FREENESS AND DISCRETENESS OF ACTIONS ON R-TREES BY FINITELY GENERATED FREE GROUPS, III

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Abstract

Suppose $G = F(x, y, z)$ is the free group generated by x, y and z, $G' = F(x, y)$, $G'' = F(z)$ are subgroups of G. G acts on an $\mathbf{R}$-tree T minimally with T', T'' be the minimal invariant subtrees of G', G'' respectively, $T_0 = T' \cap T''$. Assume Σ' is the set of partial isometries on T_0 generated by elements of G'.

We prove that the action $T \times G \to T$ is discrete provided it is free if the following condition is satisfied: For any $\sigma, \tau \in \Sigma'$, if there is an integer m such that Domain$(\sigma)\cdot z^m \cap$ Domain$(\tau) \neq \emptyset$, then one of the following is true: (a) Domain(σ)=Domain(τ); (b) One of Domain(σ), Domain(τ) consists of a single point which is an endpoint of the other.

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0. Introduction

In Part 1 and 2, we investigated minimal actions of a finitely generated free group G on an $\mathbf{R}$-tree T. We studyed the following property for such actions:

Property (DF): The action is discrete provided that it is free.

As we learn from part 1 that there is an example of a minimal action of the free group of rank 3 on an $\mathbf{R}$-tree which is free and indiscrete (Bestvina-Handel), therefore Property (DF) is not true in general. In order for the data to be sufficient for our study, in Part 1 we introduced condition **A**, **A**$'$ and **B** (Part 1, page 8, 9, 14).

In the first two parts, we decomposed G as a free product of free groups G' and G'' of smaller rank, we worked on the intersection T_0 of T' and T'', where T' and T'' are the minimal invariant subtrees of G', G'' respectively, and we translated the problems of the freeness and discreteness of the action $T \times G \to T$ to the problems of the partial action of Σ on T_0 under Condition **A**, where Σ is the set of partial isometries on T_0 defined by elements of G (see Proposition 4.2 and 4.3 of Part 1). We showed that an action satisfies Property (DF) if Condition **A** (**A**$'$) and **B** are satisfied (Theorem

4.11 of Part 1).

In Part 3, we continue the study of Property (DF) for the action of G on T. We provide another approach to see what actions satisfy this property. We assume Condition **A** and the freeness of the action $T \times G \to T$, prove, in some certain cases, that the action is discrete. The idea is to project the partial isometries on T_0 in Σ to partial isometries on one side quotient space, say on $Q' = T'/G'$. All such obtained partial isometries on Q' generate a pseudo group P', we prove that the action $T \times G \to T$ is discrete if and only if the partial action of P' on Q' has no infinite orbit (see Theorem 2.5).

Section 1 contains preliminary materials including the notation. Section 2 is devoted to the main theorems of this paper along with the proofs. In Section 3 and 4, we provide examples which are applications of the theorems in Section 2.

1. Preliminary

Throughout this paper, G always represents a finitely generated free group and T always stands for an **R**-tree. We use $T \times G \to T$ for the action of G on T, and $u \cdot g$ for the image of the pair (u, g) under the action, where $u \in T$ and $g \in G$.

We always assume, without mention everywhere, that Condition **A** is satisfied. Without loss of the generality, as in part 1 we make the following:

Assumption 1: The actions $T' \times G' \to T'$ and $T'' \times G'' \to T''$ are free and discrete.

Assumption 2: $T_0 \neq \emptyset$.

Assumption 3: $|T_0| < \infty$.

Assume $p\colon X \to Y$ is a map, S is a subset of X, we use $p|_S$ for the map p restricted on S, and $(S)^\circ$ for the interior of S with respect to X. When S is the union of a family of **R**-trees or **R**-graphs, we denote by $Y(S)$ ($E(S)$ resp.) the set of branch points (end points resp.) of connected components of S.

An **alternating word** (with respect to G' and G'') is an ordered family $\{a_1, a_2, \ldots a_n\}$ of elements of $G' \cup G'' - \{1\}$, such that $a_{2k} \in G'' - \{1\}$, $a_{2k+1} \in G' - \{1\}$ or $a_{2k} \in G' - \{1\}$, $a_{2k+1} \in G'' - \{1\}$ for all k. We allow the empty word to be an alternating word. For every element $g \in G$, there is a unique alternating word $\{a_1, a_2, \ldots, a_n\}$ such that g is the product of a_i's, i.e. $g = a_1 a_2 \cdots a_n$. ($g = 1$ if and only if the corresponding word is empty.) Call this word as the **alternating word of** g (in elements of G' and G''), call n as the (alternating) **word length** of g

and denote it by $L(g)$. Set $g_b = a_1, g_e = a_n$ and

$$g_i = \begin{cases} 1, & \text{if } i = 0; \\ a_1 \cdots a_i, & \text{if } i \leq n \text{ and } i > 0 \text{ ;} \\ g, & \text{if } i > n. \end{cases}$$

Every element $g \in G$ induces an isometry from $T_0 \cdot g^{-1} \cap T_0$ to $T_0 \cap T_0 \cdot g$, we denote this partial isometry of T_0 by σ_g, denote its domain and range by D_g and R_g respectively, which are closed subtrees of T_0.

Let

$$\Sigma' = \{\sigma_g | g \in G', D_g \neq \emptyset\}$$

$$\Sigma'' = \{\sigma_g | g \in G'', D_g \neq \emptyset\}$$

$$\Sigma = \{\sigma_g | g \in G, D_g \neq \emptyset\}$$

Σ acts from the right on T_0, with the product of elements of Σ being the composition of them in the usual sense, if this composition exists and is an elements of Σ. Notice that the identity map of T_0 is included in Σ.

Assume $\phi' \colon T' \to Q' = T'/G'$ is the quotient map. To simplify the notation, for every subset X of T_0, we denote $\phi'(X)$ by $\overline{X}$.

Suppose $g \in G'' - \{1\}$ and $D_g \neq \emptyset$, if D_g and R_g are embedded into Q' by ϕ', then σ_g induces a partial isometry of Q' from $\overline{D}_g = \phi'(D_g)$ to $\overline{R}_g = \phi'(R_g)$ denoted by $\overline{\sigma}_g$ such that

$$\phi'(u \cdot g) = (\phi'(u))\overline{\sigma}_g \tag{1.1}$$

for every point $u \in D_g$, (where $(\phi'(u))\overline{\sigma}_g$ is the image of $\phi'(u)$ under $\overline{\sigma}_g$).

In general, divide D_g and R_g into finitely many closed subtrees (of finite total measure) with disjoint interiors:

$$D_g = \bigcup_{i \in I_g} D_g^i \qquad\qquad R_g = \bigcup_{i \in I_g} R_g^i$$

where I_g is a finite index set, such that for every i, $D_g^i \cdot g = R_g^i$ and D_g^i, R_g^i are embedded into Q' by ϕ'. Write $\overline{D}_g^i = \phi'(D_g^i)$ and $\overline{R}_g^i = \phi'(R_g^i)$. σ_g induces a homeomorphism from $\overline{D}_g^i$ to $\overline{R}_g^i$ denoted by $\overline{\sigma}_g^i$, satisfying (1.1) with $\overline{\sigma}_g^i$ replacing $\overline{\sigma}_g$ for every point $u \in D_g^i$.

