b(u,g) \cap H_e(u,g) \neq \emptyset$. There is a direction $t \in H_b(u,g) \cdot g^{-1}$ such that $t \cdot g \in H_e(u,g)$. There are elements $h, l \in G'$ such that $u \cdot h \in K$ and $u \cdot gl \in K$. Because $u, u \cdot g \in Y_0 \subset Y(T') \cup \{p,q\}$, $u \cdot h, u \cdot gl \in X$.

It is clear that $C(u,T') \cdot h \subset D(u \cdot h, K) \subset \Phi$. $C(u,T'') \neq \emptyset$ if and only if $u = p$ and $C(u,T'') = \{t_p(p \cdot z^{-1})\}$ or $u = q$ and $C(u,T'') = \{t_q(q \cdot z)\}$, so $C(u,T'') \cdot h \subset \{t_p, t_q\} \subset \Phi$. Therefore, $t \cdot h \in \Phi$, symmetrically, we have $t \cdot gl \in \Phi$. Take $f = h^{-1}gl$, then $t \cdot hf = t \cdot gl$, so f carries a direction of Φ into Φ. Assume $f = 1$. Then because $h, l \in G'$, $g \in G'$. Since $g_b = g_e \in G'$, $t \in C(u,T'')$ and $t \cdot g \in C(u \cdot g, T'')$, therefore $\{t, t \cdot g\} = \{t_p(p \cdot z^{-1}), t_q(q \cdot z)\}$, it follows that $\{t \cdot h, t \cdot gl\} = \{t_p, t_q\}$. Then we have $t_p = t_q$, contradicting Condition **D** (a). If $f \neq 1$, then either $f_b \in G'$ or $u \cdot h \in T_0$, and either $f_e \in G'$ or $u \cdot gl \in T_0$. We have $\{(u \cdot h) \cdot f_i | 0 < i < L(f)\} \subset \{u \cdot g_i | i \geq 0\} \subset T_0$, therefore $R(u h, f)$ is true, contradicting Condition **D**. This proves the proposition. ◇

7. The 'freeness' of Condition A

Condition **A** is a kind of freeness condition in the following sense:

Theorem 7.1: *Given any minimal actions of G' on T' and G'' on T'', and any nonempty connected intersection $T_0 \subset T'$, $T_0 \subset T''$, there is an extended tree T of $T' \cup T''$, and a minimal action $\rho: T \times G \to T$ containing these data and satisfying Condition* **A**, *where $G = G' * G''$. The action ρ satisfies the following property: For any minimal action $\bar{\rho}$ of G on a tree $\bar{T}$ containing these data, there is a unique G-equivariant dominating map from T to $\bar{T}$ which is the identity on $T' \cup T''$. Furthermore, such T and ρ are unique up to a unique G-equivariant isomorphism.*

Proof: Assume $x \in T' \cup T''$, $g \in G$. If $x \in T'$, $g \in G'$ or $x \in T''$, $g \in G''$, then $x \cdot g$ is defined. Suppose $g = a_1 a_2 \cdots a_n$ be the alternating word of g, if $x \cdot g_i$ is defined for some $1 \leq i \leq n-1$, and if $x \cdot g_i \in T_0$, then $x \cdot g_{i+1} = x \cdot g_i a_{i+1}$ is defined. So if $x \in T'$, $a_1 \in G'$ or $x \in T''$, $a_1 \in G''$ and if $x \cdot g_i$ are defined and belong to T_0 for all i such that $1 \leq i \leq n-1$, then $x \cdot g$ is defined by the actions $T' \times G' \to T'$ and $T'' \times G'' \to T''$, it is a point of $T' \cup T''$.

Set

$$\hat{T} = \{(x,g) | x \in T' \cup T'', g \in G\}$$

Define a equivalence relation $\sim$ on elements of $\hat{T}$ in the following manner: $(x,g) \sim (y,h)$ if there are $i \geq 0$ and $j \geq 0$ such that $x \cdot g_i$ and $y \cdot h_j$ are defined by the actions $T' \times G' \to T'$ and $T'' \times G'' \to T''$, $x \cdot g_i = y \cdot h_j$ and $g_i^{-1} g = h_j^{-1} h$. It can be checked that $\sim$ is a equivalent relation. The equivalent class of (x,g) is denoted by $\{(x,g)\}$.

Set $T = \hat{T}/\sim$ the space of all the equivalent classes in $\hat{T}$. Define a map $\rho: T \times G \to T$ such that $\rho(\{(x,g)\}, h) = \{(x, gh)\}$ for all $\{(x,g)\} \in \hat{T}$ and $h \in G$. ρ is well defined, because if $(x,g) \sim (y,l)$, then $(x, gh) \sim (y, lh)$ for all $h \in G$.

Define $\Phi: T' \cup T'' \to T$ such that $\Phi(x) = \{(x,1)\}$ if $x \in T' \cup T''$. Φ is an injective map, so we regard $T' \cup T''$ as a subspace of T.

Define G_n, T_n for all n in the same way as we did before introducing Condition **A′** (see page 8).

Lemma 7.2:

$$T = \bigcup_{n=0}^{\infty} T_n$$

Proof: Take any $\{(x,g)\}$ from T, let us prove by induction on $L(g)$ that $\{(x,g)\} \in \bigcup_{n=0}^{\infty} T_n$.

If $L(g) = 0$, $g = 1$ and $\{(x,g)\} \in T' \cup T'' \subset T_2$.

Assume $g = a_1 a_2 \cdots a_m$ be the word of g. By induction, $\{(x, g_{m-1})\} \in T_n$ for some nonnegative integer n, then $\{(x,g)\} = \{(x, g_{m-1})\} \cdot a_m \in T_n \cdot (G' \cup G'') \subset T_{n+1}$. So the claim is true. ◇

Lemma 7.3: *For every $n \geq 1$, we have:*

(a) If $u \in T_n - T_{n-1}$ then $u = \{(x,g)\}$ for some $x \in T' \cup T''$ and $g \in G$ such that $g_e \in G_n$.

(b) If $u = \{(x,g)\} \in T_n$ and $g_e \in G_{n+1} - \{1\}$, then $u \in T_{n-1}$.

(c) $T_n \cdot (G_{n+1} - \{1\}) \cap T_n \subset T_{n-1}$.

Proof: (a) Because $T_n - T_{n-1} \subset (T_{n-1} - T_{n-2}) \cdot G_n$, this can be proved by induction on n.

