FREENESS AND DISCRETENESS OF ACTIONS ON R-TREES BY FINITELY GENERATED FREE GROUPS, II

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Abstract

Suppose $G = F(x, y, z)$ is the free group generated by x, y and z, $G = G' * G''$, where $G' = F(x, y)$ and $G'' = F(z)$ are subgroups of G. G acts on an $\mathbf{R}$-tree T freely and minimally, with T', T'' be the minimal invariant subtrees of G', G'' in T respectively and $T_0 = T' \cap T''$.

We prove that the action is discrete under the following conditions:

(a) If I is any nondegenerate subsegment of T_0, then $\#\{g \in G' | I \cdot g \subset T_0\} \leq 2$.

(b) $|T_0| \leq 2\tau(z)$, where $\tau(\ )$ is the translation length function for the action of G on T.

*(c) Condition **A** (see Part 1 page 9) is satisfied.*

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In part 1, we investigated the freeness and the discreteness of minimal actions on an $\mathbf{R}$-tree T by a finitely generated free group G. We decomposed G as a free product of smaller rank free groups G' and G'', i.e. $G = G' * G''$. Let T' and T'' be the minimal invariant subtrees for the groups G', G'' respectively. Set $T_0 = T' \cap T''$. We proved that if the action $T \times G \to T$ satisfies Condition **A** (see Part 1 page 9), then it is free (discrete resp.) if and only if the partial action on T_0 by the set of alternating combinations of elements of Σ' and Σ'' is free (has no infinite orbit resp.), where Σ' and Σ'' are the sets of partial isometries on T_0 defined by elements of G' and G'' respectively. (see Proposition 4.2 and 4.3 of Part 1). We showed (Part 1, Theorem 4.11) that if an action satisfies Condition **A** and **B** (see Part 1 page 14), then it satisfies the following

Property (DF): The action is discrete provided it is free.

We predict that, under Condition **A**, Property (DF) is satisfied by all actions on $\mathbf{R}$-trees of finitely generated free groups.

As applications of the theorems in Part 1, we here concentrate ourselves on the action of free group of rank 3, prove, in some certain cases, that Property (DF) is true for such actions when condition **A** is assumed, leaving the proof of this property in general cases as an open problem. The results in Part 2 are examples where the action is free and discrete or it is not free, so that Property (DF) is true.

Before giving the theorems, we need to refresh ourselves with notation and definitions introduced in Part 1 as follows:

1. Preliminary

Throughout of this paper, we assume Condition **A**, and keep notation G, T, T', $\tau(\ )$ etc. introduced in the abstract. We use $T \times G \to T$ for the action of G on T, and $u \cdot g$ for the image of the pair (u, g) under the action, where $u \in T$ and $g \in G$.

Without loss of the generality, as in Part 1, we make the following assumptions:

Assumption 1: The actions $T' \times G' \to T'$ and $T'' \times G'' \to T''$ are free and discrete.

Assumption 2: $T_0 \neq \emptyset$.

Assumption 3: $|T_0| \neq \infty$.

An **alternating word** (with respect to G' and G'') is an ordered family $\{a_1, a_2, \ldots a_n\}$ of elements of $G' \cup G'' - \{1\}$, such that $a_{2k} \in G'' - \{1\}$, $a_{2k+1} \in G' - \{1\}$ or $a_{2k} \in G' - \{1\}$, $a_{2k+1} \in G'' - \{1\}$ for all k. We allow the empty word to be an alternating word. For every element $g \in G$, there is a unique alternating word $\{a_1, a_2, \ldots, a_n\}$ such that g is the product of a_i's, i.e. $g = a_1 a_2 \cdots a_n$. ($g = 1$ if and only if the corresponding word is empty.) Call this word as the **alternating word of** g (in elements of G' and G''), call n as the (alternating) **word length** of g and denote it by $L(g)$. Set $g_b = a_1, g_e = a_n$ and

$$g_i = \begin{cases} 1, & \text{if } i = 0; \\ a_1 \cdots a_i, & \text{if } i \leq n \text{ and } i > 0\ ; \\ g, & \text{if } i > n. \end{cases}$$

If S is a subset of T, H is a subset of G, set

$$S \cdot H = \{s \cdot h | s \in S, h \in H\}$$

We use the letter d for the distance between points or sets as usual.

Assume $p: X \to Y$ is a map, S is a subset of X, we use $p|_S$ for the map p restricted on S, and use $(S)^\circ$ for the interior of S with respect to X. When S is the union of a family of **R**-trees or **R**-graphs, we denote by $Y(S)$ ($E(S)$ resp.) the set of branch points (end points resp.) of connected components of S.

Every element $g \in G$ induces an isometry from $T_0 \cdot g^{-1} \cap T_0$ to $T_0 \cap T_0 \cdot g$, we denote this partial isometry of T_0 by σ_g, denote its domain and range by D_g and R_g respectively, which are closed subtrees of T_0.

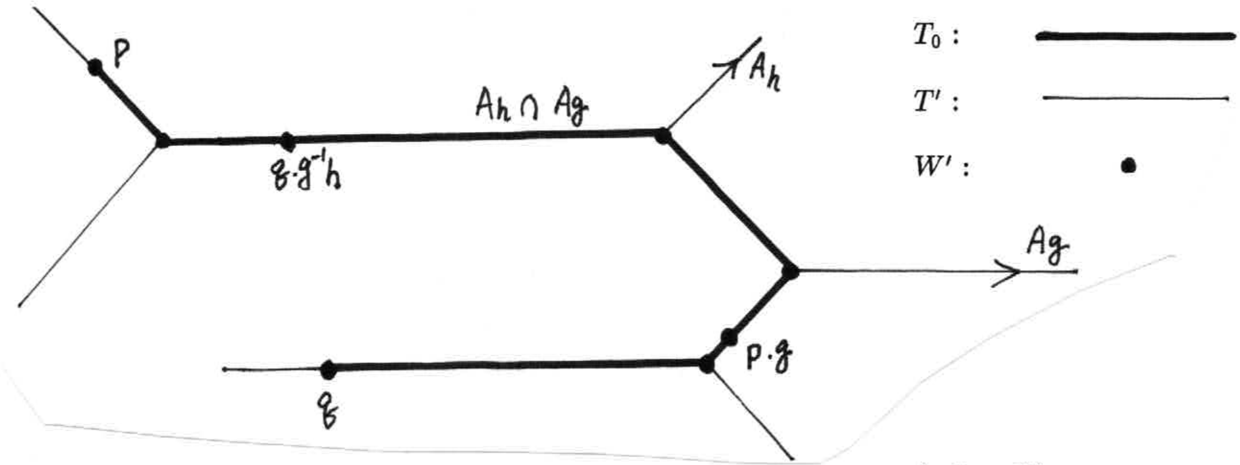
Let

$$\Sigma' = \{\sigma_g | g \in G', D_g \neq \emptyset\}$$

$$\Sigma'' = \{\sigma_g | g \in G'', D_g \neq \emptyset\}$$

$$\Sigma = \{\sigma_g | g \in G, D_g \neq \emptyset\}$$

Also let $W' = (Y(T') \cup E(T_0) \cdot G') \cap T_0$ and $W'' = (Y(T'') \cup E(T_0) \cdot G'') \cap T_0$.



$(g, h \in G')$

Fig. 1.

Σ acts from the right on T_0, the product of elements of Σ is the composition of them in the usual sense if this composition exists and is an elements of Σ. Notice that the identity map of T_0 is included in Σ.

Set

$$Y_0 = (Y(T') \cup Y(T'') \cup E(T_0) \cdot (G' \cup G'')) \cap T_0 = W' \cup W''$$

It is clear that Y_0 is a finite set.

$$S = \{(u, g) | u \in Y_0, g \in G, u \cdot g_i \in T_0 \ \forall i \geq 0\}$$

This is the set of pairs, a point u in Y_0 and an alternating word in elements of $G' - \{1\}$ and $G'' - \{1\}$ whose inductive images keep u belong to T_0.

By assumption, $G = F(x, y, z)$ and $G' = F(x, y)$, $G'' = F(z)$. Then if $\tau(z) \neq 0$, T' is the axis of z. Since $T_0 \subset T'$ and $|T_0| < \infty$, there are $p, q \in T'$ such that $T_0 = [p, q]$.

Example 1: If there is an element $g \in G'$ such that $|D_g| \geq \tau(z)$ then the action $T \times G \to T$ is not free.

Proof: We know that σ_g is a translation or a reflection restricted to D_g. If $|D_g| \geq \tau(z)$, then there is a point $u \in D_g$ such that $u \cdot z \in D_g$. We have $(u \cdot z)\sigma_g = (u)\sigma_g \cdot z \in R_g$, if σ_g is a translation, and $(u \cdot z)\sigma_g \cdot z = (u)\sigma_g \in R_g$, if σ_g is a reflection. So we have either $u \cdot zgz^{-1}g^{-1} = u$ or $u \cdot zgzg^{-1} = u$, i.e. u is a fixed point. ◊

Fig. 2.

According to the Example 8.1 of Part 1, if $|T_0| < \tau(z)$, then the action $T \times G \to T$ is free and discrete. **In view of this and Example 1, we can make the following assumptions without loss of the generality:**

Assumption 4: $|T_0| \geq \tau(z)$.

Assumption 5: For each $g \in G'$, we have $|D_g| < \tau(z)$.

From Assumption 5, D_g, R_g are embedded into G'' by ϕ'' for each $g \in G'$ then

$$\Sigma'' = \{\overline{\sigma}_g | g \in G', D_g \neq \emptyset\}$$

which is a finite set since $|T_0| < \infty$.

2. Theorems and proofs

Define

$$\varepsilon' = \{[a,b] | a \neq b, [a,b] \subset T_0, [a,b] \cap W' = \{a,b\}\}$$

$$\varepsilon'' = \{[a,b] | a \neq b, [a,b] \subset T_0, [a,b] \cap W'' = \{a,b\}\}$$

T_0 is covered by segments in ε' and by segments in ε'' respectively. Since W' is Σ' invariant, Σ' acts on ε'. Define an equivalence relation among segments in ε' as follows: If $I, J \in \varepsilon'$, $I \sim J$ if and only if I and J are in the same Σ'-orbit. Symmetrically, we define such an equivalence relation among segments in ε'', using the Σ''-orbits. If $J \in \varepsilon'$ or $J \in \varepsilon''$, we denote by $e(J)$ the equivalence class containing J.

Theorem 2: *The action $T \times G \to T$ is not free, if we assume the following conditions:*

(A) $\#e(J) \geq 2$ for each $J \in \varepsilon'$.

(B) $|T_0| \geq 2\tau(z)$

(C) $|T_0| > 2\tau(z)$ *or there is a segment* $J \in \varepsilon'$ *such that* $\#e(J) \geq 3$.

Proof: The proof of this theorem is an application of Theorem 7.3 and Proposition 1.1 of [6].

Define a map m' from $T_0 - W'$ to the set of positive integers as follows: For any $u \in T_0 - W'$, there is a unique segment $J \in \varepsilon'$ containing u, then $m'(u) = \#e(J) - 1$. Symmetrically, we can define a map m'' from $T_0 - W''$ to the set of positive integers.

The conditions of this theorem tell us the following information: $m'(u) \geq 1$ if $u \in T_0 - W'$, $m''(u) \geq 1$ if $u \in T_0 - W''$ and there is a nondegenerate subsegment of $T_0 - W' - W''$ on which either m' or m'' has value greater than 1.

Suppose the action $T \times G \to T$ is free. Let $\mathcal{F}$ be the set of alternating words $w = \sigma_1 \cdot \tau_1 \cdots \sigma_n \cdot \tau_n, n \geq 1$ with $\sigma_i \in \Sigma'' - \{id\}$ and $\tau_i \in \Sigma' - \{id\}$. Every $w \in \mathcal{F}$ corresponds to a partial isometry σ_w on T_0, if its domain is not empty, it belongs to Σ.

Applying Theorem 7.3 of [6], we have constants $a, b > 0$ and a point $x \in T_0$ such that for all n sufficiently large, the number of $w \in \mathcal{F}$ of alternating length $2n$ with σ_w defined at x is at least $a(\exp(bn))$. (Note that here T_0 is $\bar{A}$ in [6], although T_0 is not a circle, the proofs go through equally well, also m', m'' are m_1, m_0 respectively in that paper.) Then according to Proposition 1.1 of [6], there are distinct words $w, w' \in \mathcal{F}$ such that $(x)\sigma_w = (x)\sigma_{w'}$. Because the natural map from $\mathcal{F}$ to G is injective, there are different elements $g, g' \in G$ such that $x \cdot g = x \cdot g'$, so x is fixed by the nontrivial element $g(g')^{-1}$. Hence the action is not free. ◊

Set

$$\Lambda = \bigcup \{J \in \varepsilon' | \#e(J) \geq 2\}$$

$$\Lambda_0 = \{u \in \Lambda | ((u)\Sigma'' - \{u\}) \cap \Lambda \neq \emptyset\}$$

Λ_0 is the set of points in Λ whose Σ''-orbits intersect Λ at points other than themselves.

Let $W = (W')\Sigma'' = W' \cdot G'' \cap T_0$. It is easy to see that $\#W < \infty$ and $W'' \cup W' \subset W$.

Theorem 3: *Assume Condition* **A** *is satisfied, then the action* $T \times G \to T$ *is discrete if it is free when we assume the following conditions:*

(A) $\#e(J) \leq 2$ *for every* $J \in \varepsilon'$.