Set

$$\overline{\Sigma}' = \{\overline{\sigma}_g^i | g \in G'' - \{1\}, i \in I_g, D_g \neq \emptyset\}$$

Denote the pseudo-group of partial isometries generated by elements of $\overline{\Sigma}'$ by $\overline{P}'$. It is clear that $\#\overline{\Sigma}' < \infty$, so $\overline{P}'$ is finitely generated.

A **word in elements of** $\overline{\Sigma}'$ is an orderd family $\{\sigma_1, \sigma_2, \cdots \sigma_n\}$ of elements of $\overline{\Sigma}' - \{id\}$ written in the form of a production $\sigma_1 \cdot \sigma_2 \cdots \sigma_n$. If the word really involves some letters, it is said to be a **nonempty word**, 1 is the **empty word**. Suppose $\sigma \in \overline{\Sigma}'$, if $g \in G''$ be such that $\sigma = \overline{\sigma}^i_g$ for some $i \in I_g$, then g is called a **lift** of σ. Suppose $w = \sigma_1 \cdot \sigma_2 \cdots \sigma_n$ is a word in elements of $\overline{\Sigma}'$, $g_1, g_2, \ldots g_n$ are elements of G'', if g_i is a lift of σ_i for $i \leq n$, then $g_1 g_2 \cdots g_n$ is called a lift of w. If the action $T \times G \to T$ is free, every word in elements of $\overline{\Sigma}'$ has a unique lift. A nonempty word is **reduced** if no letter involved is followed by its inverse in the word. We see that every word $w = \sigma_1 \cdot \sigma_2 \cdots \sigma_n$ corresponds a partial isometry of Q, which is the composition $\sigma_1 \sigma_2 \cdots \sigma_n$ if it exists or a map with empty domain, we denote this partial isometry by σ_w. When we say that a word w fixes a point u, we mean that σ_w is defined at u and fixes it. we denote by $D(w)$ and $R(w)$ the domain and the range of σ_w.

2. Main theorems

Set $\hat{S} = \{(u, g) | u \in T_0, g \in G \text{ and } u \cdot g_i \in T_0, \forall i \geq 0\}$.

Lemma 2.1: *(a) If* $u \in T_0, g \in G$ *are such that* $(u, g) \in \hat{S}$, *then there is an element* σ *of* $\overline{P}'$ *lifting to* g, *such that* $\overline{u \cdot g} = (\overline{u})\sigma$.

(b) Suppose $w = \overline{\sigma}^{i_1}_{g_1} \cdot \overline{\sigma}^{i_2}_{g_2} \cdots \overline{\sigma}^{i_n}_{g_n}$ *is a reduced word in elements of* $\overline{\Sigma}'$, $u \in D(w)$. *Assume* $v \in T_0$ *be such that* $\overline{v} = u$, *then there is a unique set of elements* $\{s_1, s_2, \ldots s_n\} \subset G'$ *such that if* $h_j = s_1 g_1 s_2 g_2 \cdots s_j g_j$ *and* $w_j = \overline{\sigma}^{i_1}_{g_1} \cdots \overline{\sigma}^{i_j}_{g_j}$ *for* $j \leq n$, *then* $(v, h_n) \in \hat{S}$ *and* $v \cdot h_{j-1} s_j \in D^{i_j}_{g_j}$, $\overline{v \cdot h_j} = (u)\sigma_{w_j}$ *for each* j.

(c) If $u \in Q', v \in T_0$ *such that* $\overline{v} = u$, *then* $\overline{\{v \cdot g | (v, g) \in \hat{S}\}} = (u)P'$.

Proof: (a) Assume $g = a_1 a_2 \cdots a_n$ is the alternating word of g. For each $i \leq n$ both $u \cdot g_{i-1}$ and $u \cdot g_i$ belong to T_0, so either $a_i \in G'$ and then $\overline{u \cdot g_{i-1}} = \overline{u \cdot g_i}$, or $a_i \in G'', u \cdot g_{i-1} \in D_{a_i}$ and $\overline{u \cdot g_i} = (\overline{u \cdot g_{i-1}})\overline{\sigma}^j_{a_i}$ for some $j \in I_{a_i}$. By induction on n, the existence of σ is clear.

(b) We prove by induction on n. Assume there are uniquely $s_1, s_2, s_{n-1} \in G'$ such that $\overline{v \cdot h_j} = (u)\sigma_{w_j}$ and $(v, h_{n-1}) \in \hat{S}$. Because $\overline{v \cdot h_{n-1}} \in \overline{D}^{i_n}_{g_n}$ and $D^{i_n}_{g_n}$ is embedded into Q', there is a unique $s_n \in G'$, such that $v \cdot h_{n-1} s_n \in D^{i_n}_{g_n}$, then $(u)\sigma_w = (\sigma_{w_{n-1}}(u))\overline{\sigma}^{i_n}_{g_n} = (\overline{v \cdot h_{n-1}})\overline{\sigma}^{i_n}_{g_n} = \overline{v \cdot h_1 s_n g_n} = \overline{v \cdot h_n}$. Since $v \cdot h_{n-1} s_n \in D^{i_n}_{g_n} \subset T_0$ and $v \cdot h_n \in R^{i_n}_{g_n} \subset T_0$, $(v, h_n) \in \hat{S}$.

(c) This is a direct consequence of (a) and (b). Note that for every element $\sigma \in \overline{P}'$, there is at least one word w in elements of $\overline{\Sigma}'$, such that $\sigma_w = \sigma$. ◇

Corollary 2.2: $\overline{B}_0 = (\overline{Y}_0)\overline{P}'$

Proof: Suppose $u \in \overline{B}_0$, then there are $v \in Y_0, g \in G$ such that $(v, g) \in \hat{S}$ and $\overline{v \cdot g} = u$. By Lemma 2.1 (a), there is an element σ of $\overline{P}'$ such that $(\overline{v})\sigma = \overline{v \cdot g} = u$, so $u \in (\overline{Y}_0)\overline{P}'$.

Assume $u \in (\overline{Y}_0)\overline{P}'$, then there is a point $v \in Y_0$, an element $\sigma_w \in \overline{P}'$, where w is a reduced word in elements of $\overline{\Sigma}'$, such that $u = (\overline{v})\sigma_w$. By Lemma 2.1 (b), there is an element $h \in G$ such that $(v, h) \in S$ and $u = (\overline{v})\sigma_w = \overline{v \cdot h}$. We see that $v \cdot h \in B_0$ and $u \in \overline{B}_0$. ◇

Lemma 2.3: *If the action $T \times G \to T$ is not free then $\overline{P}'$ has a fixed point in $\overline{Y}_0$.*

Proof: Suppose the action is not free, by Propositions 3.1, there is a point $u \in Y_0$, an element $\sigma_g \in \Sigma$ with $g = a_1 a_2 \cdots a_n$ be an alternating word in elments of $G' - \{1\}$ and $G'' - \{1\}$ such that $(u)\sigma_g = u$, i.e. $u \cdot a_1 a_2 \cdots a_n = u$ and $u \cdot a_1 a_2 \cdots a_i \in T_0$ for $i \leq n$. We may assume that $a_1, a_3 \ldots \in G'$ and $a_2, a_4 \ldots \in G''$, it is easy to see that $\overline{\sigma}_{a_2}\overline{\sigma}_{a_4} \cdots \overline{\sigma}_{a_{2k}} \in \overline{P}'$ fixes $\overline{u} \in \overline{Y}_0$, where k is the gratest integer such that $2k \leq n$. ◇

Lemma 2.4: *Assume Condition **A** is true and the action $T \times G \to T$ is free, then*

$\#(\overline{Y}_0)\overline{P}' < \infty$ if and only if the action $T \times G \to T$ is discrete.