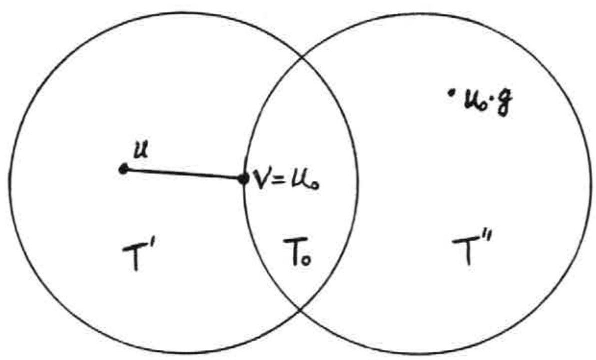
(b) Suppose $u = \{(x,g)\} \in T_n$ and $g_e \in G_{n+1} - \{1\}$. If $u \notin T_{n-1}$ then by (a), there is a point $y \in T' \cup T''$, an element $h \in G$ such that $h_e \in G_n$ and $u = \{(y,h)\}$. Then $(x,g) \sim (y,h)$, by the definition of the equivalent relation $\sim$, we see that $x \cdot g$ and $y \cdot h$ are defined by the actions $T' \times G' \to T'$ and $T'' \times G'' \to T''$, and $x \cdot g = y \cdot h$. Assume $h = 1$ and $y \in T'$, then $u \in T_1$, since $u \notin T_{n-1}$, we have $n = 1$. Then $g_e \in G_2 - \{1\} = G'' - \{1\}$ and therefore $x \cdot g \in T''$, so $x \cdot g = y \in T' \cap T'' = T_0$. Similarly, if

$h = 1$ and $y \in T''$, we have $x \cdot g \in T_0$. Now assume $h \neq 1$, then $h_e \in G_n - \{1\}$. Because one of g_e and h_e belongs to G' and the other belongs to G'', $x \cdot g = y \cdot h \in T' \cap T'' = T_0$. Then $u = x \cdot g, 1 \in T_{n-1}$, impossible.

(c) Assume $u \in T_n \cdot (G_{n+1} - \{1\}) \cap T_n - T_{n-1}$, then there is a point $v \in T_n - T_{n-1}$ and element $h \in G_{n+1} - \{1\}$, such that $u = v \cdot h$. By (a), $v = \{(x, g)\}$, with $x \in T' \cup T''$, $g \in G$ and $g_e \in G_n$, then $u = \{(x, gh)\} \in T_n - T_{n-1}$ and $(gh)_e \in G_{n+1} - \{1\}$, by (b), $u \in T_{n-1}$, impossible. This proves that $T_n \cdot (G_{n+1} - \{1\}) \cap T_n \subset T_{n-1}$. ◊

We have metrics on T' and T'' which coincide on $T_0 = T' \cap T''$, there is a unique metric d_0 on $T' \cup T''$, which agrees with the metrics on T' and T'', and under which, $T' \cup T''$ is a tree. Now, let us define a metric d on T_2 such that when restricted to $T' \cup T''$, d is d_0.

For any $u \in T_1 = T'$, there is a unique point $v \in T_0$ such that $B(u, T_0) = [u, v]$ if $u \notin T_0$, and $u = v$ if $u \in T_0$, denote this point v by u_0. For any $u \in T'$, $g \in G''$, $u_0 \cdot g \in T''$.



Assume $u, v \in T_2$, $u \neq v$. If $u, v \in T' \cup T''$, define $d(u, v) = d_0(u, v)$. If $u \notin T' \cup T''$, then $u \in T_1 \cdot G_2$, by Lemma 7.3 (c), $u = u' \cdot g$ for a unique $u' \in T'$ and a unique element $g \in G_2 = G''$. If $v \in T' \cup T''$, define $d(u, v) = d_0(u', u'_0) + d_0(u'_0 \cdot g, v)$. If $u \in T' \cup T''$, $v \notin T' \cup T''$, $d(u, v)$ is defined similarly. Finally assume $u, v \notin T' \cup T''$, then uniquely there are $u', v' \in T'$, $g, h \in G''$ such that $u = u' \cdot g$ and $v = v' \cdot h$, if $g = h$, then define $d(u, v) = d_0(u', v')$, if $g \neq h$, define $d(u, v) = d_0(u', u'_0) + d_0(v', v'_0) + d_0(u'_0 \cdot g, v'_0 \cdot h)$. Thus d is defined on $T_2 \times T_2$.

If $u, v, w \in T_2$, we can easily verify the following:

(a) $d(u, u) = 0$.

(b) $d(u, v) = d(v, u)$.

(c) $d(u, w) \leq d(u, v) + d(v, w)$.

So (T_2, d) is a metric space.

Lemma 7.4: *(a) $(T_1 - T_0)\cdot g$ is open in T_2 for any $g \in G''$.*

(b) $T' \cup T''$ is closed in T_2.

(c) If J is any subsegment of $T_2 - (T' \cup T'')$, then there is an element $g \in G'' - \{1\}$ such that $J \subset (T_1 - T_0)\cdot g$.

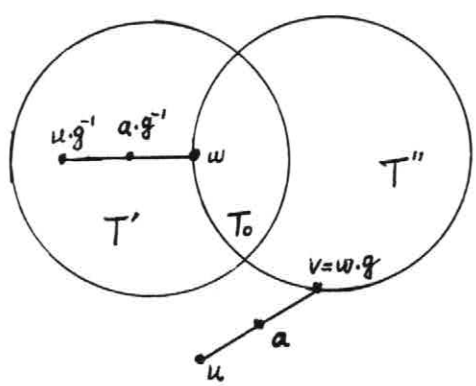
(d) Assume $J = [u, v] \subset T_2$ is a segment, $J \cap (T' \cup T'') = \{v\}$, then if $u' \in T' - T_0$, $g \in G'' - \{1\}$ are such that $u = u' \cdot g$, then $v = u'_0 \cdot g$.

Proof: (a) Fix $u \in (T_1 - T_0)\cdot g$, $u = u' \cdot g$ for some $u' \in (T_1 - T_0)$. For any $v \in T_2 - (T_1 - T_0)\cdot g$, we have $d(u, v) \geq d_0(u', u'_0) > 0$, so u is not a limit point of $T_2 - (T_1 - T_0)\cdot g$, therefore $(T_1 - T_0)\cdot g$ is an open set.

(b) $T' \cup T'' = T_2 - (\bigcup_{g\neq 1}(T_1 - T_0)\cdot g)$ is closed.

(c) According to (a), for each $g \in G'' - \{1\}$, $J \cap (T_1 - T_0)\cdot g$ is open in J. Because by lemma 7.3 (c), J is a disjoint union of $\{J \cap (T_1 - T_0)\cdot g | g \in G'' - \{1\}\}$, and J is connected, all of these open sets except one must be empty, so $J \subset (T_1 - T_0)\cdot g$ for some $g \in G'' - \{1\}$.

(d) By (c), $J - \{v\} \subset (T_1 - T_0)\cdot g$. So $(J - \{v\})\cdot g^{-1}$ is a segment in $T_1 - T_0$ with one end open, whose endpoint is denoted by w. Suppose $w \notin T_0$, since $g: T_1 - T_0 \to (T_1 - T_0)\cdot g$ preserves distance, we have that $v = w \cdot g \in (T_1 - T_0)\cdot g$, then $v \notin T' \cup T''$, which is impossible. Therefore, $w \in T_0$, so $w = (a \cdot g^{-1})_0$ for every $a \in J - \{v\}$. By the definition of the distance d, we have $d(a, w \cdot g) = d(a \cdot g^{-1}, w) = d(a, v)$ if $a \in J - \{v\}$, then it is clear that $v = w \cdot g = u'_0 \cdot g$, with $u' = u \cdot g^{-1}$. ◊



Assume $u, v \in T_2$, by a path between u and v, we mean a closed segment whose two end points are u and v.

Lemma 7.5: *There is a path between any two points of T_2.*

Proof: We know that $T' \cup T''$ is a tree, if $u, v \in T' \cup T''$, then there is a unique path $p_0(u, v)$ inside $T' \cup T''$ between these two points.