(B) For every $u \in \Lambda - W$, *there is at most one integer* $n \neq 0$ *such that* $(u)\sigma_{z^n} \in \Lambda - W$.

Note: This two conditions are equivalent to or implied by the conditions (a) and (b) in the abstract respectively.

Proof: Assume the action $T \times G \to T$ is free.

Set $\varepsilon = \{[a, b] | a \neq b, [a, b] \subset T_0, [a, b] \cap W = \{a, b\}\}$.

Assume $J = [a,b] \in \varepsilon$, $\tau \in \Sigma''$ and $D(\tau) \cap (J)^\circ \neq \emptyset$, then $J \subset D(\tau)$ since $E(D(\tau)) \subset W'' \subset W$. Because W is Σ'' invariant, $(J)\tau \cap W' \subset (J)\tau \cap W = \{(a)\tau, (b)\tau\} = E((J)\tau)$, so there is a segment $I \in \varepsilon'$, such that $(J)\tau \subset I$. It is easy to see that if $\Lambda_0 \cap (J)^\circ \neq \emptyset$, then $J \subset \Lambda_0$. Therefore, any nondegenerate component of Λ_0 is the union of a subset of segments of ε with disjoint interiors.

Because W is Σ''-invariant, Σ'' acts on ε. Λ_0 is the union of segments of ε whose Σ''-orbit has at least two segments, including its self, which are contained in Λ.

For each segment $J \in \varepsilon$ which is included in Λ_0, according to (B) and the above arguments, there is a unique $\tau_0 \in \Sigma'' - \{id\}$ such that $J \subset D(\tau_0)$ and $(J)\tau_0 \subset I$ for some segment $I \in \varepsilon'$ with $I \subset \Lambda$, by (A), there is a unique $\tau_1 \in \Sigma' - \{id\}$ such that $I \subset D(\tau_1)$. Define ρ_J to be the composition of τ_0 and τ_1.

Define a map $\rho: \Lambda_0 \to \Lambda$ as follows: $\rho|_J = \rho_J$ for every segment $J \in \varepsilon$ such that $J \subset \Lambda_0$. Define $\hat{W} = W \cap \Lambda_0$, then ρ is well defined (has a single value) on $\Lambda_0 - (\hat{W} - E(\Lambda_0))$. Each point of $\hat{W} - E(\Lambda_0)$ is an end point of two segments in ε, so ρ has two values on it. By deleting some points from $\hat{W}$ if necessary, we may assume that the two values of ρ on each point of $\hat{W} - E(\Lambda_0)$ are different from each other.

If $u \in \Lambda_0$, $v \in \Lambda$, by $v = (u)\rho$ we mean that v is one value of ρ on u. If S is a subset of T_0, define $\rho^{-1}(S)$ to be the set of points in Λ_0 which has at least one ρ-value in S.

Define $I_0 = T_0 - \Lambda_0 - \{p, q\}$, $I_{n+1} = \rho^{-1}(I_n) - \hat{W}$ for each $n \geq 0$. Then By induction on n we can see that I_n is open for every n. I_0 is the interior of $T_0 - \text{Domain}(\rho)$, I_n is the set of points on which ρ^n is well defined (has single value on each point) and whose ρ^n-images are outside the domain of ρ.

Lemma 4: *If $i, j \geq 0$ and $i \neq j$, then $I_i \cap I_j = \emptyset$.*

Proof: Suppose this is not true, assume i is the smallest integer such that $I_i \cap I_j \neq \emptyset$ for some integer $j > i$, then i must be 0, otherwise $I_{i-1} \cap I_{j_1} \supset \rho(I_i) \cap \rho(I_j) \supset \rho(I_i \cap I_j) \neq \emptyset$, this is impossible. But I_j is included in the domain of ρ, which is Λ_0, I_j dose not intersects I_i, this is a contradiction. $\diamondsuit$

Assume I is any subset of T_0, define $l(I)$ to be the minimum total measure of nondegenerate components of I if I has one, and take $l(I) = 0$ if I has no nondegenerate component.

Because $\rho(\hat{W})$ is a finite set, there is an positive integer n such that $\rho(\hat{W}) \cap I_i = \emptyset$ for every $i \geq n$.

Lemma 5: *Assume n is such an integer that $\bigcup_{i=n}^{\infty} I_i \cap \rho(\hat{W}) = \emptyset$, then for every $i \geq n$, we have $l(I_i) \geq l(i_n)$ or $l(I_i) = 0$.*

Proof: This can be proved by induction on i. Suppose J is a nondegenerate component of I_i, because $J \cap \rho(\hat{W}) = \emptyset$, it is easy to see that either $J \cap \rho(\Lambda_0) = \emptyset$, or $J \subset \rho(J')$ for some $J' \in \varepsilon$ such that $J' \subset \Lambda_0$, therefore either $|\rho^{-1}(J)| = 0$ or $|\rho^{-1}(J)| = |J|$. Compared with I_i, I_{i+1} has no nondegenerate components of smaller total measure, so we have $l(I_{i+1}) \geq l(I_i) \geq l(I_n)$ if $l(I_{I+1}) \neq \emptyset$. ◊

Lemma 6: *There is an integer $n > 0$ such that $I_i = \emptyset$ if $i \geq n$.*

Proof: By Lemma 5, if there is an integer n such that $I_n = \emptyset$, then $I_i = \emptyset$ if $i \geq n$ since I_i is open. Suppose this is not true, then from Lemma 5, we can see that there is a positive number λ such that $|I_i| \geq \lambda$. By Lemma 4, $I_i \cap I_j = \emptyset$ if $i \neq j$, so for any $n > 0$, we have $n\lambda \leq \Sigma_{i=1}^{n} |I_i| \leq |T_0| < \infty$, this is impossible. ◊

Set $K = T_0 - (\bigcup_{j=0}^{n} I_j)$, then K is the union of finitely many closed segments and $K \subset \Lambda_0 \cup \{p, q\}$. We can easily see that $\rho(K - \hat{W}) \subset K$. If u is a point of K, then either there is a nonnegative integer m such that ρ^m is defined at u and $(u)\rho^m \in \hat{W}$, or for every nonnegative integer m, ρ^m is defined at u.

Suppose $v \in T_0$, define: $n(v) = \#\{\sigma\tau | \sigma \in \Sigma'', \tau \in \Sigma', v \in D(\sigma\tau)\}$. We have $n(v) < \infty$ for every point $v \in T_0$.

A **forward orbit** (finite or infinite) is a sequence of points $\{u_0, u_1, \ldots\}$ of T_0 such that $u_i = (u_{i-1})\sigma_i\tau_i$ for some $\sigma_i \in \Sigma'' - \{id\}$ and $\tau_i \in \Sigma' - \{id\}$. If $u_i \notin W$, then either $n(u_i) = 0$ and the sequence terminates at u_i, or $n(u_i) = 1$ and $u_{i+1} = \rho(u_i)$. Since we assumed the action is free, points in a forward orbit are all distinct from each other. A forward orbit is called complete, if it is finite and its last point v satisfies $n(v) = 0$. Obviously any infinite forward orbit does not contain a complete suborbit.

Assume Φ is the set of infinite forward orbits $\{u_0, u_1, u_2, \ldots\}$ such that $u_i \notin W$ for every $i \geq 0$. If $\{u_0, u_1, \ldots\} \in \Phi$, then $u_i \notin I_n$ for every $i \geq 0$ and every $n \geq 0$, otherwise $u_{i+n} \in I_0 \subset T_0 - \Lambda_0$ and therefore $n(u_{i+n}) = 0$ so that $\{u_0, u_1, \ldots, u_{i+n}\}$ is complete, impossible. Therefore $\{u_0, u_1, \ldots\} \subset K$. This implies that if $\Phi \neq \emptyset$, then K is infinite.

Suppose K_0 is the union of all the nondegenerate components of K. Then $K_0 \subset \Lambda_0$ which is the domain of ρ.

Claim 7: $K_0 = \emptyset$.

This claim will be proved later. Assume Claim 7 is true, then K is a finite set and therefore $\Phi = \emptyset$. If the action $T \times G \to T$ is not discrete, then according to Proposition 4.3 of Part 1, there is a point $u \in Y_0$ such that $\#(u)\Sigma = \#F(u) = \infty$. Because $\#W < \infty$, there is an upper limit r of $\{n(v) | v \in T_0\}$. Clearly we can have at most $\#W \cdot r$ complete orbits starting from u. Then there must be an infinite forward orbit $\{u_0, u_1, \ldots\}$ such that $u_0 = u$. We have an integer $k > 0$ such that $u_i \notin W$ if $i \geq k$, so the subsequence $\{u_k, u_{k+1}, u_{k+2}, \ldots\}$ belongs to Φ, contradicting the fact that

$\Phi = \emptyset$. This proves the discreteness of the action $T \times G \to T$.

Proof of Claim 7: Assume this claim is not true, i.e. $K_0 \neq \emptyset$. Set

$$\varepsilon_{K_0} = \{J \cap K_0 | J \in \varepsilon, \text{ and } |J \cap K_0| \neq 0\}$$

Then K_0 is the union of the segments in ε_{K_0} with disjoint interiors .

Assume $J \in \varepsilon_{K_0}$, then J is a nondegenerate segment and there is a segment $J_1 \in \varepsilon$ such that $J = J_1 \cap K_0$. Write ρ_J for the restriction of ρ_{J_1} on J, then $\rho_J(J)$ is also a nondegenerate segment. Since $(\rho_J(J))^\circ = \rho_J((J)^\circ) \subset \rho(K_0 - \hat{W}) \subset K_0$ and K_0 is closed, we have $\rho_J(J) \subset K_0$. According to (A) and (B), $\rho|_{K_0 - \hat{W}}$ is injective, it is clear that $\rho|_{K_0}$ is an intervial exchange in the following sense that K_0 is the union of $\rho_J(J)$ for all $J \in \varepsilon_{K_0}$, with disjoint interiors.

Set $W_{K_0} = E(K_0) \cup (\hat{W} \cap K_0)$, this is the set of end points of segments in ε_{K_0}. If $u \in E(K_0)$, then there is a unique segment $J \in \varepsilon_{K_0}$ containing u. Assume $u \in W_{K_0} - E(K_0) \subset \hat{W} - E(\Lambda_0)$, u is an end point of two different segments in ε, since $u \notin E(K_0)$, the intersections of these two segments with K_0 are nondegenerate, therefore, u is an end point of two different segments in ε_{K_0}.

Choose a point $u_0 \in W_{K_0}$, and choose a segment $J_0 \in \varepsilon_{K_0}$ such that $u_0 \in J_0$, define $u_1 = \rho_{J_0}(u_0)$, then $u_1 \in E(\rho_{J_0}(J_0))$. If $u_1 \in E(K_0)$, there is a unique $J_1 \in \varepsilon_{K_0}$ such that $u_1 \in J_1$, define $u_2 = \rho_{J_1}(u_1)$; assume $u_1 \notin E(K_0)$, since $\rho|_{K_0}$ is an intervial exchange, there is a unique point $v_2 \in E(J_2)$ for some segment $J_2 \in \varepsilon_{K_0}$ such that $v_2 \neq u_0$ and $\rho_{J_2}(v_2) = u_1$, define $u_2 = v_2$. If $u_2 \in E(K_0)$, there is a unique point $v_3 \in E(J_3)$ for some $J_3 \in \varepsilon_{K_0}$, such that $u_2 = \rho_{J_3}(v_3)$, define $u_3 = v_3$; assume $u_2 \in W_{K_0} - E(K_0)$, then there is a segment $J_2' \in \varepsilon_{K_0}$ such that $J_2' \neq J_2$ and $u_2 \in E(J_2) \cap E(J_2')$, define $u_3 = \rho_{J_2'}(u_2)$.

Assume we have got $u_0, u_1, \ldots, u_n$, we choose the point u_{n+1} by the same rules as above. If $u_n \in E(K_0)$, then $u_{n+1} = \rho(u_n)$ if $u_n = \rho(u_{n_1})$ and $u_{n+1} = \rho^{-1}(u_n)$ if $u_n = \rho^{-1}(u_{n-1})$. If $u_n \in W_{K_0} - E(K_0)$, then choose $u_{n+1} \neq u_{n-1}$ and $u_{n+1} = \rho(u_n)$ if $u_n = \rho^{-1}(u_{n-1})$, $u_{n+1} = \rho^{-1}(u_n)$ if $u_n = \rho(u_{n-1})$.

This process continues for ever, so we get an infinite sequence $\{u_0, u_1, \ldots\}$. Any finite consecutive subsequence of the above sequence is called an **admissible sequence**.

Assume $s = \{u_0, u_1, \ldots u_m\}$ is an admissible sequence, m is called the **length** of s. For every $i \leq m-1$, $u_{i+1} = (u_i)\rho$ or $u_i = (u_{i+1})\rho$, so there are $g_i \in G'' - \{1\}$ and $h_i \in G' - \{1\}$ such that $u_{i+1} = (u_i)\sigma_{g_i}\sigma_{h_i} = u_i \cdot g_i h_i$ or $u_{i+1} = (u_i)\sigma_{h_i}\sigma_{g_i} = u_i \cdot h_i g_i$. Then $u_m = u_0 \cdot g_s$ where g_s is a product of g_i's and h_i's. g_s is determined by s.