Proof: If $\#(\overline{Y_0})\overline{P}' = \#\overline{B_0} < \infty$, then there are only finitely many G'-orbits which intersect B_0. Because the intersection of each G'-orbit and B_0 can have only finitely many points, we get $\#B_0 < \infty$. From Theorem 4.8 of Part 1, we see that Condition **B** is satisfied, so according to Theorem 4.11 of Part 1, the action $T \times G \to T$ is discrete. On the other hand, if $\#\overline{B_0} = \#(\overline{Y_0})\overline{P}' = \infty$, then $\#B_0 = \infty$, by the proof of Proposition 4.3 (a) of Part 1, $\#F(u) = \infty$ for some point $u \in Y_0$, as a consequence, the action $T \times G \to T$ is not discrete. ◇

Theorem 2.5: *Assume Condition **A** is true and the action $T \times G \to T$ is free, then the action $T \times G \to T$ is discrete if and only if $\#(u)\overline{P}' < \infty$ for every point $u \in \overline{Y}_0$, if and only if this is true for every point $u \in Q'$.*

Proof: Assume $\#(u)\overline{P}' < \infty$ for every $u \in \overline{Y}_0 \subset Q'$, then $\#(\overline{Y_0})\overline{P}' < \infty$ since $\#Y_0 < \infty$, therefore the action $T \times G \to T$ is discrete by Lemma 2.4. Suppose there is a point $u \in Q'$ such that $\#(u)\overline{P}' = \infty$, then clearly $u \in \overline{T_0}$. Suppose $v \in T_0$ be a preimage of u under ϕ', by Lemma 2.1 (c), $\overline{\{v \cdot g | (v, g) \in \hat{S}\}} = (u)\overline{P}'$. So if $\#(u)\overline{P}' = \infty$, $\overline{\{v \cdot g | (v, g) \in \hat{S}\}}$ is a indiscrete set. Then it is easy to see that $\{v\} \cdot G$ is also indiscrete, therefore the action $T \times G \to T$ is not discrete. ◇

3. Examples in general cases

Suppose the action $T \times G \to T$ is free. Assume that $w = \overline{\sigma}_{g_1}^{i_1} \cdot \overline{\sigma}_{g_2}^{i_2} \cdots \overline{\sigma}_{g_n}^{i_n}$ is a word in elements of $\overline{\Sigma}'$ with $D(w) \neq \emptyset$, define P(w) to be the following property of a point $v \in \phi'^{-1}(D(w))$: $\overline{v \cdot h} = (\overline{v})\sigma_w$, where h is the lift of w. Suppose v satisfies $P(w)$ and $u = \overline{v} \in D(w)$, if the set $\{s_1, s_2, \ldots s_n\} \subset G'$ is given by Lemma 2.1 (b), then we have $s_j = 1$ for every $j \geq 2$.

A nonempty word w in elements of $\overline{\Sigma}'$ is called a **trivial word** , if $D(w) \neq \emptyset$, 1 is a lift of w and there is a point of $\phi'^{-1}(D(w))$ satisfying the property P(w). If a word w is trivial, then σ_w fixes a point in its domain.

From now on we assume that for every $g \in G''$, D_g and R_g are embedded into Q' by ϕ', then $\overline{\Sigma}' = \{\sigma_g | g \in G'', D_g \neq \emptyset\}$.

Assume that there is a subset Σ_0 of $\overline{\Sigma}'$ satisfying the following properties:

(a) Elements of Σ_0 generate $\overline{P}'$, i.e. for every $\sigma \in \overline{P}'$, there is a word w in elements of $\Sigma_0 \cup (\Sigma_0)^{-1}$, where $(\Sigma_0)^{-1} = \{\sigma | \sigma^{-1} \in \Sigma_0\}$, such that σ is σ_w limited on a subset of its domain.

(b) No proper subset of Σ_0 generates $\overline{P}'$.

Then Σ_0 is called a **minimal generating subset of $\overline{\Sigma}'$**.

Lemma 3.1: *Suppose Σ_0 is a generating subset of $\overline{\Sigma}'$, $w = \tau_1 \cdot \tau_2 \cdots \tau_n$ is a nonempty reduced word in elements of $\mathcal{K}$, where $\mathcal{K} = \{\sigma | \sigma \in \Sigma_0$ or $\sigma^{-1} \in \Sigma_0\}$, $u \in D(w)$ and u is fixed by σ_w, then for $i = 1, 2, \ldots, n$, we have $D(\tau_i)$ is nondegenerate.*

Proof: Suppose there is a $j \leq n$ such that $D(\tau_j)$ consists of a single point v, then the domain of $\sigma_w = \{u\}$, so σ_w is the identity map on the set $\{u\}$. If there is a $k \neq j$ such that $\tau_j = \tau_k$, we may assume that $j < k$ and $\tau_i \neq \tau_j$ if $j < i < k$, then the subword $w_0 = \tau_j \cdot \tau_{j+1} \cdots \tau_{k-1}$ fixes the point v. Taking w_0 instead of w, we may assume that $\tau_i \neq \tau_j$ if $i \neq j$. By a similar argument, we may also assume that $\tau_i \neq (\tau_j)^{-1}$ for $i \leq n$. Because $D(\tau_j)$ and $R(\tau_j)$ both consists of one point, we have $\tau_j = \sigma_{w'}|_{\{v\}}$ here $w' = \tau_{j-1}^{-1}\tau_{j-2}^{-1} \cdots \tau_1^{-1}\tau_n^{-1} \cdots \tau_{j+1}^{-1}$. But Σ_0 is a minimal generating subset of $\overline{\Sigma}'$, this is impossible. ◇

A minimal generating subset Σ_r of $\overline{\Sigma}'$ is called a **reduced generating subset** if every reduced word w in elements of Σ_r is nontrivial.

Assume the action $T \times G \to T$ is free, then every element of $\overline{\Sigma}'$ has a unique lift. Therefore, every word in elements of $\overline{\Sigma}'$ has a unique lift. Suppose $x_1, x_2, \ldots, x_n$ is a set of free basis of G'', w is a word in elements of $\overline{\Sigma}'$, define $e_i(w)$ to be the total sum of the exponents of x_i in the lift of w.

The functions e_i is additive, i.e. $e_i(w_1 w_2) = e_i(w_1) + e_i(w_2)$ for every pair of words w_1, w_2 in elements of $\overline{\Sigma}'$.

Example 3.2: Assume that the action $T \times G \to T$ is free. Suppose there is a minimal generating subset Σ_0 of $\overline{\Sigma}'$ such that there is at most one element $\tau \in \Sigma_0$ satisfying that $e_i(\tau) = 0$ for all i, then there exists a reduced generating subset of $\overline{\Sigma}'$.