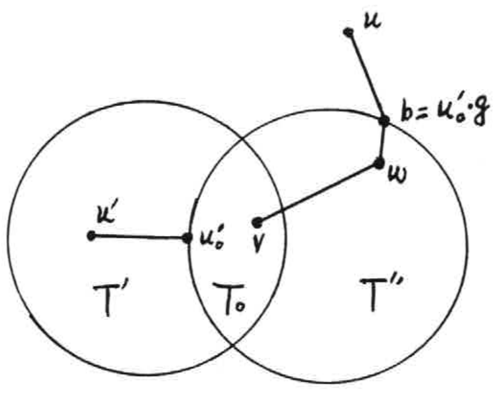
Suppose $u \in T_2 - (T' \cup T'')$, $u = u' \cdot g$ as before, then by the definition of d, $g: p_0(u', u'_0) \to p_0(u', u'_0) \cdot g$ is a isomorphism, so $p_0(u', u'_0) \cdot g$ is a path between u and $u'_0 \cdot g$. Assume v is any point in $T' \cup T''$, define $[u, v] = p_0(u', u'_0) \cdot g \cap p_0(u'_0 \cdot g, v)$. Assume $w \in p_0(u', u'_0) \cdot g$ and $q \in p_0(u'_0 \cdot g, v)$, since $w' = w \cdot g^{-1} \in p_0(u', u'_0)$ and $w'_0 = u'_0$, by the definition we have $d(w, q) = d_0(w', w'_0) + d_0(w'_0 \cdot g, q) = d(w, u'_0 \cdot g) + d(u'_0 \cdot g, q)$. This proves that $[u, v]$ is a path. If $v \in T_2 - (T' \cup T'')$, v' and h are as before, define $[u, v] = p_0(u', u'_0) \cdot g \cup p_0(u'_0 \cdot g, v'_0 \cdot h) \cup p_0(v', v'_0) \cdot h$, we can prove, similar to the above, that $[u, v]$ is a path from u to v. This proves that (T_2, d) is path connected. ◇

Lemma 7.6: *Assume I and J are two segments in T_2, then if $I \cap J$ is a single point, which is the end point of both, then $I \cup J$ is a segment in T_2.*

Proof: Assume $I \cap J = \{w\}$, u is the other end of I and v is the other end of J, we only have to prove that

$$d(u, v) = d(u, w) + d(w, v) \tag{7.1}$$

Assume $w \in T' \cup T''$, we take the case where $u \notin T' \cup T''$, $v \in T' \cup T''$ for example to prove (7.1). Assume $u' \in T' - T_0$, $g \in G'' - \{1\}$ are such that $u = u' \cdot g$, $d(u, v) = d(u', u'_0) + d(u'_0 \cdot g, v)$. Suppose $b \in [u, w]$ be such that $b \in T' \cup T''$ and $d(u, a) \geq d(u, b)$ for any $a \in T' \cup T'' \cap [u, w]$ (such point b exists since $T' \cup T''$ is closed in T_2). According to Lemma 7.4 (d), we can prove that $[b, w] \subset T' \cup T''$ and $b = u'_0 \cdot g$. Therefore, $[u'_0 \cdot g, w] \cap [w, v] \subset [u, w] \cap [v, w] = \{w\}$. Because $T' \cup T''$ is a tree, we have $d(u'_0 \cdot g, w) + d(w, v) = d(u'_0 \cdot g, v)$, then $d(u, v) = d(u', u'_0) + d(u'_0 \cdot g, w) + d(w, v) = d(u, w) + d(w, v)$.



Assume $w \in T_2 - (T' \cup T'')$, suppose $w' \in T_1 - T_0$, $g \in G'' - \{1\}$ be such that $w = w' \cdot g$.

Case 1, $I^\circ \cap (T' \cup T'') = J^\circ \cap (T' \cup T'') = \emptyset$, then $(I \cup J) \cap (T' \cup T'') \subset \{u, v\}$, by Lemma 7.4 (c) and (d), $I \cup J \subset T_1 \cdot g$, because $I \cdot g^{-1} \cap J \cdot g^{-1} = \{w \cdot g^{-1}\}$ and T_1 is a tree, $d(u \cdot g^{-1}, v \cdot g^{-1}) = d(u \cdot g^{-1}, w \cdot g^{-1}) + d(w \cdot g^{-1}, v \cdot g^{-1})$, this implies (7.1).

Case 2, $I \cap (T' \cup T'') \neq \emptyset$, there is a point $a \in I$ such that $[w, a] \cap (T' \cup T'') = \{a\}$, by Lemma 7.4 (d), $a = w'_0 \cdot g$. We claim that $J \cap (T' \cup T'') = \emptyset$, otherwise $a = w'_0 \cdot g \in J$, impossible. As in Case 1, we can prove that $d(a, w) + d(w, v) = d(a, v)$, and $[a, v] = [a, w] \cup [w, v]$ is a segment. Because $a \in T' \cup T''$ and $[a, v] \cap [a, u] = \{a\}$, we have $d(u, v) = d(a, u) + d(a, v) = d(a, u) + d(a, w) + d(w, v) = d(u, w) + d(w, v)$.

Case 3, $J \cap (T' \cup T'') \neq \emptyset$. (7.1) is proved similar to Case 2. ◊

Lemma 7.7: *Between any two points of T_2, there is a unique path.*

Proof: Suppose there are two paths p_1, p_2 between a pair of points of T_2. Assume $p_1 \neq p_2$, then there are subsegments $\hat{p}_1 \subset p_1$ and $\hat{p}_2 \subset p_2$ such that $\hat{p}_1$ and $\hat{p}_2$ share the same endpoints, $(\hat{p}_1)^\circ \cap (\hat{p}_2)^\circ = \emptyset$ and at least one of $\hat{p}_1$ and $\hat{p}_2$ is nondegenerate. According to Lemma 7.6, $\hat{p}_1 \cup \hat{p}_2$ is homeomorphic to a circle, it is called a loop.

To prove the lemma, it is enough to show that there is no loop in T_2. Suppose we have a loop l in T_2.

Case 1: $l \subset T' \cup T''$, this is impossible because $T' \cup T''$ is a tree.

Case 2: $l \cap (T' \cup T'') = \emptyset$. Then $l \subset (T_1 - T_0) \cdot (G'' - \{1\})$. By Lemma 7.4 (c), $l \subset (T_1 - T_0) \cdot g$ for some $g \in G'' - \{1\}$, then $l \cdot g^{-1}$ is a loop in $T_1 - T_0$ which is impossible.

Case 3: $l \not\subset T' \cup T''$ but $l \cap (T' \cup T'') \neq \emptyset$. Take any component J of $l - (T' \cup T'')$, J is an open subsegment of l, whose two end points u, v belong to $T' \cup T''$. Pick any point $a \in J$, consider $J_1 = [u, a]$ and $J_2 = [v, a]$, by Lemma 7.4 (d), if $a' \in T_1 - T_0$, $g \in G'' - \{1\}$ are such that $a = a' \cdot g$, then $u = a'_0 \cdot g = v$, $l \cap T' \cup T'' = \{u\} = \{a'_0 \cdot g\}$. It is clear that $l \cdot g^{-1}$ is a loop in T', this is impossible.