We have the following notions for an admissible sequence $s = \{u_0, \ldots, u_m\}$:

s is **simple**, if $u_i \notin E(K_0)$ for all $i \geq 1, i \leq m-1$. s is **u-u**, if $u_1 = (u_0)\rho$ and $u_m = (u_{m-1})\rho$; is **u-d**, if $u_1 = (u_0)\rho$ and $u_{m-1} = (u_m)\rho$; is **d-u**, if $u_0 = (u_1)\rho$ and $u_m = (u_{m+1})\rho$, and is **d-d**, if

$u_0 = (u_1)\rho$ and $u_{m+1} = (u_m)\rho$. Suppose the length of s is 1, i.e. $s = \{u_0, u_1\}$, then if $u_1 = (u_0)\rho$, s is called a **u-step**; if $u_0 = (u_1)\rho$, s is a **d-step**.

Set $\Psi = \bigcup\{D(w,\Lambda)|w \in \Lambda\}$ and $\Psi_0 = \bigcup\{D(w,\Lambda_0)|w \in \Lambda_0\}$, then Ψ is the set of directions in Λ starting from points of Λ, and Ψ_0 is that for Λ_0.

Let

$$\hat{\Sigma}' = \{\sigma \in \Sigma' || D(\sigma)| > 0\}$$

$$\hat{\Sigma}'' = \{\sigma \in \Sigma'' || D(\sigma)| > 0\}$$

where $D(\sigma)$ is the domain of σ.

If $t \in D(u,\Lambda)$ for some point $u \in \Lambda$, according to (A), there is a unique $t' \in D(v,\Lambda)$ for some $v \in \Lambda$ such that $(t)\hat{\Sigma}' = \{t, t'\}$. Denote this t' by $C'(t)$. If $t \in D(u,\Lambda_0)$ for some point $u \in \Lambda_0$, according to (B), there is a unique direction $t'' \in D(v,\Lambda_0)$ for some $v \in \Lambda_0$ such that $(t)\hat{\Sigma}'' \cap \Psi_0 = \{t, t''\}$. Denote this t'' by $C''(t)$. We have $C'(C'(t)) = t$ for every $t \in \Psi$ and $C''(C''(t)) = t$ for every $t \in \Psi_0$.

Assume $s = \{u_0, u_1, \ldots, u_m\}$ is an admissible sequence, g_i, h_i for $0 \le i \le m-1$ are as before. There is uniquely a sequence of directions $\{t_0, t_1, \ldots, t_{m-1}\}$ such that for $i \le m-1$, we have $t_i \in D(u_i, K_0)$, and $\Delta(t_i) \in D(u_{i+1}, K_0)$, where Δ is ρ or ρ^{-1} such that $u_{i+1} = \Delta(u_i)$. This sequence $\{t_0, t_1, \ldots, t_{m-1}\}$ is called the **sequence of directions associated to** s.

Lemma 8: *Assume $s = \{u_0, \ldots, u_m\}$ is a simple admissible sequence, if there is an integer i such that $0 < i < m$, $u_{i+1} = (u_i)\rho$ and $g_{i-1}g_i \ne 1$ then $u_i \cdot g_{i-1}^{-1}, u_i \cdot g_i \in E(T_0)$.*

Proof: Suppose this is not true, we may assume that $u_i \cdot g_i \notin E(T_0)$.

There is a direction $t \in D(u_i, \Lambda_0)$ such that $C''(t) \in D(u_i \cdot g_{i-1}^{-1}, \Lambda_0)$. Because $u_i \cdot g_i \notin E(T_0)$, g_i carries t to a direction in $D(u_i \cdot g_i, \Lambda_0)$. This contradicts (B), impossible. $\diamond$

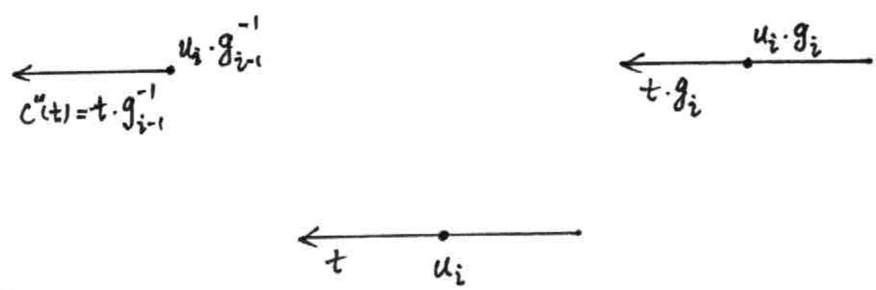


Fig. 3.

Suppose u is a point of Λ, let D_u be the set of directions $t \in D(v,\Lambda)$ such that there is a sequence $\{v_0, v_1, \ldots, v_l\}$ of points and a sequence $\{t_0, t_1, \ldots, t_{l-1}\}$ of directions satisfying the following conditions: $v_0 = u, v_l = v$, $t_i \in D(v_i, \Lambda)$, $D(v_{i+1}, \Lambda) = \{C'(t_i), t_{i+1}\}$ $(C'(t_i) \ne t_{i+1})$ for

$i = 0, 1, \ldots, l-1$, and $C'(t_{l-1}) = t$. We also include $D(u, \Lambda)$ in D_u. The set of starting points of all the directions in D_u is denoted by S_u. Obviously, in the above definition, $t_i \in D_u$, and $v_i \in S_u$ for each $i \leq l$, and $v_i \in (\Lambda)^\circ$ if $1 \leq i \leq l-1$.

Fig. 4.

Lemma 9: *(a) Assume every point of $Y(T')$ has order 3 in T', or equivalently, $Y(T')$ has two G'-orbits. If $u \in Y(T')$, then $\#(S_u) \leq 4$. If $\#(S_u) = 4$, then $p, q \in S_u$. As a consequence, if $\#(S_u) \cap (T_0)^\circ \geq 3$, then $\#(S_u) = 3$.*

(b) Assume every point of $Y(T')$ has order 4 in T', or equivalently, $Y(T')$ is a G'-orbit. If $u \in Y(T')$, then $\#(S_u) \leq 5$. If $\#(S_u) = 5$, then $p, q \in S_u$. As a consequence, if $\#(S_u) \cap (T_0)^\circ \geq 4$, then $\#(S_u) = 4$.

(c) If $u \in W' - Y(T')$, then either p or q belongs to S_u and $\#(S_u) \leq 3$. If $\#(S_u) = 3$, $p, q \in S_u$; if $\#(S_u) = 2$, then $S_u \subset E(\Lambda)$.

Proof: (a) Since u has order 3 in T', there are at most 3 $\hat{\Sigma}'$-classes of directions in D_u. By (A), each class contains not more than 2 directions, therefore, $\#D_u \leq 6$, this implies (a).

(b) Similar to (a), $\#D_u \leq 8$, so (b) is implied.

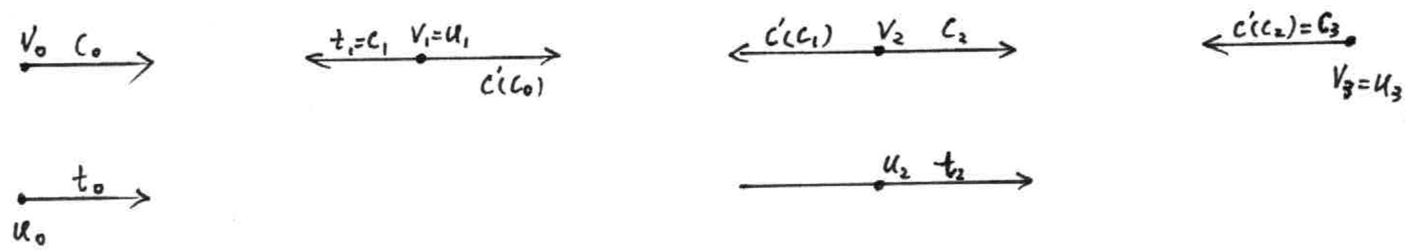
(c) Claim 1: either p or q belong to $(u)\hat{\Sigma}'$. To prove this claim, we may assume $u \notin \{p, q\}$. Suppose $u \in (p)\Sigma'$. Since u, and therefore p, is not a branch point of T', $u \in (p)\hat{\Sigma}'$, or equivalently $p \in (u)\hat{\Sigma}' \subset S_u$. Similarly, if $u \in (q)\Sigma'$, then $q \in (u)\hat{\Sigma}' \subset S_u$.

Take any $u \in W' - Y(T')$, let $v \in (u)\Sigma'$. Claim 2: $\{u, v\} \cap \{p, q\} \neq \emptyset$. Assume this claim is not true, then $u, v \in (T_0)^\circ$. Because u and v are not branch points of T', we have $(D(u, \Lambda))\Sigma' = D(v, \Lambda)$. This contradicts Claim 1, impossible.

By the claim 2, if $u \in W' - Y(T') - \{p, q\}$, then $S_u - \{p, q\} = \{u\}$. So $\#(S_u) \leq 3$. Actually, $\#D(u, \Lambda) = 2$ if and only if both p and q belong to $(u)\hat{\Sigma}' = S_u$, if and only if $\#(S_u) = 3$. This implies all the claims of (c) for S_u.

Suppose $p \notin Y(T')$, and $p \in \Lambda$, t is the unique direction in $D(p,\Lambda)$. Assume $C'(t) \in D(u,\Lambda)$ for some $u \in T_0$. If $u = q$, then $S_p = (p)\hat{\Sigma}' = \{p,q\}$. If $u \neq q$, then $S_p = S_u$, by the last paragraph, the claims of (c) for S_p, and symmetrically for S_q if $q \notin Y(T')$, are true. ◊.

A simple admissible sequence s is called **illegal** if $h_0 h_1 \cdots h_{m-1} = 1$, and any pair of consecutive g_i's in g_s are inverse of each other. For example, if $g_s = g_0 h_0 h_1 g_1 g_2 h_2 h_3 g_3 g_4 h_4$, then s is illegal if and only if $g_1 g_2 = g_3 g_4 = h_0 h_1 h_2 h_3 h_4 = 1$. If $s = \{u_0, u_1\}$ is a step, since $h_0 \neq 1$, s can not be illegal. For a sequence $s = \{u_0, u_1, \ldots, u_m\}$ satisfying that any pair of consecutive g_i's in g_s are inverse of each other, we define a sequence $\{v_0, v_1, \ldots, v_m\}$ of points, and a sequence $\{c_0, c_1, \ldots, c_m\}$ of directions as follows: For each i, if $u_i = (u_{i-1})\rho$, take $v_i = u_i$ and take $c_i = t_i$ if $i < m$, $c_m = C'(C''(t_{m-1})) = C'(c_{m-1})$; if $u_{i-1} = (u_i)\rho$, let $v_i = u_i \cdot g_{i-1}^{-1} = u_i \cdot g_i$ and let $c_i = t_i \cdot g_i$ if $i < m$, $c_m = t_{m-1} \cdot h_{m-1}$. We see that $v_i = v_0 \cdot h_0 h_1 \cdots h_{i-1}$ for all i. For each i, $c_i \in D(v_i, \Lambda_0)$, $C'(c_i) = c_i \cdot h_i \in D(v_{i+1}, \Lambda_0)$, and if $i \leq m-2$, $c_i \cdot h_i \neq c_{i+1}$. $\{v_0, \ldots, v_m\}$ and $\{c_0, \ldots, c_m\}$ are called the **lift** of $\{u_0, \ldots, u_m\}$ and $\{t_0, \ldots, t_{m-1}\}$ respectively. From now on, unless mention of contrary, $\{t_0, t_1, \ldots, t_{m-1}\}$, $\{v_0, \ldots, v_m\}$ and $\{c_0, \ldots, c_m\}$ are always as defined above.



$(m = 3)$

Fig. 5.

Lemma 10: *For any illegal admissible sequence $s = \{u_0, u_1, \ldots u_m\}$, we have $v_0 = v_m$ and $c_0 \neq c_m$. Therefore, $v_0 = v_m \in (\Lambda_0)^\circ$.*

Proof: Suppose $c_0 = c_m$, then $C'(c_0) = C'(c_m)$, i.e. $c_0 \cdot h_0 = c_m \cdot h_{m-1}^{-1} = c_{m-1}$. Then $c_1 = c_{m-2} \cdot h_{m-2} = C'(c_{m-2})$ and therefore, $c_1 \cdot h_1 = C'(c_1) = c_{m-2}$. Inductively, for any $k < m$, we have $c_k \cdot h_k = C'(c_k) = c_{m-k-1}$. If m is even, we have $c_{\frac{m}{2}-1} \cdot h_{\frac{m}{2}-1} = C'(c_{\frac{m}{2}-1}) = c_{\frac{m}{2}}$; if m is odd, we have $c_{\frac{m-1}{2}} \cdot h_{\frac{m-1}{2}} = c_{\frac{m-1}{2}}$, these are all impossible. ◊

Lemma 11: *Assume $s = \{u_0, u_1, \ldots, u_m\}$ is a u-u or u-d simple admissible sequence, suppose one of the following three conditions is true:*

(a) $\{u_0, u_m\} \cap E(K_0) \neq \emptyset$.

(b) s is legal.

(c) s is u-u.

then $(g_s)_b \in G'' - \{1\}$, *in particular,* $g_s \neq 1$. *If* $g_s \in G''$, *then* s *is illegal.*

Proof: Case 1: s is u-u and such that any pair of consecutive g_i's in g_s are inverse of each other, then $g_s = g_0 h_0 h_1 h_2 \cdots h_m$, so $(g_s)_b = g_0$ and $g_s \in G''$ if and only if $h_0 h_1 \cdots h_m = 1$, i.e. if and only if s is illegal.