Proof: Notice that if we change any element in Σ_0 to its inverse, we do not change the minimal generating property of Σ_0. We now construct a set Σ_r from Σ_0 by changing part of its elements to their inverses in the following way: For every element $\sigma \in \Sigma_0$, if $e_i(\sigma) = 0$ for all i, then we keep σ in Σ_r, if this is not true, assume that i is the smallest integer such that $e_i(\sigma) \neq 0$, then if $e_i(\sigma) > 0$, we keep σ in Σ_r, otherwise change σ to σ^{-1}. We know that Σ_r is still a minimal generating subset of $\overline{\Sigma}'$ and for every element $\sigma \in \Sigma_r$, either $e_i(\sigma) = 0$ for all i and by the assumption $\sigma = \tau$, or

$e_i(\sigma) > 0$ if i is the smallest integer such that $e_i(\sigma) \neq 0$. We claim that Σ_r is a reduced generating subset.

Suppose not, then there is a reduced word $w = \sigma_1 \cdot \sigma_2 \cdots \sigma_k$ in elements of Σ_r which is trivial. We have $\Sigma_{j=1}^k e_i(\sigma_j) = e_i(w) = 0$ for each i. If $e_i(\sigma_j) = 0$ for all $i \leq n, j \leq k$, then $\sigma_j = \tau$ for $j = 1, 2, \ldots, k$ and therefore $w = \tau^k$, then the lift of w can not be 1, impossible. Assume this is not true, i is the smallest integer such that $e_i(\sigma_j) \neq 0$ for some $j \leq k$, then $e_i(\sigma_j) > 0$ and $e_i(\sigma_l) \geq 0$ for all $l \leq k$, so $e_i(w) > 0$, this is a contradiction. ◊

Assume r is the following relation of two finite closed subtrees I_1 and I_2 of T: One of I_1, I_2 consists of a single point which is an end point of the other. If I_1, I_2 have the relation r, we write $r(I_1, I_2)$.

Example 3.3: Assume that there is a reduced generating subset Σ_r satisfies the following properties: For every pair of elements σ, τ of Σ_r such that $\sigma \neq \tau$, we have $r(D(\sigma), D(\tau))$ if $D(\sigma) \cap D(\tau) \neq \emptyset$, and $r(R(\sigma), R(\tau))$ if $R(\sigma) \cap R(\tau) \neq \emptyset$. Then (DF) is true for the action $T \times G \to T$.

Proof: Assume the action $T \times G \to T$ is free. Suppose $\Sigma_r = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$. The domain and the range of σ_i are denoted by D_i and R_i for each $i \leq k$. Set $\mathcal{K} = \Sigma_r \cup (\Sigma_r)^{-1} = \{\sigma_1, \ldots, \sigma_m, \sigma_1^{-1}, \ldots, \sigma_m^{-1}\}$.

Set $D = \bigcup_{i=1}^m D_i$, $R = \bigcup_{i=1}^m R_i$ and $\hat{D} = Q' - D, \hat{R} = Q' - R$, and $X = E(\hat{D}) \cup E(\hat{R})$. Then from the assuptions, for each i, $E(D_i) \cup E(R_i) \subset X$.

Also, let $\Sigma_s = \{\sigma_i | i \leq m,\ |D_i| = |R_i| = 0\}$. Assume $w = \tau_1 \cdot \tau_2 \cdots \tau_n$ is a word in elements of $\mathcal{K}$, if there is an $i \leq n$ such that $\tau_i \in \Sigma_s$, then we say that w **intersects** Σ_s.

Define $\sigma: D \to R$ be such that its restriction to each D_i is σ_i. σ is multivalued at intersections of some domains. Every such intersection consists of a single point which belong to X, so σ is well defined (i.e. has a single value) in the interior of each domain D_i. Similarly, σ^{-1} is defined on R and is well defined in the interior of each range R_i.

According to Theorem 2.5, it is enough to prove that for every $u \in \overline{Y}_0 \subset Q', (u)\overline{P}'$ is a finite set. Now, fix a point $u \in \overline{Y}_0$. Assume that $\mathcal{F}_u$ is the space of finite reduced word w in elements of $\mathcal{K}$ such that $u \in D(w)$. Then $(u)\overline{P}' = (u)\mathcal{F}_u$, because Σ_r is a generating subset of $\overline{\Sigma}'$. It is enough to prove that $\#\mathcal{F}_u < \infty$ for every $u \in Q'$.

Suppose $\tau \in \mathcal{K}$, for simplicity we say that $\tau = \sigma$, if $\tau \in \{\sigma_1, \ldots, \sigma_m\}$, and $\tau = \sigma^{-1}$, if $\tau \in \{\sigma_1^{-1}, \ldots, \sigma_m^{-1}\}$.

Suppose $w = \tau_1 \cdot \tau_2 \cdots \tau_n$ with each $\tau_i \in \mathcal{K}$, if there is an i such that either (a), $\tau_i = \sigma, \tau_{i+1} = \sigma^{-1}$, or (b) $\tau_i = \sigma^{-1}, \tau_{i+1} = \sigma$, then $R(\tau_i) \cap R(\tau_{i+1}^{-1}) \neq \emptyset$, so either $R(\tau_i)$ or $D(\tau_{i+1}) = R(\tau_{i+1}^{-1})$ consists of a single point $v \in X$, we say that w has a **negative turn** at v in case (a), and a **positive turn** at v in case (b).

Lemma 3.4: *Assume $w = \tau_1\tau_2\cdots\tau_n$ is a reduced word in elements of $\mathcal{K}$, if w has a turn, then σ_w fixes no point in Q'.*

Proof: If w has a turn, then one of τ_i's must belong to Σ_s, by Lemma 3.1, σ_w can not fix any point of Q'. ◊

According to Lemma 3.4, a reduced word w in elements of $\mathcal{K}$ can not have three turns at the same point, so it can only have at most $2\#X < \infty$ turns. If a word w has no turn, it is called a **straight word**. For every point $u \in Q'$, let $\hat{\mathcal{F}}_u^+$ be the subset of $\mathcal{F}_u$ consists of straight words in elements of Σ_r, $\hat{\mathcal{F}}_u^-$ be the subset of $\mathcal{F}_u$ consists of straight words in elements of Σ_r^{-1} and $\hat{\mathcal{F}}_u$ be the union of $\hat{\mathcal{F}}_u^+$ and $\hat{\mathcal{F}}_u^-$.

Lemma 3.5: *The following two statments are equivalent:*

(a) $\#\mathcal{F}_u < \infty$ for every $u \in Q'$.

(b) $\#\hat{\mathcal{F}}_u < \infty$ for every $u \in Q'$.

Proof: The proof of (a) $\Longrightarrow$ (b) is trivial.

(b) $\Longrightarrow$ (a): If two words w and w' in $\mathcal{F}_u$ have exactly the same positive and negative turns in the same order, assume they have the last turn at $v \in X$, then by Lemma 3.4, the subwords of w and w' before this last turn are the same and the subwords of them after this last turn both belong to $\hat{\mathcal{F}}_v$. Because $\#\hat{\mathcal{F}}_v < \infty$ for every $v \in X$ and $\#\hat{\mathcal{F}}_u < \infty$, we have $\#\mathcal{F}_u < \infty$. ◊

Now, Let us prove that for every point $u \in Q'$, $\#\hat{\mathcal{F}}_u^+ < \infty$ and $\#\hat{\mathcal{F}}_u^- < \infty$.