This proves the uniqueness of the path. ◊

Then according to Proposition II. 1.13 of [8], T_2 is a tree.

We have extended the metric d_0 of $T' \cup T''$ to d of T_2, and made T_2 a tree, we adopt the above process repeatedly to extend d further. Now suppose d has been extended to T_n for an integer $n \geq 2$, such that (T_n, d) is a tree containing $T' \cup T''$, $T_2, T_3, \ldots T_{n-1}$ as subtrees. Assume $w \in T_n - T_{n-1}$, since T_{n-1} is a subtree of T_n, there is a point $v \in T_{n-1}$ such that $[w, v]$ (the path between w and v in T_n) is the bridge between w and T_{n-1}, we denote this point v by w_{n-1}. Suppose u is a point of $T_{n+1} - T_n$, then $u \in (T_n - T_{n-1}) \cdot G_{n+1}$, by Lemma 7.3 (c), there is a unique point $u' \in T_n - T_{n-1}$ and a unique element $g \in G_{n+1}$ such that $u = u' \cdot g$. The point $u'_{n-1} \cdot g$ belong to T_{n-1}. In the process of extending d to T_{n+1}, we do similarly as we extend it from $T' \cup T''$ to T_2, taking $T_n - T_{n-1}$ for $T_1 - T_0$, T_n for $T' \cup T''$, G_{n+1} for $G_2 = G''$ and for each $u \in T_{n+1} - T_n$, taking u'_{n-1} for u'_0. Lemma 7.4, 7.5 and 7.6 remain valid, so T_{n+1} is a tree.

In this way, we extend d to T and make T a tree. Next we prove that for any $g \in G$, the map $g: T \to T$ induced by ρ preserves distance.

Lemma 7.8: *Assume $g \in G' \cup G''$, n is a nonnegative integer, $u, v \in T_n$, then if $u \cdot g, v \cdot g \in T_n$, we have*

$$d(u \cdot g, v \cdot g) = d(u, v) \tag{7.2}$$

Proof: We prove this lemma by induction on n. As before, in the induction process, for $n = 1$, we use $T' \cup T''$ instead of $T_1 = T'$.

(7.2) is obviously true for $u, v, u\cdot g, v\cdot g$ in T_0 or in $T' \cup T''$ because the actions of G', G'' on T', T'' preserve distance, so the lemma is true when $n = 0$ or $n = 1$.

Assume $n \geq 2$, $u, v, u\cdot g, v\cdot g \in T_n$, without loss of generality, we assume that $g \in G' - \{1\}$.

First assume that n is even. If $u \in T_n - T_{n-1}$, since $g \in G' - \{1\} = G_{n+1} - \{1\}$, by Condition $\mathbf{A}'$, $u \in T_{n+1} - T_n$, contradicting the assumption, therefore, $u \in T_{n-1}$. Similarly, $v \in T_{n-1}$. Because T_{n-1} is invariant under g, $u\cdot g, v\cdot g \in T_{n-1}$, then by induction, (7.2) is true.

Assume n is odd, we take the following case as an example to prove (7.2) that $u \in T_n - T_{n-1}$, $v \in T_{n-1} - T_{n-2}$. In all the other cases, the proof is easier, similar and is omited here.

There is a point $u' \in T_{n-1} - T_{n-2}$, an element $h \in G_n - \{1\} = G' - \{1\}$, such that $u = u'\cdot h$ and $hg \neq 1$, then $u\cdot g = u'\cdot hg$. Since $v \in T_{n-1} - T_{n-2}$, $v\cdot g \in T_n - T_{n-1}$, by the way we extend the metric d from T_{n-1} to T_n, we get $d(u\cdot g, v\cdot g) = d(u', (u')_{n-2}) + d((u')_{n-2}\cdot hg, v_{n-2}\cdot g) + d(v, v_{n-2})$, by induction, $d((u')_{n-2}\cdot hg, v_{n-2}\cdot g) = d((u')_{n-2}\cdot h, v_{n-2})$. Since $(u')_{n-2}\cdot h \in T_{n-2}$, $d((u')_{n-2}\cdot h, v_{n-2}) + d(v_{n-2}, v) = d((u')_{n-2}\cdot h, v)$, so $d(u\cdot g, v\cdot g) = d(u', (u')_{n-2}) + d((u')_{n-2}\cdot h, v) = d(u, v)$, this proves (7.2). ◊

Corollary 7.9: *For every $g \in G$, the map $g: T \to T$ induced by ρ preserves distance.*

Proof: Using Lemma 7.8, we can easily prove this by induction on $L(g)$, which is the alternating word length of g. ◊

This proves that the map $\rho: T\times G \to T$ is an action of G on T. It is clear that the action $\rho: T\times G \to T$ is minimal, T', T'' are the minimal invariant subtrees of G', G'' respectively under ρ and T_n is a closed subtree of T for all $n \geq 0$. By Lemma 7.3 (c), the action ρ satisfies Condition $\mathbf{A}'$, therefore, satisfies Condition $\mathbf{A}$.

Assume $\bar{\rho}: \bar{T} \times G \to \bar{T}$ is an action of G on a tree $\bar{T}$, which contains T' and T'' as the minimal invariant subtrees of G' , G'' respectively.

We extend the identity map on $T' \cup T''$ to a map ϕ from T to $\bar{T}$ in the following way:

Assume $u = \{(x, g)\} \in T$, where $x \in T' \cup T''$ and $g \in G$, define $\phi(u) = \phi(x)\cdot g$. $\phi(u)$ is well defined, since $(x, g) \sim (y, h)$, there are $i \leq L(g)$, $j \leq L(h)$, such that $x\cdot g_i$ and $y\cdot h_j$ are defined by the actions $T' \times G' \to T'$ and $T'' \times G'' \to T''$, $x\cdot g_i = y\cdot h_j \in T' \cup T''$ and $g_i^{-1}g = h_j^{-1}h$, then $\phi(x)\cdot g = \phi(x\cdot g_i)\cdot g_i^{-1}g = \phi(y\cdot h_j)\cdot h_j^{-1}h = \phi(y)\cdot h$.

Then ϕ is defined on T, it is clearly an G-equivariant map and is the only G-equivariant extension of the identity on $T' \cup T''$ from T to $\bar{T}$.

Define $\bar{T}_1 = \bar{T}'$, $\bar{T}_2 = T'' \cup \bar{T}_1 \cdot G_2$ and for $n \geq 3$, $\bar{T}_n = \bar{T}_{n-1} \cdot G_n$.

Inductively we see that $\phi(T_n) = \bar{T}_n$ for all $n \geq 0$.

Lemma 7.10: *For any two points $u, v \in T$, we have that*

$$d(\phi(u), \phi(v)) \leq d(u, v) \tag{7.3}$$

*Furtheremore, if the action $\bar{\rho}: \bar{T} \times G \to \bar{T}$ satisfies Condition **A**, then the map ϕ preserves distances.*

Proof: It is enough to prove that for every $n \geq 2$, (7.3) is true if $u, v \in T_n$. We prove this by induction on n.

We know that if $u, v \in T' \cup T''$, then $d(\phi(u), \phi(v)) = d(u, v)$.