Suppose Case 1 is not true, there is at lest one odd integer j such that $g_j g_{j+1} \neq 1$ or $j+1 = m$. Suppose $j_1, j_2, \ldots, j_k$ are all the odd integers with the above property (in their original order), set $j_0 = -1$. For each l, the u-d subsequence $s_l = \{(u_{(j_{l-1}+1)}, \ldots, u_{(j_l+1)}\}$ is such that any pair of consecutive g_i's in g_{s_l} are inverse of each other. We have $g_{s_l} = g_{(j_{l-1}+1)} h_{(j_{l-1}+1)} h_{(j_{l-1})} \cdot h_{j_l} g_{j_l}$ for each $l \leq k$ and $g_s = g_{s_1} \cdots g_{s_k} r$, where $r = 1$ if $j_k = m-1$ and $r = g_{(j_k+1)} h_{(j_k+1)} \cdots h_m$ if $j_k \neq m-1$ (Since j_k is the last integer with the property described above, the subsequence $\{u_{(j_k+1)}, \ldots, u_m\}$ is u-u).

Case 2, $k > 1$ or $k = 1$ but $j_1 + 1 \neq m$. Then for each l either the initial point or the terminal point of s_l, is not in $E(\Lambda_0)$. Denote the lifts of s_l and its associated sequence of directions by $\{v_{(j_{l-1}+1)}, \ldots, v_{(j_l+1)}\}$ and $\{c_{(j_{l-1}+1)}, \ldots, c_{j_l}, c_{j_l} \cdot h_{j_l}\}$ respectively. If s_l is illegal, by Lemma 10 we have $v_{(j_l+1)} = v_{(j_{l-1}+1)} \in (\Lambda_0)^{\circ}$. On the other hand, according to Lemma 8, at least one of this two points belong to $E(\Lambda_0)$, this is impossible. This proves that s_l is not illegal, i.e. $h_{(j_{l-1}+1)} \cdots h_{j_l} \neq 1$, then $(g_{s_l})_b = g_{(j_{l-1}+1)}$ and $(g_{s_l})_e = g_{j_l}$. It is clear that $(g_s)_b = (g_{s_1})_b = g_0$ and $g_s \notin G''$.

Assume, from now on, that $k = 1$ and $j_1 + 1 = m$, then s is u-d.

Case 3, s is legal. Then $h_0 h_1 \cdots h_{m-1} \neq 1$ and $g_s = g_0 h_0 h_1 \cdots h_{m-1} g_{m-1}$. We have $(g_s)_b = g_0$ and $g_s \in G''$.

Case 4, s is illegal and $\{u_0, u_m\} \cap E(K_0) \neq \emptyset$. We have $g_s = g_0 g_{m-1}$. If $g_0 g_{m-1} = 1$, then $u_m = u_0$. By the assumption this point belong to $E(K_0)$, we have $t_0 = t_{m-1} \cdot h_{m-1} g_{m-1}$ and therefore, $c_0 = t_0 \cdot g_0 = t_{m-1} \cdot h_{m-1} g_{m-1} g_0 = t_{m-1} \cdot h_{m-1} = c_m$, contradicting Lemma 10. So this case can not happen.

It is clear that Case (a), (b) and (c) are included in Case 1, 2 and 3. The Lemma is thus proved. $\diamondsuit$

Lemma 12: *Assume* $s = \{u_0, u_1, \ldots, u_m\}$ *is a d-d or u-d simple admissible sequence, suppose one of the following three conditions is true:*

(a) $\{u_0, u_m\} \cap E(K_0) \neq \emptyset$.

(b) s *is legal.*

(c) s *is d-d.*

then $(g_s)_e \in G'' - \{1\}$, *in particular,* $g_s \neq 1$. *If* $g_s \in G''$, *then* s *is illegal.*

Proof: The inverse sequence s^{-1} of s is a u-u or u-d simple admissible sequence. We have $g_{s^{-1}} = (g_s)^{-1}$. This Lemma follows from Lemma 11. ◇

Lemma 13: *There is no d-u illegal admissible sequence* $s = \{u_0, u_1, \ldots, u_m\}$ *such that* $u_0 = u_m \in E(K_0)$.

Proof: Suppose s is a d-u illegal sequence, we have $u_0 = u_m$. If this point belong to $E(K_0)$, then $c_0 = c_m$, contradicting Lemma 10. ◇

Lemma 14: *Assume* $s = \{u_0, \ldots, u_m\}$ *is a simple admissible sequence, if* $u_i \neq u_j$ *for* $0 \leq i < j \leq m-1$, *then* $u_m \neq u_n$ *for every* $0 < n < m$.

Proof: Suppose there is an integers n such that $0 < n < m$ and $u_m = u_n$. Look at the subsequence $s_0 = \{u_n, u_{n+1}, \ldots, u_m\}$. According to Lemma 11, 12 and 13, if s_0 is legal or if s is u-u or d-d, $g_{s_0} \neq 1$, but u_n is fixed by g_{s_0}, impossible. So s_0 is illegal and is either u-d or d-u. Let us take the case where s_0 is u-d for example.

Because $u_{m-1} = (u_m)\rho = (u_n)\rho$, $u_{m-1} = u_{n-1}$ or $u_{m-1} = u_{n+1}$. By Lemma 10, $t_n \cdot g_n \neq t_{m-1} \cdot h_{m-1}$, so $u_{m-1} \neq u_{n+1}$, we have $u_{m-1} = u_{n-1}$, contradicting the assumption. ◇

A **loop** is a simple admissible sequence of positive length with the initial point and the terminal point coincide.

Corollary 15: *Assume* $s = \{u_0, \ldots, u_m\}$ *is a simple admissible sequence,* $u_0 \in E(K_0)$. *If* s *contains a loop* l, *i.e. if a consecutive subsequence* l *of* s *is a loop, then* $l = s$.

Proof: Suppose j is the smallest positive integer such that $u_j = u_i$ for some integer $i < j$, (such j exists, because s contains a loop). By Lemma 14 we have $i = 0$, then $u_j = u_0 \in E(K_0)$, so $j = m$. This implies that $s = l$. ◇

Corollary 16: *There is an upper limit for the lengths of simple admissible sequences starting from points of* $E(K_0)$.

Proof: If $s = \{u_0, u_1, \ldots, u_m\}$ is a simple admissible sequence and $u_0 \in E(K_0)$, then either $\{u_{2k} | k \leq [\frac{m}{2}]\} \subset \hat{W}$, or $\{u_{2k+1} | k \leq [\frac{m-1}{2}]\} \subset \hat{W}$. According to Corollary 15, points in s are distinct from each other, except possibly $u_0 = u_m$, we have $[\frac{m}{2}] \leq \#\hat{W} + 1$, so $m \leq 2\#\hat{W} + 2 < \infty$. ◇

Assume $\{u_0, u_1, \ldots\}$ is a infinite admissible sequence with $u_0 \in E(K_0)$. According to Corollary 16, it can be divided into infinitely many simple subsequences $s_1, s_2, \ldots$, such that each s_i starts and ends at points of $E(K_0)$ and the terminal point of s_i equal to the initial point of s_{i+1} for $i \geq 1$. We can make the length of each s_i positive. Since $\#E(K_0) < \infty$, there are nonnegative integers $m < n$ such that the subsequence s which is the union of $s_m, s_{m+1}, \ldots, s_n$ in their original order is a loop. We may assume that $m = 1$ and the terminal points of s_i for $i = 1, \ldots, n$ are different from each other. For $i \leq n$, write s_i as $\{u_0^i, u_1^i, \ldots, u_{m_i}^i\}$ where m_i is the length of s_i. Denote the sequence of directions associated to s_i by $\{t_0^i, t_1^i, \ldots, t_{m_i-1}^i\}$. Also the notation g_j^i and h_j^i for $j = 0, 1, \ldots, m_i - 1$ are naturally used.

The remaining part of this chapter is devoted to prove that $g_s \neq 1$ for the loop s, which contradicts our assumption that the action $T \times G \to T$ is free and proves that $K_0 \neq \emptyset$ is impossible.

Lemma 17: *Assume s and s_i for $i = 1, 2, \ldots, n$ are as in the last paragraph. If $i \leq n$, and s' is the union of s_i's in the order of $s_i, s_{i+1}, \ldots, s_n, s_1, \ldots s_{i-1}$, then s' is also an admissible sequence, and $g_{s'}$ is conjugate to g_s.*

Proof: The lemma is implied by the following claim: $u_1^1 \neq u_{m_n-1}^n$. Since if this claim is true, then s_n is u-u or d-u, if s_1 is u-u or u-d, and s_n is u-d or d-d, if s_1 is u-d or d-d. So the union of s_n and s_1, and therefore s', is an admissible sequence.

Proof of the claim: Assume $u_1^1 = u_{m_n-1}^n$. Take s_n^{-1} to be the inverse sequence of s_n, then the initial two points of s_n^{-1} are those of s_1. By the rules for admissible sequences, we have $s_1 = s_n^{-1}$. This implies $u_{m_1}^1 = u_0^n = u_{m_{n-1}}^{n-1}$, contradicting our assumption that the terminal points of $s_1, s_2, \ldots, s_n$ are distinct from each other. $\diamond$

Lemma 18: *Assume $s = \{u_0, \ldots, u_m\}$ is a u-u illegal sequence, $\{t_0, t_1, \ldots, t_{m-1}\}$ is the sequence of directions associated to s, then*

(a) $m = 3$ and the lift $\{v_0 = v_3, v_1, v_2\}$ of s belong to $Y(T')$.

Assume further that $u_0, u_3 \in E(K_0)$, then we have:

(b) $v_0 = v_3 \in E(K_0)$, $v_1 \in (K_0)^\circ$.

(c) $t_2 \cdot g_2 = c_2$ is not in K_0 and as a consequence, $\{v_0, v_1, v_2,\} \cap (K_0)^\circ = \{v_1\}$.

(d) If $D(v, T_0) = \{t, t'\}$, with $v \in \{v_0, v_1, v_2\}$, then $C'(t), C'(t')$ are directions from the other two points of the lift $\{v_0, v_1, v_2\}$, one from each.

Proof: (a) From Lemma 10 we know the two directions in $D(v_0, T_0)$ are c_0 and c_m. It is clear that $S_{v_0} = \{v_0 = v_m, v_1, \ldots, v_{m-1}\}$, so $m = \#S_{v_0}$ and $S_{v_0} \subset (T_0)^\circ$. By Lemma 9, $\#S_{v_0} \leq 4$, and if $v_0 \notin Y(T')$, $\#S_{v_0} \leq 2$. But m is odd, and a u-step can not be illegal, we can only have that $m = 3$ and $v_0 \in Y(T')$. Therefore, $\{v_0, v_1, v_2, v_3\} \subset Y(T')$.

(b) $v_3 = u_3 \in E(K_0)$, $v_1 = u_1 \in (K_0)^\circ$.

(c) According to Lemma 10, $c_3 = c_2 \cdot h_2 \neq c_0 = t_0 \cdot g_0$. Because $u_0 \notin E(\Lambda_0)$, $c_3 \cdot g_0^{-1} \in D(u_0, \Lambda_0)$. But $u_0 \in E(K_0)$, so $c_3 \cdot g_0^{-1} \notin D(u_0, K_0)$.

It is clear that $(c_3 \cdot g_0^{-1})\rho = c_2$. Because the restriction of ρ on $\Lambda_0 - \hat{W}$ is 1-1, each direction can have exactly one preimage under ρ, since $(K_0)\rho = K_0$, c_2 can not be in K_0.

(d) This can be easily checked. ◊

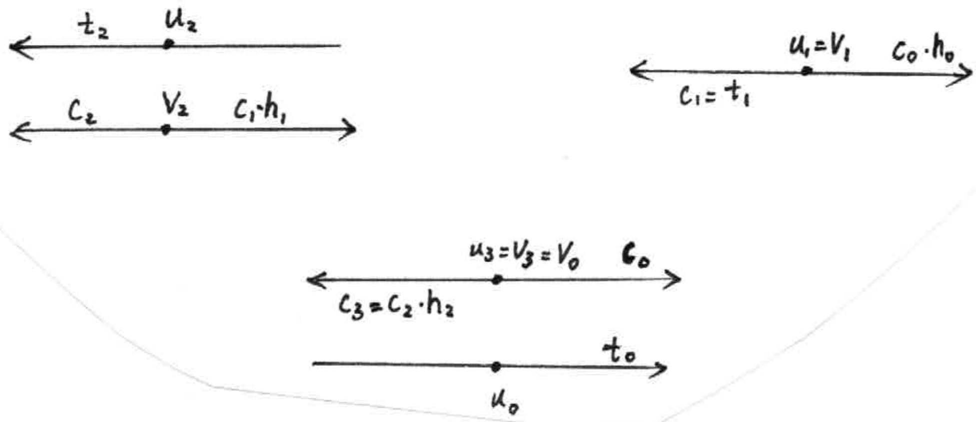


Fig. 6.

Lemma 19: *Assume $s = \{u_0, u_1, \dots, u_m\}$ is a u-u or d-u simple admissible sequence, suppose there is a positive integer k such that the subsequence $\{u_k, u_{k+1}, \dots, u_m\}$ is u-u illegal, then either $k = 0$ or $g_{k-1}g_k = 1$. As a consequence, if s is not illegal, $(g_s)_e \in G' - \{1\}$.*

Proof: Assume $k > 0$. If $g_{k-1}g_k \neq 1$, by Lemma 8, $u_k \cdot g_k \in E(T_0)$. But since $\{u_k, \dots, u_m\}$ is illegal, Lemma 10 tells us that $u_k \cdot g_k \in (\Lambda_0)^\circ$, impossible.