Suppose $u \in D$, if there is a composition $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ (with each $\sigma_{i_j} \in \Sigma_r$) defined at u, we say that σ^k is defined at u and we write $(u)\sigma^k$ for $(u)\sigma_1\sigma_2\cdots\sigma_k$. Note $(u)\sigma^k$ may have more that one values. If S is a subset of Q', $(S)\sigma^k$ is defined as $\{(u)\sigma^k | u \in S \cap D, \sigma^k \text{ is defined at } u\}$. $(u)\sigma^{-k}$ and $(S)\sigma^k$ are similarly used for $u \in D$ and $S \subset Q'$.

Lemma 3.6: *Assume $u \in Q'$, w is a word in elements of $\overline{\Sigma}''$ and u is fixed by w. Then*

(a) The lift of w is 1.

(b) w can not be a nonempty word in elements of Σ_r, i.e. $\sigma_w \neq \sigma^k$ for any positive integer k.

Proof: (a) Suppose $w = \tau_1\cdot\tau_2\cdots\tau_k$ with $\tau_i \in \overline{\Sigma}''$ for each i and $v \in T_0$ is such that $\overline{v} = u$. Assume the lift of τ_i is $g_i \in G''$, by Lemma 2.1 (b), there is a unique set of elements $\{s_1, s_2, \ldots, s_k\} \subset G'$ such that if $h = s_1g_1\cdots s_kg_k$, then $(v,h) \in \hat{S}$ and $\overline{v\cdot h} = (u)\sigma_w = u = \overline{v}$. Since the action $T \times G \to T$ is free, we must have $h \in G'$. This implies that $g_1g_2\cdots g_k = 1$.

(b) Suppose w is a word in elements of Σ_r. Assume $v \in T_0$, $g_i \in G''$ and $s_i \in G'$ for $i = 1, 2, \ldots k$

are as in (a). Assume that $s_j = 1$ for all $j \geq 2$, then $v \cdot s_1$ satisfies the property $P(w)$, so w is trivial. Assume there is a $j \geq 2$ such that $s_j \neq 1$, we may assume that for any $l < j, l \geq 2$, $s_j = 1$, then $g_1 g_2 \cdots g_{j-1} = 1$, so the reduced word $\tau_1 \cdot \tau_2 \cdots \tau_{j-1}$ is trivial, this is impossible since Σ_r is a reduced generating subset of $\overline{\Sigma}''$. ◊

Set $I_0 = \hat{D}$, $I_{k+1} = \cup_{i=1}^{m} \{(I_k)\sigma_i^{-1} | \sigma_i \notin \Sigma_s\}$.

Lemma 3.7: *If $i, j \geq 0$ and $i \neq j$, then $(I_i)^\circ \cap (I_j)^\circ = \emptyset$.*

Proof: Suppose this is not true, assume i is the smallest integer such that $(I_i)^\circ \cap (I_j)^\circ \neq \emptyset$ for some integer $j > i$. Then i must be 0, otherwise $(I_{i-1})^\circ \cap (I_{j-1})^\circ \neq \emptyset$. But $I_j \subset D$ which does not intersect $I_0 = \hat{D}$, this is a contradiction. ◊

Assume I is a subset of Q', define $l(I)$ to be the minimum total measure of nondegenerate components of I if I has one, and take $l(I)$ to be 0 if I has no nondegenerate component.

Since X is a finite set, there is an positive integer n such that $X \cap (I_i)^\circ = \emptyset$ for every $i \geq n$.

Lemma 3.8: *Assume n is such an integer that $\bigcup_{i=n}^{\infty} (I_i)^\circ \cap X = \emptyset$, then for every $i \geq n$, we have $l(I_i) \geq l(I_n)$ or $l(I_i) = 0$.*

Proof: This can be proved by induction on i. Suppose J is a nondegenerate component of I_i, because $X \cap J^\circ = \emptyset$, either $J^\circ \subset \hat{R}$ or $J^\circ \subset R$ so that $J \subset R$ since R is closed. Then if $\sigma_i \notin \Sigma_s$ and $J \cap R_i \neq \emptyset$, $(J)\sigma_i^{-1}$ either consists of one or two points or is isomorphic to J, therefore either $|(J)\sigma_i^{-1}| = 0$ or $|(J)\sigma_i^{-1}| = |J|$. Compared with I_i, I_{i+1} has no nondegenerate component of smaller total measure, so we have $l(I_{i+1}) \geq l(I_i) \geq l(I_n)$ if $l(I_{i+1}) \neq \emptyset$. ◊

Lemma 3.9: *There is an integer $n > 0$ such that for any $k \geq n$, $l(I_k) = 0$.*

Proof: Suppose $l(I_i) \neq \emptyset$ for all $i \geq 0$, then by Lemma 3.8, there is a positive number λ such that $|I_k| \geq \lambda$ for every $k \geq 0$. By lemma 3.7, $(I_i)^\circ \cap (I_j)^\circ = \emptyset$ if $i \neq j$, so for any $n \geq 0$ we have $n\lambda \leq \Sigma_{i=1}^{n} |I_i| \leq |Q'|$. But $|Q'| < \infty$, this is impossible.

Suppose there is an $n > 0$ such that $l(I_n) = 0$, then $l(I_i) = 0$ for all $i \geq n$. ◊

Suppose $\overline{X}$ is the set of end points of all the open ends of I_i for $i \geq 0$, $\overline{X}$ is a finite set.

Lemma 3.10: *If $u \in \overline{X} \cap R_i$, then $(u)\sigma_i^{-1} \in \overline{X}$.*

Proof: If $\sigma_i \in \Sigma_s$, then $(u)\sigma_i^{-1} \in E(D_i) \subset \overline{X}$. Assume $\sigma_i \notin \Sigma_s$, suppose u is an end point of J which is a component of I_k for some $k \geq 0$, with the corresponding end open. Suppose the only direction in $D(u, J)$ is not in R_i, since we assumed that $u \in R_i$, we have $u \in E(R_i)$ and then $(u)\sigma_i^{-1} \in E(D_i) \subset \overline{X}$. If this direction is in R_i, then it is carried by σ_i^{-1} to a direction in $D((u)\sigma_i^{-1}, (J)\sigma_i^{-1})$ and $(u)\sigma_i^{-1}$ is the end point of a component of $(J)\sigma_i^{-1}$ with the corresponding end open. Clearly $(J)\sigma_i^{-1} \subset I_{k+1}$, so $(u)\sigma_i^{-1} \in \overline{X}$. ◊

Lemma 3.11: *If for some $k > 0$, I_k has a component J of 0 total measure, then it must consists of a single point u, with $u \in \overline{X}$.*