Assume $u \in T_2 - (T' \cup T'')$ then $u = u' \cdot g$ for some $u' \in T_1 - T_0$ and $g \in G'' - \{1\}$, for any $v \in T' \cup T''$, $d(\phi(u), \phi(v)) \leq d(\phi(u), \phi(u'_0) \cdot g) + d(\phi(u'_0) \cdot g, \phi(v)) = d(\phi(u'), \phi(u'_0)) + d(\phi(u'_0 \cdot g), \phi(v)) = d(u', u'_0) + d(u'_0 \cdot g, v) = d(u, v)$. If $v \in T_2 - (T' \cup T'')$, then $v = v' \cdot h$ for some $v' \in T_1 - T_0$, $h \in G'' - \{1\}$. If $g = h$, we can easily prove that $d(\phi(u), \phi(v)) = d(u, v)$. Assume $g \neq h$, then $d(\phi(u), \phi(v)) \leq d(\phi(u), \phi(u'_0) \cdot g) + d(\phi(u'_0) \cdot g, \phi(v'_0) \cdot h) + d(\phi(v'_0) \cdot h, \phi(v)) = d(\phi(u'), \phi(u'_0)) + d(\phi(u'_0 \cdot g), \phi(v'_0 \cdot h)) + d(\phi(v'_0), \phi(v')) = d(u', u'_0) + d(u'_0 \cdot g, v'_0 \cdot h) + d(v'_0, v') = d(u, v)$. This proves that (7.3) is true for every pair of elements $u, v \in T_2$.

Assume that (7.3) is true for every points in T_n, to prove this inequality for any pair of points in T_{n+1}, the inductive process is very similar to the above. Take $T_n - T_{n-1}$ for $T_1 - T_0$, T_n for $T' \cup T''$, G_{n+1} for $G_2 = G''$ and for each point $u \in T_{n+1} - T_n$, take u'_{n-1} for u'_0, as we did when extending the metric d from T_n to T_{n+1}, then the proof proceeds similarly.

If the action $\bar{\rho}: \bar{T} \times G \to \bar{T}$ satisfies Condition **A**, then all the '$\leq$' signs in the above equations can be changed to '$=$', so ϕ preserves distance. ◊

Lemma 7.10 proves that the map ϕ is a contraction, so it is an G-equivariant dominating map. When the action $\bar{\rho}: \bar{T} \times G \to \bar{T}$ is minimal and satisfies Condition **A**, according to Lemma 7.10, ϕ is a G-equivariant isomorphism, this implies the uniqueness of T and ρ, and completes the proof of Theorem 7.1. ◊

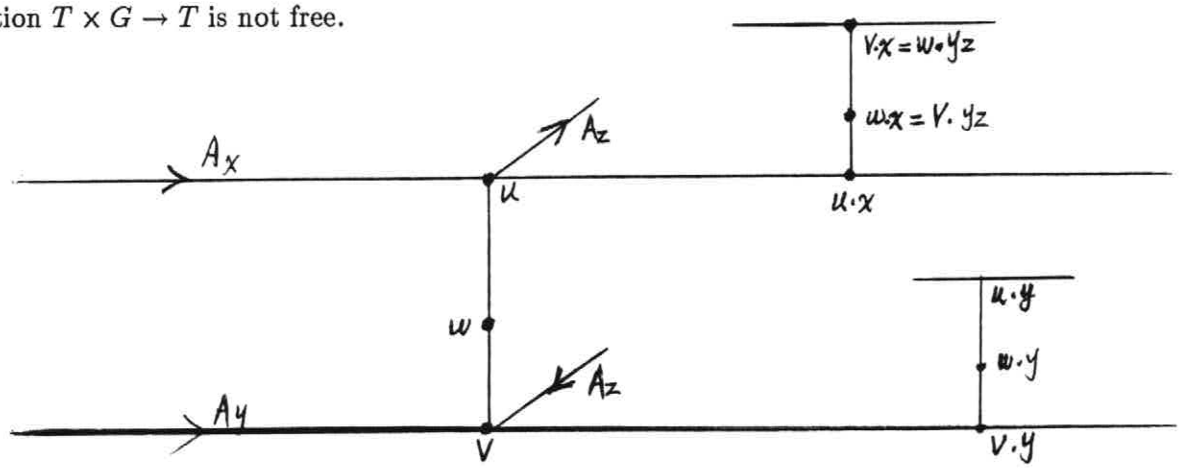
8. Condition A makes things different

We show this by giving the following example:

Example 8.1: (a) Assume that $G' = F(x, y)$, $G'' = F(z)$, so that $G = G' * G'' = F(x, y, z)$. Suppose that $A_x \cap A_y = \emptyset$, and $T_0 = B(A_x, A_y) = [u, v]$ with $u \in A_x$ and $v \in A_y$, $0 < \lambda = \tau(y) - \tau(x) < \tau(z) = |B(A_x, A_y)| = \text{dis}(u, v)$, and $w \in [u, v]$ is such a point that $\text{dis}(u, w) = \lambda$. Assume further that the direction of A_z is $t_v(u)$, i.e. $v \cdot z = u$ and $w \cdot yz = v \cdot x$, then the action $T \times G \to T$ is not free.

(b) Assume Condition **A** is satisfied, then if T_0 is embedded into $Q' = T'/G'$ or is embedded into $Q'' = T''/G''$, then the action $T \times G \to T$ is free and discrete.

Proof: (a) It is clear that $[v, w{\cdot}y]{\cdot}z = [u, v{\cdot}x]$. Since $v{\cdot}y \in [v, w{\cdot}y]$, we have $v{\cdot}yz \in [u, w{\cdot}yz] = [u, v{\cdot}x]$. Because $d(v \cdot yz, u) = d(v \cdot y, v) = \tau(y)$ and $d(u, w \cdot x) = d(u, u \cdot x) + d(u \cdot x, w \cdot x) = \tau(x) + d(u, w) = \tau(x) + \tau(y) - \tau(x) = \tau(y)$, we have $v{\cdot}yz = w{\cdot}x$. Then $w{\cdot}yzx^{-1} = w{\cdot}xz^{-1}y^{-1}$ or $w{\cdot}(yzx^{-1})^2 = 1$. Hence the action $T \times G \to T$ is not free.



(b) Assume that T_0 is embedded into Q', then $\Sigma' = \{id\}$, where id is the identity map of T_0, so $\Sigma = \Sigma''$. According to Proposition 4.1 (a), if the action $T \times G \to T$ is not free, then $\Sigma'' = \Sigma$ has a fixed point in $\Omega \subset T_0$. This is impossible since the action $T'' \times G'' \to T''$ is free. Because $\#\Sigma = \#\Sigma'' < \infty$, any Σ-orbit is finite. Then by Proposition 4.2 (a), this action is discrete also. The other case is proved symmetrically. ◇

In (a), Condition **A** is not satisfied, because $(v, z) \in S$ and $v{\cdot}z = u \in Y_0$, but $t_u(u{\cdot}x) = t_v(v{\cdot}y){\cdot}z \in H(v, z)$, so $H(v, z) \neq \emptyset$. Although we have that T_0 is embedded into the quotient $Q' = T'/G'$ by the quotient map from T' to Q', the action $T \times G \to T$ is not free. From this example, we can see the power of Condition **A**.