Assume s is not illegal. If any pair of consecutive g_i's in g_s are inverse of each other, then $(g_s)_e = h_0 h_1 \cdots h_{m-1} \in G' - \{1\}$. Assume the contrary, suppose j is the greatest integer such that $\{u_j, u_{j+1}\}$ is a u-step, and $g_{j-1}g_j \neq 1$. Then $\{u_j, \dots, u_m\}$ can not be illegal, so $h_j h_{j+1} \cdots h_{m-1} \neq 1$. As in the proof of Lemma 11, we have $g_{j-1}g_j$ can not be canceled in g_s from left, therefore, $(g_s)_e = h_j h_{j+1} \cdots h_{m-1}$. ◊.

Lemma 20 : *If s is a legal d-u or d-d simple admissible sequence, then $(g_s)_b \in G' - \{1\}$.*

Proof: As in Lemma 12, this lemma can be proved by taking the inverse sequence of s and by applying Lemma 19. ◊

Assume s' is the subsequence of s consisting of $s_i, s_{i+1}, \dots, s_j$ for some $1 \leq i \leq j \leq n$. Suppose s' satisfies the following properties: if $s' = s$, then $g_s = g_{s'} \neq 1$; when $s' \neq s$, we have $(g_{s'})_b \in G'' - \{1\}$ if $i > 1$ and s' is u-u or u-d; $(g_{s'})_b \in G' - \{1\}$ if $i > 1$ and s' is d-u or d-d; $(g_{s'})_s \in G'' - \{1\}$ if $j < n$ and s' is d-d or u-d and $(g_{s'})_s \in G' - \{1\}$ if $j < n$ and s' is d-u or u-u. Then s' is said to be **ideal** in s.

Lemma 21: *Assume s is the union of $s_1, s_2, \dots, s_n$ as before.*

(a) If s_i is not u-u or d-d illegal for some $i \leq n$, then s_i is ideal in s.

(b) If s can be divided into subsequences $t_1, t_2, \dots, t_r$, such that each t_j is the union of some s_i's which are consecutive in s, and all the t_j are ideal in s, for example, if no s_i is a u-u or d-d

illegal subsequence (we can take $t_i = s_i$ for $i = 1, 2, \ldots, n$), then $g_s \neq 1$.

Proof: (a) It follows from Lemma 11, 12, 13, 19 and 20.

(b) If all t_j are ideal in s, then when we write $g_s = g_{t_1} g_{t_2} \cdots g_{t_r}$, there is no cancellation of letters between subwords $g_{t_1}, g_{t_2}, \ldots, g_{t_r}$, therefore, $g_s \neq 1$. ◊

We assumed that the action $T \times G \to T$ is free, Lemma 21 tells us the situation described in (b) can not happen.

If there are exactly two simple subsequences s_i and s_j of s which are u-u illegal, and all the remaining simple subsequences are u-steps, then s is called a **u-u double illegal circle.** The definition of d-d double illegal circle is symmetric to the above with u replaced by d.

If s is a double illegal circle, s_i and s_j are as above, we may assume $i < j$. Since $g_s = 1$ (we don't have fixed points), the number of u-steps between s_i and s_j should be the same as the number of u-steps before s_i or after s_j. So s consists of $2n$ simple subsequences whose initial and terminal points belong to $E(K_0)$ and $j - i = n$ for some integer n. We may assume $i = n$ and $j = 2n$.

For simplicity, we use 0 for $2n$, so $s_{2n} = s_0$. Denote $u_0^k = v_0$, $g_0^k = g_k$ for $0 \leq k \leq 2n-1$, and $h_0^k = h_k$ for $0 < k < n$ and $n < k \leq 2n-1$. From now on, for a double illegal circle s we always keep the assumptions and notation given above.

It can be seen that $g_s = g_0 g_1 h_1 g_2 h_2 \cdots g_n g_{n+1} h_{n+1} \cdots g_{2n-1} h_{2n-1}$, since $g_s = 1$, we must have $g_0 g_1 = 1$, and $g_{n-k+1} g_{n+k} = 1$, $h_{n+k} h_{n-k} = 1$ for $k = 1, 2 \ldots n-1$. From this we get $v_1 \cdot g_1 = (u_0^0 \cdot g_0) \cdot g_0^{-1} = u_0^0 = v_0$, and similarly, $v_i \cdot g_i = v_{2n-i+1}$ for $i = 2, 3, \ldots, n$.

Set $\Delta' = \{u \in T_0 \cap Y(T') | u \in D_g,$ for some $g \in G'$ with $\sigma_g \in \hat{\Sigma}'\}$.

Lemma 22 : *Assume s is a double illegal circle, then*

(a) Δ' consists of the lifts L_{s_0}, L_{s_n} of s_0 and s_n and L_{s_0}, L_{s_n} are in the different G'-orbits.

(b) $W_{K_0} = \{v_i | 0 \leq i \leq 2n-1\} \cup \{u_2^0, u_2^n\}$.

Proof: (a) Obviously $L_{s_0}, L_{s_n} \subset \Delta'$. Because $v_1 \cdot g_1 h_1 \cdots g_{n-1} h_{n-1} g_n = u_1^n$, v_1 and u_1^n are not in the same G'-orbit, otherwise v_1 is fixed by a nontrivial elements of G impossible. since $v_1 = u_3^0 \in L_{s_0}$ and $u_1^n \in L_{s_n}$, this two lifts can not be in the same G'-orbit. (This implies that $Y(T')$ has two G'-orbits .)

Suppose u is any point in Δ'. Because each G'-orbit contains one of L_{s_0} or L_{s_n}, we may assume that $u \in O_0$, where O_0 is the G'-orbit containing L_{s_0}. Then as in the proof of Lemma 9 (a), we can prove that $u \in \Delta' \cap O_0 = L_{s_0}$.

(b) Suppose this is not true, take $u_0 \in W_{K_0} - \{v_i | 0 \leq i \leq 2n-1\} \cup \{u_2^0, u_2^n\}$, and take any ρ-image of u_0 as u_1, beginning from u_0 and u_1, we have an infinite admissible sequence which contains a subloop s'.

Suppose $s' \cap E(K_0) = \emptyset$. Assume s' is u-u or d-d, by Lemma 11 and 12, $g_{s'} \neq 1$, this is impossible. If s' is u-d or d-u, then $m \geq 4$, the lift of s' contains at least 4 points which are in $(T_0)^\circ$, this contradicts Lemma 9 (a). Therefore $s' \cap E(K_0) \neq \emptyset$.

If s_0 or s_n are contained in s', then $s = s'$, so $u_0 \in s \cap E(K_0) = \{v_i | 0 \leq i \leq 2n-1\} \cup \{u_2^0, u_2^n\}$, contradicting our assumption. By the proof of (a), there is no u-u illegal admissible sequence other than s_0 and s_n whose initial and terminal points belong to $E(K_0)$. Therefore, s' has no simple subsequence which is u-u or d-d illegal and whose initial and terminal points belong to $E(K_0)$, then by Lemma 21 (b), $g_{s'} \neq 1$, but $g_{s'}$ fixes a point of T_0, this is impossible. ◊

Lemma 23: *s can not be a double illegal circle.*

Proof: Let us first set up some notation, which will be convenience in the later discussion.

Suppose C is the set of components of K_0. Set $C_0 = C \cap \varepsilon_{K_0}$.

There are J_s, $J_{s'} \in \varepsilon_{K_0}$ such that $u_2^0 \in E(J_s)$, $u_1^0 \in E(\rho(J_s))$, $u_2^n \in E(J_{s'})$ and $u_1^n \in E(\rho(J_{s'}))$. Let $J_1^0, J_2^0, J_1^n, J_2^n$ be the segments in $\varepsilon_{K_0} \cup \rho(\varepsilon_{K_0})$ such that $u_i^j \in E(J_i^j)$ for each pair of i, j and $J_1^0 \neq \rho(J_s)$, $J_2^0 \neq J_s$, $J_1^n \neq \rho(J_{s'})$ and $J_2^n \neq J_{s'}$.

We have

$$\{u_1^0\} = E(\rho(J_s)) \cap E(J_1^0)$$

$$\{u_2^0\} = E(J_s) \cap E(J_2^0)$$

$$\{u_1^n\} = E(\rho(J_{s'})) \cap E(J_1^n)$$

$$\{u_2^n\} = E(J_{s'}) \cap E(J_2^n)$$

There are points $a_i^l \in T_0$ such that $J_i^l = [a_i^l, u_i^l]$ for $i = 1, 2$ and $l = 1, n$, and $b, b' \in T_0$ such that $J_s = [u_2^0, b], J_{s'} = [u_2^n, b']$.

For every point $u \in W_{K_0}$, there is a $m > 0$ such that $\rho^m(u) = u_1^0$ or $\rho^m(u) = u_1^n$, assume m is the smallest positive integer satisfying this.

Assume $t \in D(u, K_0)$ if $u \in v_i$ for some i, or $t \in D(u_2^j, J_2^j)$ for $j = 0, n$, define $\phi(t) = \rho^m(t)$, then $\phi(t) \in D(u_1^j, J_1^j)$ for $j = 0$ or n.

Denote the component of K_0 containing u_i^j by c_i^j for $i = 1, 2$ and $j = 0, n$. If u is an end point of a segment J, we denote by $t(u, J)$ the only direction in $D(u, J)$.

Claim 1: $J_2^0 \neq J_2^n$ and $J_1^0 \neq J_1^n$.

Proof of Claim 1: Suppose $J = J_2^0 = J_2^n$, since $u_1^0 \neq u_1^n$, we have $J = [u_2^0, u_2^n]$. If there is a positive integer $k \leq n$ such that u_2^0 or u_2^n belongs to $\rho^k(J)$, we may assume that k is the smallest integer satisfying this, then since $\rho^k(J) \in C$, we would have $\rho^k(J) = c_2^0 = c_2^n$, but this is impossible because $|\rho^k(J)| = |J| = |J_2^0| < |c_2^0|$. Therefore u_2^0 and u_2^n do not belong to $\rho^k(J)$ for $k \leq n$, so $\rho^k(J) \in C_0$ for $1 \leq k \leq n$. It can be easily checked that the union of end points of $\rho^k(J)$ for $0 \leq k \leq n$ is exactly the set W_{K_0}, which does not include the end points of c_2^0, this is impossible. Hence $J_2^0 \neq J_2^n$.

The proof of $J_1^0 \neq J_1^n$ is similar.

Claim 2: If $c \in C_0$, or $c = J_2^j$ for $j = 0$ or n with $a_2^j \in E(K_0)$, one of the following facts is true for $l = 0$ or n:

(a) There is an integer k such that $0 < k \leq n$ and $\rho^k(c) = c_2^l$.

(b) There are integers k_1, k_2 such that $0 < k_1 \leq n$, $k_2 \geq 0$ satisfying that $\rho^{k_1}(c) = J_1^{l_1}$ and $\rho^{k_2}(c_1^{l_1}) = c_2^l$ where $l_1 = 0$ or n.

(c) There are integers $0 < k_1 \leq n$ and $k_2, k_3 \geq 0$ such that $\rho^{k_1}(c) = J_1^{l_1}$, $\rho^{k_2}(c_1^{l_1}) = J_1^{l_2}$ and $\rho^{k_3}(c_1^{l_2}) = c_2^l$ where l_1, l_2 are 0 or n and $l_1 \neq l_2$.

Proof of Claim 2: Suppose k is the smallest positive integer such that $\rho^k(c) \cap c_i^l \neq \emptyset$ for some $i = 1$ or 2 and $l = 0$ or n, such k exists because positive powers of ρ carries the end points of c to u_1^0 or u_1^n. This power is less or equal to n, so $k \leq n$. We have $E(\rho^k(c)) \subset E(K_0) \cup \{u_1^0, u_1^n\}$ and $\rho^k(c)$ is isometric to c, therefore, connected, consequently, we have $\rho^k(c) \subset c_i^l$.

Assume $\rho^k(c) \subset c_2^l$ for $l = 0$ or n, if both ends of $\rho^k(c)$ are ends of K_0, we have $\rho^k(c) = c_2^l$, then (a) is true. Suppose one of the end points is $u_1^{l_1}$ for $l_1 = 0$ or n, then the case is included in the following one:

$\rho^k(c) \subset c_1^{l_1}$ for $l_1 = 0$ or n. Since $\rho^k(c) \in \rho(\varepsilon_{k_0})$ we may assume that $u_1^{l_1} \in E(\rho^k(c))$. Because c is not J_s or $J_{s'}$, we have $\rho^k(c) = J_1^{l_1}$. There is an $a \in E(c) \cap E(K_0)$ such that $\rho^k(a) = u_1^{l_1}$.

Consider $c_1^{l_1}$, as for c we can prove that there is a nonnegative integer $k_2 \leq n$ such that $\rho^{k_2}(c_1^{l_1}) \subset c_i^{l_2}$ for $i = 1$ or 2 and $l_2 = 0$ or n.

If $i = 1$, then $\rho^{k_2}(c_1^{l_1}) = J_1^{l_2}$. Since $|\rho^{k_2}(c_1^{l_1})| > |J_1^{l_1}|$, we have $l_2 \neq l_1$. Obviously $c_1^{l_2}$ can not be carried to J_1^1 or J_1^2 by a nonnegative power of ρ because it has longer length, so there is an nonnegative integer $k_3 \leq n$ such that $\rho^{k_3}(c_1^{l_2}) \subset c_2^l$ for $l = 0$ or n. Because $E(\rho^{k_3}(c_1^{l_2})) \subset E(K_0)$, we have $\rho^{k_3}(c_1^{l_2}) = c_2^l$. Take $k_1 = k$, then (c) is true.