Proof: There is a sequence $\{J_0, J_2, \dots J_k = J\}$ of closed subtrees such that J_j is a component of I_j and $J_{j+1} \subset (J_j)\sigma_{i_j}^{-1}$ for some $\sigma_{i_j} \in \Sigma_r - \Sigma_s$. Notice that $|J_{j+1}| \leq |J_j|$, since $|J_0| > 0$, there is a $j \geq 0$ such that $|J_j| > 0$, $|J_{j+1}| = 0$. Then $|J_j \cap R_{i_j}| = 0$, so $J_j \cap R_{i_j} \subset E(R_{i_j})$. If $J_{i+1} = \{v\}$, then $v \in (E(R_{i_j}))\sigma^{-1} = E(D_{i_j}) \subset \overline{X}$. By Lemma 3.10, $u \in \overline{X}$. ♢

For every point $u \in Q'$, let $(\hat{\mathcal{F}}_u^+)_0$ be the subset of $\hat{\mathcal{F}}_u^+$ consists of all the words which do not intersect Σ_s. The subset $(\hat{\mathcal{F}}_u^-)_0$ of $\hat{\mathcal{F}}_u^-$ is defined in the same way. Because for every point $v \in Q'$, there is at most one $\sigma_i \in \Sigma_r - \Sigma_s$ which is defined at v, it can be seen that for any positive integer n, there is at most one word in $(\hat{\mathcal{F}}_u^+)_0$ whose word length is n, if $w, w' \in (\hat{\mathcal{F}}_u^+)_0$ and w has longer word length than that of w', then w' is a subword of w. The same is true for words in $(\hat{\mathcal{F}}_u^-)_0$.

Lemma 3.12: *For every $u \in \overline{X}$, there is an integer k such that all the words in $(\hat{\mathcal{F}}_u^-)_0$ have word length less that k.*

Proof: Suppose this is not true for a point $u \in \overline{X}$, then $\#(\hat{\mathcal{F}}_u^-)_0 = \infty$. Set $U = \{(u)\sigma_w | \in (\hat{\mathcal{F}}_u^-)_0\}$, then by Lemma 3.6 (b), $(u)\sigma_w \neq (u)\sigma_{w'}$ if $w, w' \in (\hat{\mathcal{F}}_u^-)_0$ and $w \neq w'$, so $\#U = \infty$. According to Lemma 3.10, $U \subset \overline{X}$, but $\overline{X}$ is a finite set, this is impossible. ♢

Lemma 3.13: *There is an integer $n > 0$ such that $I_n = \emptyset$.*

Proof: Suppose n' is such a number that for every $i \geq n'$ $l(I_k) = 0$, as in Lemma 3.9. Then I'_n consists of several single element components, i.e. $I'_n = \cup_{j=1}^{l}\{u_j\}$. By Lemma 3.11 and Lemma 3.12, for each u_j, there is an integer $n_j > 0$ such that all the words in $(\hat{\mathcal{F}}_{u_j}^-)_0$ have word length less than n_j. Take $n = n' + \max\{n_j | 0 \leq j \leq l\}$, then it is easy to see that $I_n = \emptyset$. ♢

Set $M = \bigcup_{k=0}^{\infty} I_k$, $K = Q' - M$. Then by Lemma 3.13, M consists of finitely many components, so is K if it is not empty. We have $(K)\sigma_w \subset K$ if $w \in (\hat{\mathcal{F}}_u^+)_0$.

Lemma 3.14: $|K| = 0$.

Proof: Suppose this is not true. Assume J is a component of K of maximum total measure then $|J| > 0$. Because $J \subset Q' - I_0 = Q' - \hat{D}$, $J \subset D_i$ for some $\sigma_i \in \Sigma_r - \Sigma_s$. Then $(J)\sigma_i$ is isomorphic to J, so it is also a component of K of maximum total measure. Consequently, for every $i \geq 0$, there is a word $w_i \in (\hat{\mathcal{F}}_u^+)_0$ whose word length is i such that $J \subset D(w_i)$ and $(J)\sigma_{w_i}$ is a component of K of maximum total measure. But K consists of finitely many components, so there is a pair of integers n, l such that $n \neq l$ and the two components $(J)\sigma_{w_n}$ and $(J)\sigma_{w_l}$ have nonempty intersection. Then these two components must equal to each other. We may assume that $n = 0$, then $(J)\sigma_{w_l} = J$. $\sigma_{w_l}|_{E(J)}$ is a permutation with $E(J)$ being a finite set, so there is an integer p such that $\sigma_{w_l}^p$ has a fixed point u in $E(J)$. Then $(u)\sigma^{lp} = u$, contradicting Lemma 3.6 (b), impossible. ♢

Lemma 3.14 implies that $K = \{k_1, k_2, \ldots, k_t\}$ for some integer t.

Proposition 3.15: *For every point $u \in Q'$, we have $\#(\hat{\mathcal{F}}_u^+)_0 < \infty$.*

Proof: If $u \in I_0 = \hat{D}$, then $(\hat{\mathcal{F}}_u^+)_0 = \emptyset$. If $u \in I_k$, then there is a word $w \in (\hat{\mathcal{F}}_u^+)_0$ whose word length is k such that $(u)\sigma_w \in I_0 = \hat{D}$, then $(\hat{\mathcal{F}}_u^+)_0$ contains no word of length greater than k. So $\#(\hat{\mathcal{F}}_u^+)_0 < \infty$. Suppose $u \in K$, then $(u)(\hat{\mathcal{F}}_u^+)_0 \subset K$. By Lemma 3.4, if $w, w' \in (\hat{\mathcal{F}}_u^+)_0$ and $w \neq w'$, then $(u)\sigma_w \neq (u)\sigma_{w'}$, because $\#K < \infty$, we have $\#(\hat{\mathcal{F}}_u^+)_0 < \infty$. ◊

For each $i \leq m$ such that $\sigma_i \in \Sigma_s$, we denote the unique point in R_i by p_i. Fix a point $u \in Q'$, if there is a word $w = \tau_1\tau_2\cdots\tau_k \in \hat{\mathcal{F}}_u^+$ such that $\tau_k = \sigma_i$ for some $\sigma_i \in \Sigma_s$, then by Lemma 3.2, such word is unique, denote this unique word by w_i.

Proposition 3.16: *For every point $u \in Q'$, we have $\#\hat{\mathcal{F}}_u^+ < \infty$.*

Proof: If $\sigma_i \in \Sigma_s$ for some $i \leq m$, define $H_i = \{w = \tau_1\cdots\tau_k \in \hat{\mathcal{F}}_u^+ | \exists j \leq k \text{ such that } \tau_j = \sigma_i,\ \tau_l \notin \Sigma_s \text{if } j < l \leq k\}$.

If $w \in H_i$, $\tau_j = \sigma_i$ is as above, then $\tau_1\tau_2\cdots\tau_j = w_i$ and $\tau_{j+1}\tau_{j+2}\cdots\tau_k \in (\hat{\mathcal{F}}_{p_i}^+)_0$, therefore, it is easy to see that $\#H_i \leq \#(\hat{\mathcal{F}}_{p_i}^+)_0 < \infty$. Because $\hat{\mathcal{F}}_u^+ = (\hat{\mathcal{F}}_u^+)_0 \cup \bigcup_{\sigma_i \in \Sigma_s} H_i$ we have $\#\hat{\mathcal{F}}_u^+ \leq \#(\hat{\mathcal{F}}_u^+)_0 + \Sigma_{\sigma_i \in \Sigma_s} \#H_i < \infty$. ◊

Because the conditions on σ, D and on σ^{-1}, R are symmetric, we also have:

For any point $u \in Q'$, $\#\hat{\mathcal{F}}_u^- < \infty$.