Recall in Section 2, we made the following assumptions:

Assumption 2: $T_0 \neq \emptyset$.

Assumption 3: $|T_0| \neq \infty$.

The examples in the following Section 9 and Section 10 need no extra condition (for example, they do not need Condition **A**), they show us that Assumption 2 and 3 are reasonable when we consider the question of whether the freeness implies the discreteness for a minimal action of finitely generated free group on an **R**-tree.

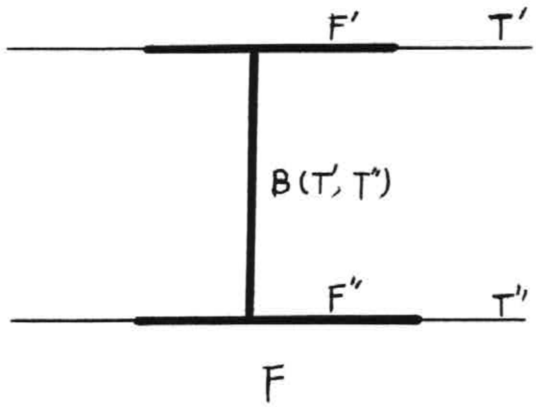
9. The case when T_0 is empty

Example 9.1: Assume $T_0 = T' \cap T'' = \emptyset$, then the action $T \times G \to T$ is free and discrete.

Proof: We know the bridge $B(T', T'')$ is a segment, $B(T', T'') = [u', u'']$ for some $u' \in T'$ and $u'' \in T''$. Because the actions $T' \times G' \to T'$ and $T'' \times G'' \to T''$ are free and discrete, by Lemma 1.1 (a), there are fundamental domains F' and F'' of T' and T'' mod G' and G'' containing u' and u'' respectively in their interiors. Set

$$F = F' \cup B(T', T'') \cup F''$$

$$\bar{T} = F \cdot G \subset T$$



For every element $g \in G$ set

$$\Lambda(g) = \begin{cases} B(F \cdot g, F) \cup F \cdot g, & \text{if } F \cdot g \cap F = \emptyset; \\ F \cdot g, & \text{if } F \cdot g \cap F \neq \emptyset. \end{cases}$$

We shall prove that $\bar{T}$ is connected. To this end, let us denote the component of $\bar{T}$ containing F by C. It is easy to see that $T' \subset C$ and $T'' \subset C$.

Lemma 9.2: *For every $g \in G - \{1\}$*

(a) $F \cdot g \subset C$.

(b) If $g_e \in G'$, then $\Lambda(g) \cap F \subset F'$, and if $g_e \in G''$, then $\Lambda(g) \cap F \subset F''$.

(c) $(F \cdot g)^\circ \cap F^\circ = \emptyset$. If $F \cdot g \cap F \neq \emptyset$, then it consists of a single point and $g \in G' \cup G''$.

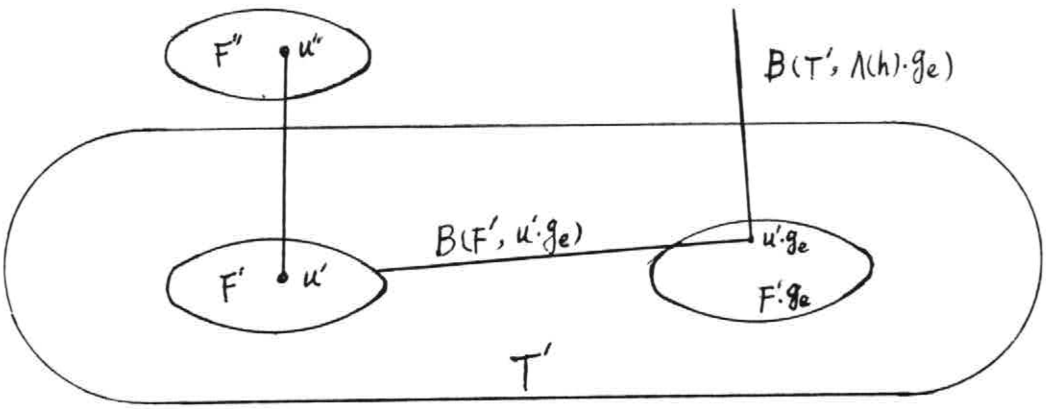
Proof: Without lose of generality, let us assume that $g_e \in G'$. We prove this lemma by induction on $L(g)$.

Suppose $L(g) = 1$, then $g \in G' - \{1\}$, $F \cdot g \cap T' \supset F' \cdot g \neq \emptyset$, therefore $F \cdot g \subset C$ because $T' \subset C$. It is clear that $F \cdot g \cap F = F' \cdot g \cap F'$, if this set is not empty, then $F \cap \Lambda(g) = F \cap F \cdot g = F' \cap F' \cdot g \subset F'$. Assume $F' \cap F' \cdot g \neq \emptyset$, then clearly $B(F', F' \cdot g) = B(F, F \cdot g)$, so $\Lambda(g) \cap F = B(F, F \cdot g) \cap F = B(F', F' \cdot g) \cap F \subset T' \cap F = F'$. Because F' is a fundamental domain of T' mod G', $(F)^\circ \cap (F \cdot g)^\circ = (F')^\circ \cap (F' \cdot g)^\circ = \emptyset$ and $F \cap F \cdot g = F' \cap F' \cdot g$ is at most a single point set.

Now, assume $L(g) > 1$. Set $h = gg_e^{-1}$, then $L(h) = L(g) - 1$. By induction hypotheses, $F \cdot h \subset C$, therefore $\Lambda(h) \subset C$.

(a) Because $(\Lambda(h) \cup F) \cdot g_e$ is connected and its intersection with T' contains the nonempty set $F' \cdot g_e$, it is contained in C, so $F \cdot g = (F \cdot h) \cdot g_e \subset (\Lambda(h) \cup F) \cdot g_e \subset C$.

(b) Because $g_e \in G'$, $h_e \in G''$. Since $\Lambda(h) \cap F \subset F''$, $[u', u''] = B(T', T'') \subset B(T', \Lambda(h))$ and then $[u', u''] \cdot g_e \subset B(T', \Lambda(h)) \cdot g_e = B(T', \Lambda(h) \cdot g_e)$, so $T' \cap B(T', \Lambda(h) \cdot g_e) = \{u' \cdot g_e\}$ and $u' \cdot g_e \in E(B(T', \Lambda(h) \cdot g_e))$. Since $u' \in (F')^\circ$, $u' \cdot g_e \notin F'$, because $B(F', u' \cdot g_e) \subset T'$, $B(F', u' \cdot g_e) \cap B(T', \Lambda(h) \cdot g_e) = \{u' \cdot g_e\}$, then $B(F', u' \cdot g_e) \cup B(T', \Lambda(h) \cdot g_e)$ is a segment, it is clearly the bridge between F and $\Lambda(h) \cdot g_e$, but $F \cdot g \subset \Lambda(h) \cdot g_e$, therefore $B(F', u' \cdot g_e) \subset B(F, F \cdot g) \subset \Lambda(g)$. Then $F \cap \Lambda(g) = B(F', u' \cdot g_e) \cap F' \subset F'$.