Assume $i = 2$. We may assume that $c_2^{l_2} \neq c_1^l$ if $l = 0$ or n. Then $E(\rho^{k_2}(c_1^{l_1})) \subset E(K_0)$, so $\rho^{k_2}(c_1^{l_1}) = c_2^{l_2}$. Take $k_1 = k$ and $l = l_2$, then (b) is true. This proves Claim 2.

Remark: From the above proof, it can be seen that when $c = J_2^j$ for $j = 0$ or n, in case (b) and (c), we have $\rho^{k_1}(a_2^j) = u_1^{l_1}$.

Claim 3: Assume $l = 0$ or n, k is the smallest positive integer such that $\rho^k(J_2^l) \cap c_1^{l_1} \neq \emptyset$ for $l_1 = 0$ or n, then $u_2^0, u_2^n \notin (\rho^j(J_2^l))^\circ$ if $0 < j < k$.

Proof of Claim 3: Suppose this is not true, we may assume that $l = 0$ and there is a positive integer $j < k$ such that either u_2^0 or u_2^n belongs to $(\rho^j(J_2^0))^\circ$, and we assume j is the smallest one. Since $j < k$, $E(\rho^j(J_2^0)) \subset E(K_0)$, so $\rho^j(J_2^0)$ is the component of K_0 containing u_2^0 or u_2^n. Because $|\rho^j(J_2^0)| < |c_2^0|$, $\rho^j(J_2^0) \neq c_2^0$, we have $\rho^j(J_2^0) = c_2^n$.

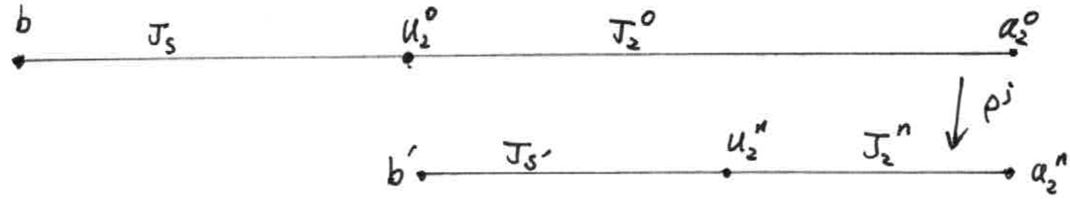


Fig. 7.

For J_2^n, (a), (b) or (c) of Claim 2 is true. Because $|J_2^n| < \min\{|c_2^0|, |c - 2^n|\}$, it can be easily seen that (a) of the Claim 2 is not true. Suppose (b) of that claim is true, $\rho^{k_1}(J_2^n) = J_1^{l_1}$ and $\rho^{k_2}(c_1^{l_1}) = c_2^l$ with $0 < k_1 \leq n, k_2 \geq 0$ and $l_1, l = 0$ or n. Because $c_2^n = \rho^j(J_2^0)$, we have $l = 0$. We have $|c_1^{l_1}| = |c_2^0| = |J_s| + |J_{s'}| + |J_2^n| = |\rho(J_s)| + |\rho(J_{s'})| + |J_1^{l_1}|$, this implies that u_1^0 and u_1^n are in the same component, i.e. $c_1^0 = c_1^n$. Because $J_2^n \neq J_s$, $J_2^n \neq J_{s'}$ and $J_s \neq J_{s'}$, we have $J_1^{l_1} \neq \rho(J_s)$, $J_1^{l_1} \neq \rho(J_{s'})$ and $\rho(J_s) \neq \rho(J_{s'})$, so c_1^0 consists of 3 parts: $J_1^{l_1}, \rho(J_s), \rho(J_{s'})$.

Assume J' is the union of $J_1^{l_1}$ and either $\rho(J_s)$ or $\rho(J_{s'}$, which share the end $u_1^{l_1}$ with $J_1^{l_1}$. If we use J' instead of $J_1^{l_2}$ and take $k_2 = 0$, then (c) of Claim 2 is true, so we may assume (c). Then $\rho^{k_1}(J_2^n) = J_1^{l_1}$, $\rho^{k_2}(c_1^{l_1}) = J_1^{l_2}$ and $\rho^{k_3}(c_1^{l_2}) = c_2^l$ with $0 < k_1 \leq n$, $k_2, k_2 \geq 0$, $l_1, l_2, l = 0$ or n and $l_1 \neq l_2$. As before we can prove that $l = 0$.

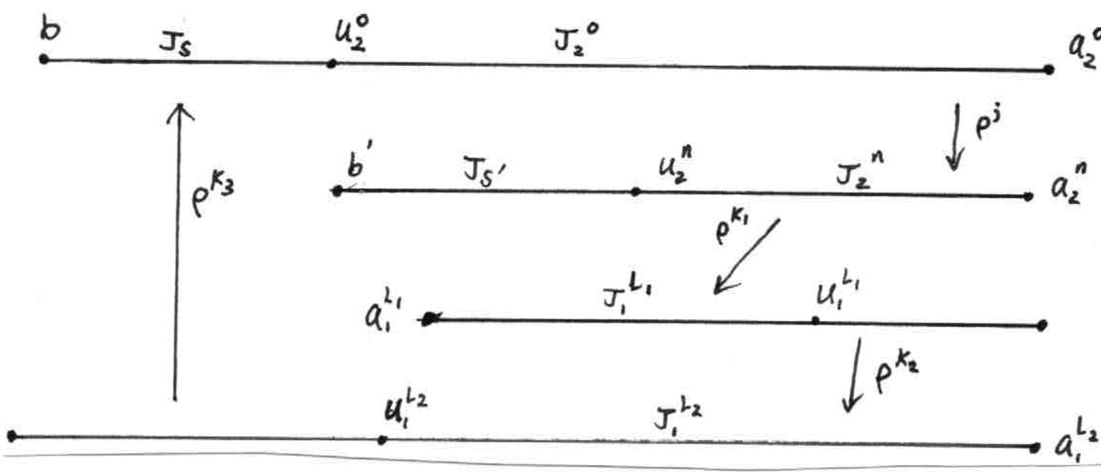


Fig. 8.

Case 1: $\rho^{k_2}(a_1^{l_1}) = a_1^{l_2}$ and $\rho^{k_3}(a_1^{l_2}) = b$.

No matter $l_1 = 0$ or n, and no matter $\rho^j(u_2^0) = a_2^n$ or $\rho^j(u_2^0) = b'$, we can always check that, $\phi(t(u_2^n, J_2^n)) = t(u_1^n, J_1^n)$, this is impossible.

Case 2: $\rho^{k_2}(a_1^{l_1}) = a_1^{l_2}$ and $\rho^{k_3}(a_1^{l_2}) = a_2^0$.

Assume $\rho^j(a_2^0) = a_2^n$, then it is easy to check that $\rho^{k_1+k_2+k_3+j}(J_2^n) = J_2^n$, so $\rho^{k_1+k_2+k_3+j}$ has a fixed point in J_2^n, this is impossible. Assume $\rho^j(u_2^0) = a_2^n$, then $\phi(t(u_2^0, J_2^0)) = \rho^{j+k_1}(t(u_2^0, J_2^0)) = t(u_1^{l_1}, J_1^{l_1})$, so $l_1 = n$, $l_2 = 0$. Then $\rho^{1+k_3}(u_2^n) = \rho^{k_3}(u_1^0) = u_2^0$, so u_2^0 is fixed by a nontrivial element of G, impossible.

Case 3: $\rho^{k_2}(a_1^{l_1}) = u_1^{l_2}$ and $\rho^{k_3}(a_1^{l_2}) = b$.

We can check that $\phi(t(u_2^n, J_2^n)) = \rho^{k_1+k_2}(t(u_2^n, J_2^n)) = t(u_1^{l_2}, J_1^{l_2})$, it follows that $l_2 = 0$ and so $l_1 = n$. Assume $\rho^j(a_2^0) = a_2^n$, we have $\rho^{j+1+k_2+k_3+1+k_3+j+k_1}(t(u_2^0, J_2^0)) = t(u_1^n, J_1^n)$, then $k_1 + k_2 = n + 1 = j + 1 + k_2 + k_3 + 1 + k_3 + j + k_1$, this is impossible. Assume $\rho^j(u_2^0) = a_2^n$, then $\rho^{1+k_3+j+1+k_2+k_3}(b) = b$, impossible.

Case 4: $\rho^{k_2}(a_1^{l_1}) = u_1^{l_2}$ and $\rho^{k_3}(a_1^{l_2}) = a_2^0$.

Like in Case 3 we can deduce that $l_2 = 0$ and if $\rho^j(a_2^0) = a_2^n$, then $k_1+k_2 = j+1+k_2+k_3+j+k_1$, if $\rho^j(u_2^0) = a_2^n$, we have $\rho^{1+k_3}(b) = b$. These are all impossible.

Up to now, we have discused all the possible cases. Claim 3 is thus proved.

By Claim 3, (a) of Claim 2 can not be true for J_2^0 and J_2^n, if (c) is true for one of these segments, (a) must be true for the other, this is impossible, so (b) of Claim 2 is true for both J_2^0 and J_2^n. We have for $l = 0$ or n, $\rho^j(J_2^l) \in C_0$ if $j < k_l$ and $\rho^{k_l}(J_2^l) = J_1^{e_l}$ for some $0 < k_l \le n$ and $e_l = 0$ or n and $\rho^{j_l}(c_1^{e_l}) = c_2^{r_l}$ for some $j_l \ge 0$ and $r_l = 0$ or n. We have $e_0 \ne e_n$. If $r_0 = r_n$, then one of c_1^0 or c_1^n is the image of the other under a nonnegative power of ρ, this power must be 0 because $c_1^e \notin \rho(\varepsilon_{K_0})$ for $e = 0$ or n, so $c_1^0 = c_1^n$. If $c_2^0 \ne c_2^n$, then we can deduce that $J_1^0, J_1^n, \rho(J_s), \rho(Js')$ are different from each other, since they are all contained in $c_1^0 = c_1^n$, and $c_1^0 = c_1^n$ contains only 3 different subsegments which belong to $\rho(\varepsilon_{K_0})$, this is impossible, therefore we have $c_2^0 = c_2^n$. It is easy to see that $c_1^0 = c_1^n$ if and only if $c_2^0 = c_2^n$.

First let us assume that $c_1^0 \ne c_1^n$ and $c_2^0 \ne c_2^n$.

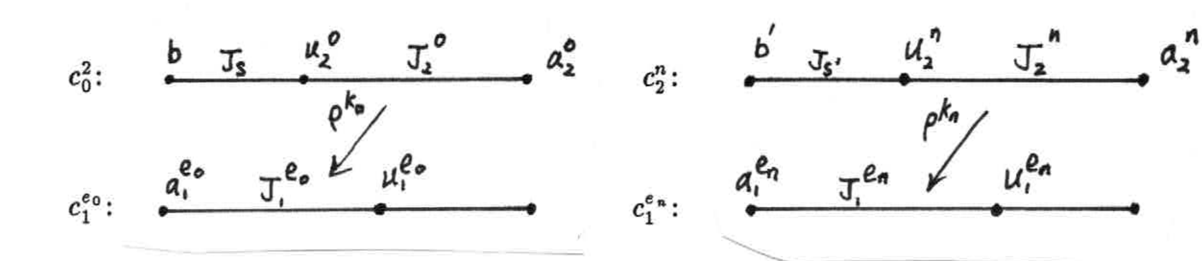


Fig. 9.

Case 1: $e_0 = r_0 = 0$ and $e_n = r_n = n$.

No matter $\rho^{j_0}(a_1^0)$ is a_2^0 or b, we can check that $\phi(t(u_2^0, J_2^0)) = t(u_1^0, J_1^0)$, this is impossible.

Case 2: $e_0 = r_n = 0$ and $e_n = r_0 = n$.

If $\rho^{j_0}(a_1^0) = b'$ and $\rho^{j_n}(a_1^n) = b$, then it can be checked that $\phi(t(u_2^0, J_2^0)) = t(u_1^0, J_1^0)$. If $\rho^{j_0}(a_1^0) = b'$ and $\rho^{j_n}(a_1^n) = a_2^0$, then $\phi(t(u_2^0, J_2^0)) = \rho^{k_0+j_0+1+j_n+1+j_0+k_n}(t(u_2^0, J_2^0)) = t(u_1^n, J_1^n)$ and $\phi(t(u_2^n, J_2^n)) = \rho^{k_n+j_n+k_0}(t(u_2^n, J_2^n)) = t(u_1^0, J_1^0)$, so $k_0 + j_0 + 1 + j_n + 1 + j_0 + k_n = k_n + j_n + k_0$, i.e. $2j_0 + 2 = 0$. If $\rho^{j_0}(a_1^0) = a_2^n$ and $\rho^{j_n}(a_1^n) = b$, we similarly have $2j_n + 2 = 0$. If $\rho^{j_0}(a_1^0) = a_2^n$ and $\rho^{j_n}(a_1^n) = a_2^0$, then $\rho^{1+j_0+1+j_n}(b) = b$. These are all impossible.

Case 3: $e_0 = r_n = n$ and $e_n = r_0 = 0$.