Then $\#\hat{\mathcal{F}}_u < \infty$ for every point $u \in Q'$ and therefore, the action $T \times G \to T$ is discrete. This completes the proof of this example. ◊

Example 3.17: If there is a reduced generating subset Σ_r of $\overline{\Sigma}'$ such that $\#\Sigma_r = 1$, then the action $T \times G \to T$ satisfies Property (DF).

Proof: This can be easily proved by applying Example 3.3. ◊

Remark: By symmetry we have $Q'' = T''/G''$, $\phi'': T_0 \to Q''$, $\overline{\Sigma}''$ and $\overline{P}''$ etc. All the results in Section 2 and 3 remain true if we replace $Q', \phi', \overline{\Sigma}'$ and $\overline{P}'$ by $Q'', \phi'', \overline{\Sigma}''$ and $\overline{P}''$.

4. Applications to actions by the free group of rank 3

In this Section, we focus on minimal actions of $G = F_3$ (the free group of rank 3) on an $\mathbf{R}$-tree T. We provide examples of actions which satisfy the Property (DF) or which are not free.

Assume that $\{x, y, z\}$ is a free basis of G, i.e. $G = F(x, y, z)$. Take $G' = F(x, y)$, $G'' = F(z)$. We may assume that $|A_x \cap A_y| < \min\{\tau(x), \tau(y)\}$, where $\tau(\ )$ is the translation length function for the action $T \times G \to T$ (cf. Part 1, page 4).

As before, T', T'' are the minimal invariant subtrees of G', G'' respectively. From the materials of Part 1 (page 4-5) we know that $Q' = T'/G'$ is one of the following:

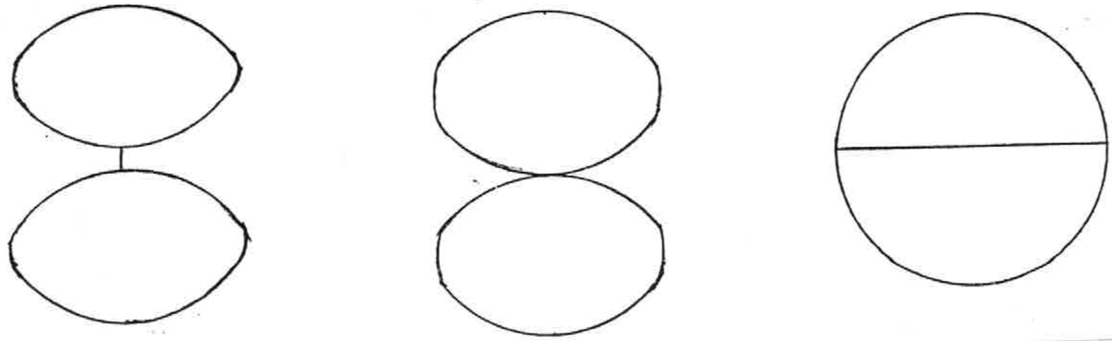

Fig. 1.

T'' is the axis A_z, so Q'' is just a circle. Since $T_0 \subset A_z$, we have $T_0 = [p, q]$ for some $p.q \in A_z$. We assume further that the direction represented by the arrow from p to q is the direction of A_z.

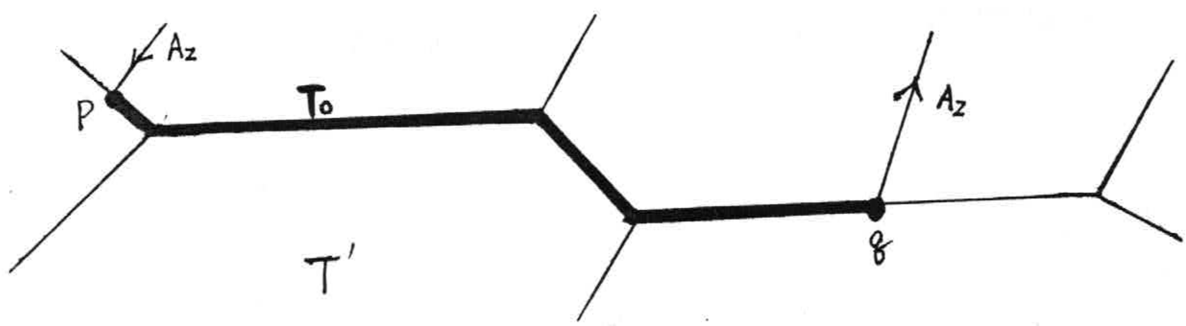

Fig. 2.

Example 4.1: Assume there is an element $g \in G'$, such that $|T_0 \cap A_g| \geq \tau(z) + \tau(g)$, then the action $T \times G \to T$ is not free.

Proof: Because $T_0 \subset A_z$, $|A_z \cap A_g| \geq \tau(z) + \tau(g)$ so there is a point $u \in A_z \cap A_g$ which is fixed by the commutator $zgz^{-1}g^{-1}$. ◊

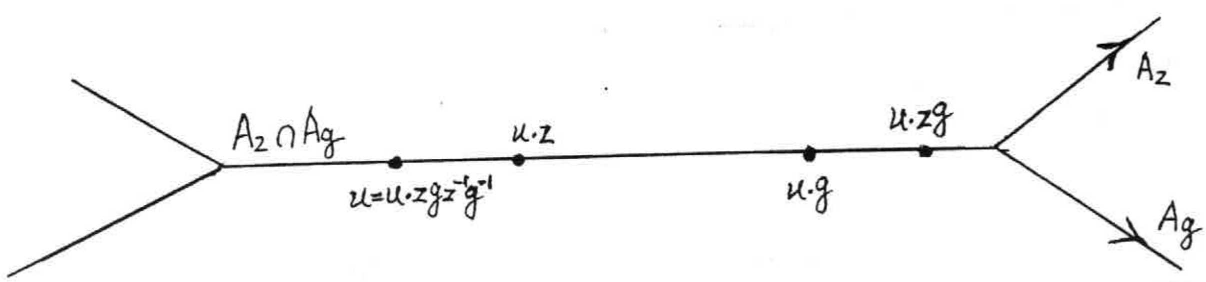

Fig. 3.

Example 4.2: Set $\omega = T_0 \cap T_0 \cdot z^{-1}$, assume that ω and $\omega \cdot z$ are both embedded into Q' by ϕ', then the action $T \times G \to T$ satisfies (DF).

Proof: We see that $\omega = D_z$ and $\omega \cdot z = R_z$. For any integer $m \neq 0$, D_{z^m} and R_{z^m} are contained in ω and $\omega \cdot z$ respectively, so they are embedded into Q' by ϕ'. Then $\overline{\Sigma}' = \{\overline{\sigma}_{z^m} | D_{z^m} \neq \emptyset\}$. Since we only have finitely many m such that $D_{z^m} \neq \emptyset$, $\overline{\Sigma}'$ is a finite set. For any positive integer n, set $w_n = (\overline{\sigma}_z)^n$, then $\overline{\sigma}_{z^n} = \sigma_{w_n}|_{\overline{D}_{z^n}}$. Therefore $\{\overline{\sigma}_z\}$ is a reduced generating subset of $\overline{\Sigma}'$. Applying Example 3.16, we see that (DF) is true for the action $T \times G \to T$.