(c) From the proof of (b), we see that $\Lambda(h) \cdot g_e \cap F = \emptyset$, in particular, $F \cdot g \cap F = \emptyset$ since $F \cdot g \subset \Lambda(h) \cdot g_e$. $\diamond$

According to Lemma 9.2 (a), $C = \bar{T}$, so $\bar{T}$ is connected, it is an G invariant subtree of T. But we assumed that T is minimal, so $T = \bar{T} = F \cdot G$. Then according to Lemma 9.2 (c), F is a fundamental domain of T mod G, so the action $T \times G \to T$ is discrete if it is free by Lemma 9.2 (b).

Suppose $u \in T$, $g \in G - \{l\}$ be such that $u \cdot g = u$, we may assume that $u \in F$. Then $u \in F \cap F \cdot g$, by lemma 9.2 (c) and its proof, $g \in G' \cup G''$ and $u \in T'$ if $g \in G'$, $u \in T''$ if $g \in G''$. Because the actions $T' \times G' \to T'$ and $T'' \times G'' \to T''$ are free, this is impossible. So the action $T \times G \to T$ is free. $\diamond$

Now, let us consider a slightly more complicated case, but again, where the extra conditions are unnecessary.

Example 9.3: If $T_0 \neq \emptyset$ and for every element $h \in G' \cup G'' - \{1\}$, $T_0 \cdot h \cap T_0 = \emptyset$, then the action $T \times G \to T$ is free and discrete.

Proof: By the condition of this example, T_0 is embedded into T'/G' and into T''/G'' respectively under the quotient maps. According to Lemma 1.1 (a), there are fundamental domains F' of T' mod G' and F'' of T'' mod G'', which contain T_0 in their interiors. Set $F = F' \cup F''$. Define $\bar{T}$, C and $\Lambda(g)$ for every $g \in G$ just as we did in Example 9.1. The arguments used above in Lemma 9.2 work as well in this case with the following slight changes in (b): We want $\Lambda(g) \cap F$ to belong to $F' - T_0$ or $F'' - T_0$, depending on whether g belongs to G' or G''. We prove that $T' \cap B(T', \Lambda(h) \cdot g_e) = \{u\}$ for some point $u \in T_0 \cdot g_e$, we use u instead of $u' \cdot g_e$ in Example 9.1. $u \notin F'$ since $T_0 \cdot g_e \cap F' = \emptyset$. It follows that F is a fundamental domain of T mod G, so the action $T \times G \to T$ is free and discrete. ◇

10. The case when T_0 has infinite total measure

Define

$$\varepsilon' = \{[a,b] | a \neq b, [a,b] \subset T_0, [a,b] \cap W' = \{a,b\}\}$$

$$\varepsilon'' = \{[a,b] | a \neq b, [a,b] \subset T_0, [a,b] \cap W'' = \{a,b\}\}$$

T_0 is covered by segments in ε' and by segments in ε'' respectively. Since W' is Σ' invariant, Σ' acts on ε'. Define an equivalence relation among segments in ε' as follows: If $I, J \in \varepsilon'$, $I \sim J$ if and only if I and J are in the same Σ'-orbit. Symmetrically, we define such an equivalence relation among segments in ε'', using the Σ''-orbits. If $J \in \varepsilon'$ or $J \in \varepsilon''$, we denote by $e(J)$ the equivalence class containing J.

Example 10.1: If the rank of G' or the rank of G'' is 1, then there is a positive number λ, which only depends on the actions $T' \times G' \to T'$ and $T'' \times G'' \to T''$, such that if $|T_0| > \lambda$, then the action $T \times G \to T$ is not free.

Proof: Without loss of generality, assume that $\text{rank}(G'') = 1$, then $G'' = F(z)$, the cyclic group generated by a letter z. Since T'' is a minimal tree under the action of G'', $T'' = A_z$.

If n is the number of edges of Q', M is the maximum length of these edges, since Q' is a finite $\mathbf{R}$-graph, we have $n < \infty$ and $M < \infty$. Set $\lambda = 2(n+2)(\tau(z) + M)$, where $\tau(z)$ is the translation length of z.

Suppose $|T_0| > \lambda$. We may assume that $\lambda < |T_0| < \infty$ (If $|T_0| = \infty$, we choose a subsegment $\hat{T}$ of T_0 such that $\lambda < |\hat{T}| < \infty$ and work with $\hat{T}$ instead of T_0). Then $T_0 = [p, q]$, for some $p, q \in A_z$, their images in Q' under the quotient map $\phi': T' \to Q'$ are denoted by p' and q' respectively.

Let

$$\bar{\varepsilon}' = \{[a,b] | a \neq b, [a,b] \subset Q', [a,b] \cap (Y(Q') \cup \{p', q'\}) = \{a,b\}\}$$

Because n is the number of edges of Q' and $|T_0| > 2(n+2)M$, we have the following inequalities:

(10.1) $$\#\bar{\varepsilon}' \leq n+2$$

(10.2) $$\#\varepsilon' > 2(n+2)$$

We have $\phi'(\varepsilon') \subset \bar{\varepsilon}'$ and the preimage of each segment of $\bar{\varepsilon}'$ under ϕ' is an equivalence class if it is not empty, by (10.1) and (10.2), there are at most $n+2$ equivalence classes and at least one equivalence class which has cardinality greater than 2.

Define

$$R = \bigcup\{J | J \in \varepsilon', \#e(J) \leq 2\}$$

Lemma 10.2: *For every point $u \in T_0$, there is an integer $m \neq 0$ such that $u \cdot z^m \in T_0 - R$. i.e. $(u)\Sigma'' \cap (T_0 - R) \neq \emptyset$.*

Proof: From the above remark, there are at most $n+1$ equivalence classes of segments in ε' whose cardinality less than 3, so $|R| \leq 2(n+1)M$, and R has at most $2(n+1)$ components. Then $T_0 - R$ has at most $2n+3$ components, and $|T_0 - R| > \lambda - |R| > 2(n+2)\tau(z)$, so there is a component I of $T_0 - R$ such that $|I| > 2\tau(z)$ or there are two components I, J of it such that $|I| > \tau(z)$ and $|J| > \tau(z)$. Hence for every $u \in T_0$, there is an integer $m \neq 0$ such that $u \cdot z^m \in I \cup J \subset T_0 - R$. ◇

Suppose that $\mathcal{F}$ is the set of all finite alternating words in elements of Σ' and Σ''. If $w \in \mathcal{F}$, the associated composition of partial isometries on T_0 is denoted by σ_w if it exists, its domain and the range are denoted by $D(w)$ and $R(w)$. We see that there is an element $g \in G$, such that $\sigma_w = \sigma_g$ for every alternating word w in elements of Σ' and Σ''. Suppose w and w' are two words, $\sigma_w = \sigma_g$, $\sigma_{w'} = \sigma_h$ for some $g, h \in G$, if $w \neq w'$, then we have $g \neq h$.

Fix a point $u \in T_0 - R$, for every positive integer n, we denote by $\mathcal{F}_n$ the set of words w in $\mathcal{F}$ such that the alternating word length of w is n, the first letter of w belong to Σ', and $u \in D(w)$. Set $N_n(\mathcal{F}) = \#\mathcal{F}_n$.