If $\rho^{j_0}(a_1^n) = a_2^0$, then $\rho^{k_0+j_0}(J_2^0) = J_2^0$, then there is a point of J_2^0 fixed by a nontrivial element of G, impossible. Similarly, $\rho^{j_n}(a_1^0)$ can not be a_2^n. So $\rho^{j_0}(a_1^n) = b$ and $\rho^{j_n}(a_1^0) = b'$, then $\phi(t(u_2^0, J_2^0)) = t(u_1^0, J_1^0)$, this is not true.

Case 4: $e_0 = r_0 = n$ and $e_n = r_n = 0$.

If $\rho^{j_0}(a_1^n) = a_2^n$ or $\rho^{j_n}(a_1^0) = a_2^0$, then b or b' will be fixed by a positive power of ρ, this is impossible. So $\rho^{j_0}(a_1^n) = b'$ and $\rho^{j_n}(a_1^0) = b$. Then $\phi(t(u_2^0, J_2^0)) = t(u_1^0, J_1^0)$, impossible.

Next, we assume that $c_1^0 = c_1^n$ and $c_2^0 = c_2^n$.

$c_2^0 = c_2^n$ is divided into 3 subsegments by u_2^0 and u_2^n which are belong to ε_{K_0}, since J_2^0, J_2^n, J_s and $J_{s'}$ are all contained in c_2^0, two of them must be the same. By Claim 1, $J_2^0 \neq J_2^n$, so we have $J_2^0 = J_{s'}$, or $J_2^n = J_s$, or $J_s = J_{s'}$. We have $j_0 = j_n$ and denote it by j.

Case 5: $J_s = J_2^n$.

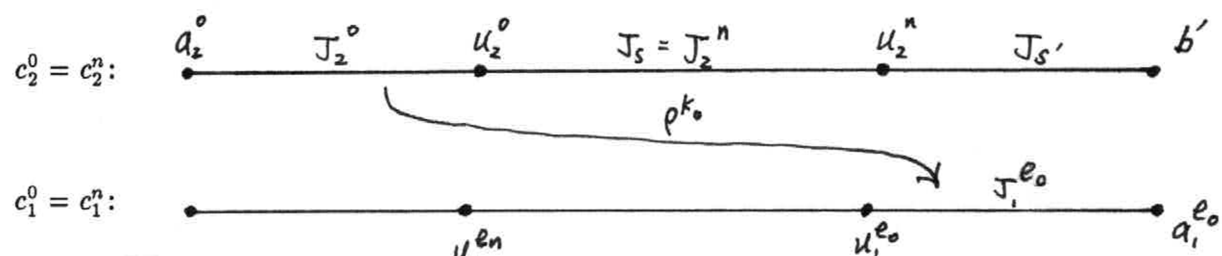


Fig. 10.

Since $J_s \neq J_{s'}$, $\rho(J_s) \neq \rho(J_{s'})$. Because $k_0 \leq n$, $a_1^{e_0} = \rho^{k_0}(u_2^0) \in E(K_0)$. If $\rho^j(a_1^{e_0}) = a_2^0$, then $\rho^{k_0+j}(J_2^0) = J_2^0$, which is impossible, so we have $\rho^j(a_1^{e_0}) = b'$.

If $e_0 = 0$, we must have $J_1^n = \rho(J_s)$, then $\phi(t(u_2^0, J_2^0)) = t(u_1^0, J_1^0)$, if $e_0 = n$, we have $\rho(J_{s'}) = J_1^0$, then $\phi(t(u_2^n, J_2^n)) = t(u_1^n, J_1^n)$, these are impossible.

Case 6: $J_{s'} = J_2^0$. Similar to Case 5, this is impossible.

Case 7: $J_s = J_{s'}$. Then $\rho(J_s) = \rho(J_{s'})$.

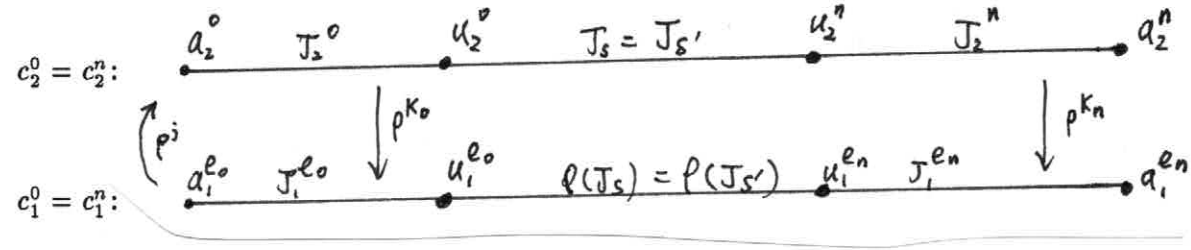


Fig. 11.

Case 7 (a): $e_0 = n, e_n = 0$. Then $\rho^j(a_1^0) = a_2^0$, otherwise $\rho^{k_0+j}(J_2^0) = J_2^0$, impossible. We have $\phi(t(u_2^0, J_2^0)) = t(u_1^0, J_1^0)$, this can not be true.

Case 7 (b) $e_0 = 0, e_n = n$. First like in Case 7 (a), we deduce that $\rho^j(a_1^0) = a_2^n$.

Define a map α from c_2^0 to its self in the following way: $\alpha|_{J_2^0} = \rho^{k_0+j}$, $\alpha|_{J_2^n} = \rho^{k_n+j}$ and $\alpha|_{J_s} = \rho^{1+j}$. α has two values on both u_2^0 and u_2^n, it translates J_2^0 and J_2^n and reflects $J_s = J_{s'}$.

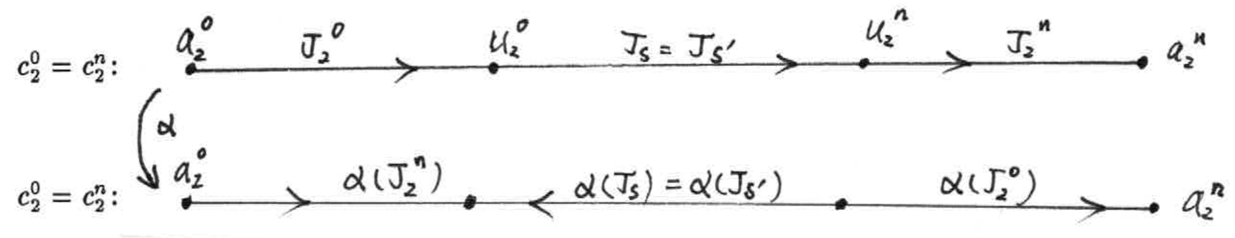


Fig. 12.

Consider the sets $\{\alpha^m(J_s)|m > 0\}$. If there are integers $m_1 < m_2$ such that $\alpha^{m_1}(J_s) = \alpha^{m_2}(J_s)$, then $\alpha^{m_2-m_1}$ fixes a point of $\alpha^{m_1}(J_s)$, so we assume this is not true. We claim that there is an integer m such that $\alpha^m(J_s) \cap J_s \neq \emptyset$. Suppose this is not true, assume $m_1 < m_2$ be such that $\alpha^{m_1}(J_s) \cap \alpha^{m_2}(J_s) \neq \emptyset$, then there is an $a \in E(\alpha^{m_1}(J_s))$ such that $[\alpha^{m_2-m_1}(a), a) = \alpha^{m_2}(J_s) - \alpha^{m_1}(J_s)$.

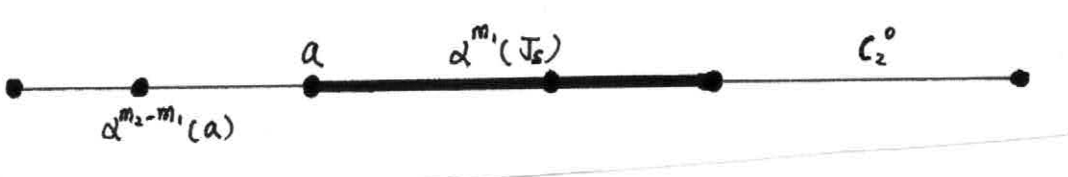


Fig. 13.

$\alpha^{m_2-m_1}$ translates $\alpha^{m_1}(J_s)$ towards a by $\lambda = \text{dis}(\alpha^{m_2-m_1}(a), a)$. Because $\alpha^m(J_s) \cap J_s = \emptyset$ for all m, $\alpha^{m_2-m_1}$ translates $\alpha^{m_1+j}(J_s)$ towards $\alpha^j(a)$ by λ for all $j \geq 0$ (this can be proved by induction on j). Then it is easy to see that $\cup\{\alpha^{k(m_2-m_1)+m_1}(J_s)|k \geq 0\}$ is connected and has an infinite total measure, this is impossible, so the claim is proved.

Assume m is the smallest positive integer such that $\alpha^m(J_s) \cap J_s \neq \emptyset$, then α^m acts on J_s by an inversion, i.e. $\alpha^m(J_s)$ and J_s have different orientations. We deduce that α^m fixes a point of J_s, since α^m is a positive power of ρ, this is impossible.

Up to now we have covered all the possible cases, and thus proved that a double illegal circle does not exists. ◇

Lemma 24: *Assume s is as before, we always have $g_s \neq 1$.*

Proof: Assume the loop s is the union of simple subsequences $s_1, s_2, \ldots, s_n$ in the previous sense.

If s contains no u-u or d-d illegal subsequence, by Lemma 21 (b), $g_s \neq 1$. As in the proof of Lemma 22 (a), we know that among $s_1, s_2, \ldots, s_n$, there are at most two u-u illegal subsequences.

Assume s contains a u-u illegal subsequence s_i. Suppose j, l are integers satisfying that

(a) $j < i < l$.

(b) s_k is u-u illegal or is a u-step if $j < k < l$.

(c) j is the smallest integer, l is the greatest integer satisfying (a) and (b).

For $j < k < l$, if s_k is a u-step, $g_{s_k} = g_k h_k$ with $g_k \in G'' - \{1\}$ and $h_k \in G' - \{1\}$; if s_k is u-u illegal, $g_{s_k} = g_k$ with $g_k \in G'' - \{1\}$. i.e. $g_k = g_0^k$ for $j < k < l$, and $h_k = h_0^k$ for $j < k < i$ and $i < k < l$.

Let s' be the subsequence of s which is the union of $s_j, \ldots, s_l$, if s_j or s_l does not exist, (i.e. if $j = 0$ or $l = n + 1$), it is the union of the remaining simple subsequences. $g_{s'} = g_{s_j} g_{s_2} \cdots g_{s_l}$. When we write each g_{s_k} as an alternating word in elements of G' and G'', $g_{s'}$ is a word in letters of elements of G' and G''. We want to prove that s' is ideal in s.

Case 1, $j = 0$ and $l = n + 1$. Then $s' = s$. We have $g_s = g_1 h_1 g_2 h - 2 \cdots g_i g_{i+1} h_{i+1} \cdots g_n h_n$. It is clear that either h_n or g_1 can not be canceled in g_s, so $g_s \neq 1$.

Case 2, $j = 0$ and $l \leq n$.

If g_{s_l} is not fully canceled in $g_{s'}$, i.e. if not all the letters in the alternating word g_{s_l} are canceled in $g_{s'}$, then $(g_{s'})_e = (g_{s_l})_e$. Since s begins with s', we have s' is ideal in s.

Suppose g_{s_l} is fully canceled in $g_{s'}$. Because s_l is u-u or u-d whose initial and terminal points are in $E(K_0)$, by Lemma 11, we have $(g_{s_l})_b \in G'' - \{1\}$.

Case 2 (a), s_i is the only u-u illegal subsequence of s'. From our assumptions, all the letters between g_k and $(g_{s_l})_b$ are canceled for some integer $k \leq i$, i.e. $h_k g_{k+1} h_{k+1} \cdots g_{l-1} h_{l-1} = 1$, then $g_{s'} = g_{s_1} \cdots g_{s_{k-1}} g_k g_{s_l}$. We have $u_0^k \cdot g_k = u_0^k \cdot g_k h_k \cdots g_{l-1} h_{l-1} = u_0^l$. By Lemma 10, $t_0^i \cdot g_i \neq t_0^{i+1}$, consequently $t_0^k \cdot g_k = (t_0^i \cdot g_i) \cdot (h_k g_{k+1} h_{k+1} \cdots g_{i-1} h_{i-1} g_i)^{-1} = (t_0^i \cdot g_i) \cdot g_{i+1} h_{i+1} \cdots g_{l-1} h_{l-1} \neq t_0^{i+1} \cdot g_{i+1} h_{i+1} \cdots g_{l-1} h_{l-1} = t_0^l$. So $u_0^k \cdot g_k = u_0^l \in (\Lambda_0)^\circ$.

Fig. 14.

We assumed that s_l is not u-u illegal. Suppose s_l is u-d illegal, then $u_0^l \cdot g_0^l = u_{m_l-1}^l \cdot h_{m_l-1}^l$ and $t_0^l \cdot g_0^l \neq t_{m_l-1}^l \cdot h_{m_l-1}^l$. According to Lemma 11, $g_0^l g_{m_l-1}^l = g_{s_l} \neq 1$, so $u_0^l \neq u_{m_l}^l$. Then similar to the proof of Lemma 8, we have $u_0^l \in E(\Lambda_0)$, this contradicts the result of the last paragraph. Therefore, we conclude that s_l is legal. So $(g_{s_l})_b = g_0^l$, and $g_{s_l} \neq g_0^l$. In fact g_s can be written as a alternating word in elements of $G' - \{1\}$ and $G'' - \{1\}$, we denote the first two letters of this word by a_1 and a_2 so $g_s = a_1 a_2 r$ with $a_1 = g_0^l, a_2 = h_0^l h - 1^l \cdots h_c^l$ for some positive integer c and r is such that $r_b \in G''$ of $r = 1$. Since we assumed that g_{s_l} can be fully canceled, we have $g_k g_0^l = 1$, $k > 1$ and $h_{k-1} a_2 = h_{k_1} h_0^l h_1^l \cdots h_c = 1$.