Lemma 10.3: *$N_n(\mathcal{F})$ grows exponentially with n. Actually for all $k > 0$ we have $N_{2k}(\mathcal{F}) \geq N_{2k-1}(\mathcal{F}) \geq 2^k$.*

Proof: We proof the formula by induction on k. Assume $N_{2k-1}(\mathcal{F}) \geq 2^k$, Suppose $w \in \mathcal{F}_{2k-1}$, then the last letter of w belongs to Σ'. By Lemma 10.2, there is an element $\sigma \in \Sigma''$ such that $(u)\sigma \in T_0 - R$. Suppose $(u)\sigma$ is is contained in a segment $J \in \varepsilon'$, then $\#e(J) \geq 3$, so there are different nonidentity τ_1 , $\tau_2 \in \Sigma'$, such that $J \subset D(\tau_1) \cap D(\tau_2)$. Set $w_1 = w \cdot \sigma \cdot \tau_1$, $w_2 = w \cdot \sigma \cdot \tau_2$. Then $w \cdot \sigma \in \mathcal{F}_{2k}$ and $w_1, w_2 \in \mathcal{F}_{2k+1}$. In this way, different words in $\mathcal{F}_{2k-1}$ correspond to different words in $\mathcal{F}_{2k}$ and in $\mathcal{F}_{2k+1}$, therefore, $N_{2k}(\mathcal{F}) \geq N_{2k-1}(\mathcal{F})$ and $N_{2k+1}(\mathcal{F}) \geq 2N_{2k-1}(\mathcal{F}) \geq 2^{K+1}$.

This formula implies that $N_n(\mathcal{F})$ grows exponentially with n. ◇

Finally, apply Proposition 1.1 of [7], we get two different words $w, w' \in \mathcal{F}$ such that $u \in D(w) \cap D(w')$ and $(u)\sigma_w = (u)\sigma_{w'}$. Suppose $\sigma_w = \sigma_g$, $\sigma_{w'} = \sigma_h$ for some $g, h \in G$, then we have $u \cdot g = u \cdot h$, or equivalently, $u \cdot gh^{-1} = u$. Since $w \neq w'$, $gh^{-1} \neq 1$. Therefore the action $T \times G \to T$ is not free.

This completes the study of Example 10.1. ◇

Example 10.4: If $|T_0| = \infty$, then the action $T \times G \to T$ is not free.

Proof: Case 1: $\#Y(T_0) = \infty$.

$Y(T')$ and $Y(T'')$ are divided into finitely many equivalence classes mod G' and G'' respectively. Since $Y(T_0) \subset Y(T')$, there is an infinite subset S of $Y(T_0)$ such that the points of S are equivalent to each other mod G'. Because $S \subset Y(T_0) \subset Y(T'')$, there are two points $u, v \in S$ which are equivalent to each other mod G''. There are $g \in G'$, $h \in G''$ such that $u = v \cdot g = v \cdot h$, then $v \cdot gh^{-1} = v$, since $gh^{-1} \neq 1$, the action $T \times G \to T$ is not free.

Case 2: $Y(T_0) < \infty$. There is an edge e of T_0 whose length is ∞. e contains a subset R which is homeomorphic to the half line $[0, \infty)$. Let us take R as $[0, \infty)$. There is a number $c > 0$ such that if J is a segment of ε' or ε'', $\#\{J' \in e(J) | J' \cap R \neq \emptyset\} < \infty$, then $J \cap [c, \infty) = \emptyset$. There is a number $d > c$ such that if $J \in \varepsilon' \cup \varepsilon''$ and $J \cap [c, d] \neq \emptyset$, then $\#\{J' \in e(J) | J' \subset [c, d]\} \geq 3$. Then for every point $u \in [c, d]$, there are at least two points in $[c, d]$ which are equivalent to u mod G' but not equal to u, and there are at least two points in $[c, d]$ which are equivalent to u mod G'' but not equal to it.

Suppose Υ' is the set of partial isometries on $[c, d]$ induced by elements of G', Υ'' is defined symmetrically. As in Lemma 2.1 (b) of Part 1, we can prove that $\#\Upsilon' < \infty$ and $\#\Upsilon'' < \infty$.

Fix a point $u \in [c, d]$, define F to be the set of all forward orbits starting from u of the form $\{u, u_1, u_2, \ldots\}$, such that $u_i \in [c, d]$ for each i and $u_{n+1} = (u_n)\sigma$ where

$$\sigma \in \begin{cases} \Upsilon', & \text{if } n \text{ is odd;} \\ \Upsilon'', & \text{if } n \text{ is even.} \end{cases}$$

Assume for each $n > 0$, F_n is the set of all the arrays $\{u, u_1, \ldots, u_n\}$ of the first $n+1$ entries of the forward orbits in F.

Assume that $\{u, u_1, \ldots, u_n\} \in F_n$, from the above arguments, there are two different points u_{n+1} and u'_{n+1} which are equivalent to u mod G' or G'' (according to n is even or odd), and which make $\{u, u_1, \ldots, u_n, u_{n+1}\}$, $\{u, u_1, \ldots, u_n, u'_{n+1}\}$ belong to F_{n+1}. Then it is easy to see that $\#F_{n+1} \geq 2\#F_n$. Therefore $\#F_n \geq 2^n$ for every $n > 0$. Applying Proposition 1.1 of [7], we can fined a fixed point as we did in Example 10.1, so the action $T \times G \to T$ is not free. ◊

ACKNOWLEDGMENTS

This paper is a part of my Ph.D. thesis written under the direction of Professor John W. Morgan. I am grateful to him for his invaluable suggestions and comments, his patience, as well as his constant encouragement.

COLUMBIA UNIVERSITY

REFERENCES

[1]. Alperin, R. and Bass, H. (1984). Length functions of group actions on Λ-trees. *Ann. Math.* **111**, 265-378.

[2]. Culler, M. and Morgan, J. W. (1987). Group actions on R-trees. *Proc. London Math. Soc. (3)* **55**, #3, 571-604.

[3]. Lee. L. (1990). Freeness and discereteness of actions on **R**-trees by Finitely Generated free groups, II. *Preprint*

[4]. Lee. L. (1990). Freeness and discereteness of actions on **R**-trees by Finitely Generated free groups, III. *Preprint*

[5]. Morgan, J. W. (1986). Group actions on R-trees and the compactification of the space of classes of $SO(n,1)$-representations. *Topology* **25**, 1-33.

[6]. Morgan, J. W. (1986). *Trees and Hyperbolic Manifolds.* CBMS Lecture Series, UCLA (preprint)

[7]. Morgan, J. W. (1988). Ergodic theorem and free actions of groups on R-trees. *Invent. Math.* **94**, #3, 605-622.

[8]. Morgan, J. W. and Shalen, P. B. (1984). Valuations, trees and degenerations of hyperbolic structures. I *Ann. Math. (2)* **120**, #3, 401-476.

[9]. Morgan, J. W. and Shalen, P. B. (1985). An introduction to compactifying spaces of hyperbolic structures by actions on trees. 228-240. Springer-Verlag Lecture Notes in Mathematics Vol. **1167**

[10]. Serre, J-P. (1980). *Trees.* Springer-Verlag, New York.