As before, we can prove $u_0^k = u_0^l \cdot g_0^l \in (\Lambda_0)^\circ$. Because s_l is not a u-step, $u_1^l \in (K_0)^\circ \subset (\Lambda_0)^\circ$, then $\#S_{u_0^l \cdot g_0^l} \cap (T_0)^\circ \geq 2$, by Lemma 9, $S_{u_0^l \cdot g_0^l} \subset Y(T')$. On the other hand, according to Lemma 18 (a), the lift L_{s_i} of s_i is also contained in $Y(T')$. As in the proof of Lemma 22 (a), we deduce that $S_{u_0^l \cdot g_0^l}$ and L_{s_i} are in different G' -orbits. Then by Lemma 9, $\#S_{u_0^l \cdot g_0^l} \leq 4$, and $\#S_{u_0^l \cdot g_0^l} \cap (T_0)^\circ \leq 3$.

Assume c' is the smallest nonnegative integer such that $h_{k-1} h_0^l \cdots h_{c'}^l = 1$. Starting from the point $u_0^{k-1} \cdot g_{k-1} = u_0^l g_0^l h_0^l \cdots h_{c'}^l$, there are two different directions: $t_0^{k-1} \cdot g_{k-1} = C'(t_0^{k-1} \cdot g_{k-1} h_{k-1}) = C'(t_0^k)$ and $t \cdot h_{c'}^l = C'(t)$ where $t = t_{c'}^l$ if c' is odd, and $t = t_{c'}^l \cdot g_{c'}^l$ if c' is even. Therefore, $\{u_0^l \cdot g_0^l, u_0^l \cdot g_0^l h_0^l, \ldots, u_0^l \cdot g_0^l h_0^l \cdots h_{c'}^l\} \subset S_{u_0^l \cdot g_0^l} \cap (\Lambda_0)^\circ$, this set can not have cardinality more than 3. Then we can only have $c' = 1$. If $c > c' = 1$, we have $h_2^l = h_{k-1}$ and $u_3^l = u_0^l \cdot g_0^l = u_0^k \in E(K_0)$. Then $s_l = \{u_0^l, u_1^l, u_2^l, u_3^l\}$ is u-u illegal, this contradicts our assumption. Therefore, $c = c' = 1$.

Assume the length m_l of s_l is more than 2, then $g_1^l g_2^l \neq 1$, by Lemma 8, $u_0^l \cdot g_0^l h_0^l h_1^l = u_2^l \cdot (g_1^l)^{-1} \in E(\Lambda_0)$, contradicting the last paragraph. Therefore $m_l = 2$.

By the assumption g_1^l is canceled with g_{k-1} in $g_{s'}$, so $u_0^{k-1} = u_2^l$, therefore $k - 1 = 1$ and $s' = s$.

But since $t_0^1 \cdot g_1 = t_0^{k-1} \cdot g_{k-1} \neq t_1^l \cdot h_1^l$, we have $t_0^1 \neq t_1^l \cdot h_1^l g_1^l$, therefore, $u_0^1 = u_2^l \notin E(K_0)$, impossible.

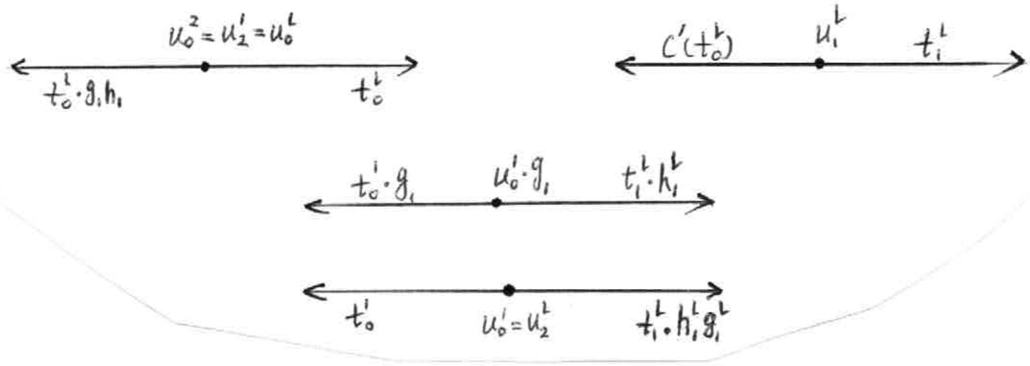


Fig. 15.

Now assume that there is an $f \neq i$, $f < l$ such that s_f is u-u illegal. We may assume that $i < f$.

If $f = i + 1$, then $g_i g_f \neq 1$, otherwise the union of s_i, s_f is a double illegal circle, contradicting Lemma 23. Then by a proof similar to that of Lemma 8, we have $u_0^f \cdot g_f \in E(\Lambda_0)$. But according to Lemma 10, $u_0^f \cdot g_f = u_3^f \in (\Lambda_0)^\circ$, this is a contradiction. So $f \neq i + 1$.

Case 2 (b), $g_{s_{f-1}}$ is fully canceled in $g_{s_1} g_{s_2} \cdots g_{s_{f-1}}$. Then there is a $k < i$ such that all the letters in $g_{s'}$ between g_k and $g_f = g_{s_f}$ are canceled, then $g_{s'} = g_1 h_1 \cdots g_k g_f g_{f+1} h_{f+1} \ldots g_{s_l}$. As in Case 2 (a), we have $u_0^k \cdot g_k = u_0^f$ and $t_0^k \cdot g_k \neq t_0^f$. Because $t_0^f \cdot g_f \neq t_2^f \cdot h_2^f$, it can be seen that g_f must carry $t_0^k \cdot g_k$ to g_k, therefore $t_2^f \cdot g_2^f = c''(t_0^k \cdot g_0^k) = t_0^k$, so $u_0^k = u_3^f$. Then the union of $s_k, \ldots, s_f$ is a double illegal circle, impossible.

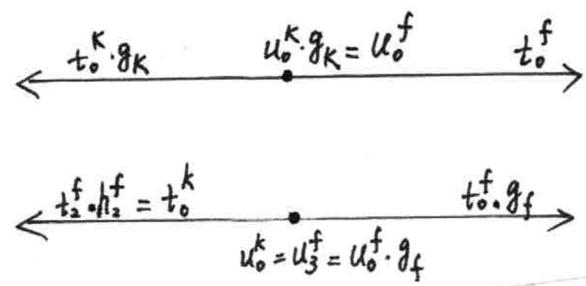


Fig. 16.

Case 2 (c), $g_{s_{f-1}}$ is not fully canceled in $g_{s_1} \cdots g_{s_{f-1}}$. Then there are integers $k < i$ and $r < f$, $r > i$ such that either (i) all the letters in $g_{s'}$ between g_k and g_r are canceled and $g_k g_r \neq 1$ or (ii) all the letters in $g_{s'}$ between h_k and h_r are canceled and $h_k h_r \neq 1$.

(i) As before, we have $g_s = g_{s_1} \cdots g_{s_{k-1}} g_k g_r h_r g_{s_{r+1}} \cdots g_{s_l}$, $u_0^k \cdot g_k = u_0^r$ and $t_0^k \cdot g_k \neq t_0^r$. If there is an integer t such that $f < t \leq l$, and all the letters in $g_{s'}$ between $g_k g_r$ and g_t are canceled, then as in the Case 2 (b), we can prove that $s_k, \ldots, s_{t-1}$ is a double illegal circle, impossible. Suppose there is no such t exists, then there is an integer o such that $r < o < f$ and all the letters in $g_{s'}$ between g_o and g_0^l are canceled. Since between s_{o-1} and s_l, there is only one u-u illegal simple subsequence, the situation is the same as in Case 2 (a), by our discussions in that case, we know that it is impossible for g_{s_l} to be fully canceled.

(ii) Suppose there is an integer c such that $f < c < l$ and all the letters in $g_{s'}$ between $h_k h_r$ and h_c are canceled. Since g_{s_l} is fully canceled in s', we have $h_k h_r h_c = 1$. Then $u_0^k \cdot g_k = u_1^c = u_0^{c+1}$ and $t_0^k \cdot g_k = C'(t_0^k \cdot g_k h_k) = C'(t_0^{k+1}) \neq C'(t_0^c \cdot g_c) = t_0^c \cdot g_c h_c = t_0^{c+1}$. From this point, we can adopt the proof of Case 2 (a) and deduce that it is impossible for g_{s_l} to be fully canceled. If no such c exists, the proof goes like in (i).

Case 3, $j > 0$ and $l = n + 1$. As in Case 2, we prove that g_{s_j} is not fully canceled in $g_{s'}$ and therefore, s' is ideal in s.

Because s_j is u-u or d-u and is not illegal if it is u-u, we have $(g_{s_j})_e \in G' - \{1\}$.

Case 3 (a), s_i is the only u-u illegal subsequence in s'. Suppose there is an integer $k > i$ such that all the letters in $g_{s'}$ between $(g_{s_j})_e$ and h_k are canceled. Then $g_{s'} = g_{s_j} h_k g_{k+1} h_{k+1} \cdots g_n l_n$.

If either g_{s_j} or h_k is fully canceled in $g_{s'}$, as in the proof of Case 2 (a), we have $m_j = 2$, $g_0^j g_1^j = 1$ and $u_0^j = u_1^k = u_0^{k+1} \in (\Lambda_0)^\circ$, $t_0^j \neq t_0^k \cdot g_k h_k = t_0^{k+1}$. Then $u_0^j \in (K_0)^\circ$, this is impossible.

Case 3 (b), there is an integer f such that $j < f \leq n + 1$ and s_f is also u-u illegal. Similar to Case 2 (b) and Case 2 (c), we can proof that g_{s_j} can not be fully canceled in $g_{s'}$.

Case 4, $j > 0$ and $l \leq n$.

Case 4 (a), s_i is the only u-u illegal subsubsequence of s'. Assume g_{s_l} is fully canceled in $g_{s'}$. As in Case 2 (a), we can prove it impossible that all the letters in $g_{s'}$ between g_k and g_0^l are canceled, for some integer k such that $j + 1 < k < i$. So either (i), letters between g_{j+1} and g_0^l or (ii), letters between g_r^j and g_0^l, for some $r \leq m_j$, are all canceled.

(i), $g_{s'} = g_{s_j} g_{j+1} g_{s_l}$. Because s_j is u-u or d-u and it is not illegal if it is u-u, we have $(g_{s_j})_e \in G' - \{1\}$. Since we assumed that g_{s_l} is fully canceled, we have $g_{j+1} g_0^l = 1$.

As in Case 2 (a), we can prove that $u_{m_j}^j = u_0^j \cdot g_0^j \in (\Lambda_0)^\circ$, $\#S_{u_{m_j}^j} \leq 4$, $g_0^l g_1^l = 1$ and $g_{m_j-2}^j g_{m_j-1}^j = 1$. But then $\{u_{m_j-2}^j, u_{m_j}^j \cdot (h_{m_j-1}^j)^{-1}, u_{m_j}^j = u_0^l \cdot g_0^l, u_1^l, u_1^l \cdot h_1^l\} \subset S_{u_{m_j}^j}$, contradicting the fact that $\#S_{u_{m_j}^j} \leq 4$.

(ii), In this case, there is an integer k such that $i < k < l$ and all letters between $(g_{s_j})_e$ and h_k are canceled. By the proof of Case 3 (a), both $(g_{s_j})_e$ and h_k are not fully canceled in $g_{s'}$, therefore g_{s_l} can not be fully canceled either, contradicts with our assumption.

This proves that g_{s_l} can not be fully canceled in $g_{s'}$. Similarly, we have g_{s_j} is not fully canceled in $g_{s'}$. Then s' is ideal in s.

Case 4 (b), there is an integer f such that $f \neq 1$, $j < f < l$ and s_f is also u-u illegal. We may assume that $f > i$. As in the Case 2, $f \neq i+1$.

By the proof of Case 4 (a), if all letters between $(g_{s_j})_e$ and h_r are canceled for some integer r between i and f, then $(g_{s_j})_e h_r \neq 1$. It can be seen that there are integers k, r such that $j \leq k < i$, $i < r < f$ and either (i) $k > j$, all the letters in $g_{s'}$ between g_k and g_r are canceled and $g_k g_r \neq 1$, or (ii) all the letters in $g_{s'}$ between $(g_{s_k})_e$ and h_r are canceled and $(g_{s_k})_e h_r \neq 1$. In case (ii), it can be proved that $k > j$, so $(g_{s_k})_e = h_k$. As in Case 2 (c), we can proof that g_{s_l} is not fully canceled in $g_{s'}$. Then $(g_{s'})_b = (g_{s_j})_b$ and $(g_{s'})_e = (g_{s_l})_e$ and therefore, s' is ideal in s.

Up to now, we have proved that every u-u illegal subsequence s_i of s is contained in a subsequence (with initial and terminal points belong to $E(K_0)$) which is ideal in s. The same statment for d-d illegal subsequence can be proved similarly. From this and Lemma 21, $g_s \neq 1$. The proof of this lemma is completed now. $\diamond$.

Lemma 24 implies that if the action is free, then $K_0 = \emptyset$. This completes the proof of Theorem 3. $\diamond$

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