Example 4.3: If $T_0 \subset A_g$ for some element $g \in G' - \{1\}$ such that $\overline{A}_g$ is a loop (i.e. a subspace homeomorphic to a circle) in Q' (this is true for example when g is conjugate to x or y), then (DF) is true for the action $T \times G \to T$.

Proof: Assume $T_0 \subset A_g$ for some $g \in G' - \{1\}$. By assumption, $\overline{A}_g$ is a loop in Q', its circumference c is a number dividing the translation length $\tau(g)$ of g. There is an element $h \in G' - \{1\}$ and there are points $u, v \in A_g$ such that $u \cdot h = v$ and $\mathrm{dis}(u, v) = c$. It is clear that $[u, v] \subset A_h$ and $c = \tau(h)$. Then $\overline{T_0} \subset \overline{A}_g = \overline{[u, v]} = \overline{A}_h$.

There is an element $s \in G'$ such that $T_0 \cdot s \cap A_h \neq \emptyset$. Suppose $T_0 \cdot s \not\subset A_h$, then there is a point $u \in E(T_0 \cdot s \cap A_h)$ such that $D(u, T_0 \cdot s) - D(u, A_h) \neq \emptyset$. Assume $t \in D(u, T_0 \cdot s) - D(u, A_h)$, since $D(\overline{u}, \overline{T_0 \cdot s}) \subset D(\overline{u}, \overline{A}_h)$, we have $\overline{t} \in D(\overline{u}, \overline{A}_h) = \phi'(D(u, A_h))$. But ϕ' maps T' to Q' locally isometrically, this is impossible. Therefore $T_0 \cdot s \subset A_h$. Then $T_0 \subset A_h \cdot s^{-1} = A_{shs^{-1}}$.

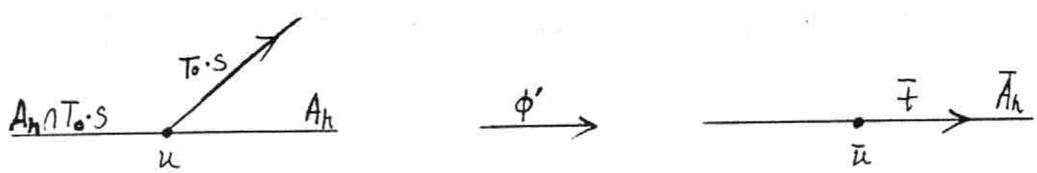

Fig. 4.

If $|T_0| \geq \tau(z) + c = \tau(z) + \tau(shs^{-1})$, then by Example 4.1, the action $T \times G \to T$ is not free. Assume $|T_0| < \tau(z) + c$, then $|\omega| = |\omega \cdot z| = |T_0| - \tau(z) < c$. So ω and $\omega \cdot z$ are embedded into $\overline{A}_g = \overline{A}_h$, therefore into Q'. Applying Example 4.2, we see that the action $T \times G \to T$ satisfies the Property (DF). ◇

Example 4.4: If there is an element $g \in G'$ such that $|D_g| \geq \tau(z)$ then the action $T \times G \to T$ is not free.

Proof: We know that σ_g is a translation or a reflection restricted to D_g. If $|D_g| \geq \tau(z)$, then there is a point $u \in D_g$ such that $u \cdot z \in D_g$. We have $(u \cdot z)\sigma_g = (u)\sigma_g \cdot z \in R_g$, if σ_g is a translation, and $(u \cdot z)\sigma_g \cdot z = (u)\sigma_g \in R_g$, if σ_g is a reflection. So we have either $u \cdot zgz^{-1}g^{-1} = u$ or $u \cdot zgzg^{-1} = u$, i.e. u is a fixed point. ◇

Fig. 5.

According to the Example 8.1 of Part 1, if $|T_0| < \tau(z)$, then the action $T \times G \to T$ is free and discrete. In view of this and Example 4.4, we can make the following

Assumption 4: $|T_0| \geq \tau(z)$.

Assumption 5: For each $g \in G'$, we have $|D_g| < \tau(z)$.

From Assumption 5, D_g, R_g are embedded into G'' by ϕ'' for each $g \in G'$ then

$$\Sigma'' = \{\overline{\sigma}_g | g \in G', D_g \neq \emptyset\}$$

which is a finite set since $|T_0| < \infty$.

Recall that in Section 2 we defined a relation r for two closed subtrees I and J as follows: $r(I, J)$ if and only if one of them consists of a single point, which is an endpoint of the other.

Assume σ, τ and θ are elements of $\overline{\Sigma}''$, $\sigma = \tau\theta$, then τ is called a **f-factors** of σ, and θ a **t-factor** of σ.

Example 4.5: Assume that any pair of elements $\sigma, \tau \in \overline{\Sigma}''$, $\sigma \neq \tau$ satisfy the following properties: if $D(\sigma) \cap D(\tau) \neq \emptyset$, then $r(D(\sigma), D(\tau))$ or σ, τ are f-factors of each other. If $D(\sigma) \cap D(\tau) \neq \emptyset$ and $R(\sigma) \cap R(\tau) \neq \emptyset$ then $r(D(\sigma), D(\tau))$ and $r(R(\sigma), R(\tau))$ or $\sigma = \tau$. Then the action $T \times G \to T$ satisfies (DF).

Corollary: If domains of any pair of elements of $\overline{\Sigma}''$ are disjoint or have the relation r, then (DF) is true for the action $T \times G \to T$.

Proof: We assume that the action $T \times G \to T$ is free.

According to Example 3.3, we only have to construct a reduced generating subset Σ_r satisfying the following properties: for $\sigma, \tau \in \Sigma_r$, $\sigma \neq \tau$, we have $r(D(\sigma), D(\tau))$ if $D(\sigma) \cap D(\tau) \neq \emptyset$ and $r(R(\sigma), R(\tau))$ if $R(\sigma) \cap R(\tau) \neq \emptyset$. To this end, we need the following three lemmas:

Lemma 4.6: *(a) If $w = \sigma_1 \sigma_2 \cdots \sigma_k$ is any word in elements of $\overline{\Sigma}''$, $D(w) \neq \emptyset$, $D(\sigma_i)$ is nondegenerate for each $i \leq k$, then $D(w) = D(\sigma_1)$ and σ_w is either an element of $\overline{\Sigma}''$ or the identity map on $D(\sigma_1)$.*

(b) For every element $\sigma \in \overline{\Sigma}''$, we have that $D(\sigma) \cap R(\sigma) = \emptyset$ and $D(\sigma^2) = \emptyset$.

(c) Suppose σ, τ are two elements of $\overline{\Sigma}''$, if $R(\sigma) \cap R(\tau) \neq \emptyset$, then either $r(R(\sigma), R(\tau))$ or σ, τ are t-factors of each other.

(d) w is as in (a), then $R(w) = R(\sigma_k)$.

Proof: (a) Suppose that $\sigma, \tau \in \overline{\Sigma}''$ be such that $D(\sigma)$ and $D(\tau)$ are both nondegenerate and $D(\sigma\tau) \neq \emptyset$, then $D(\sigma^{-1}) \cap D(\tau) \neq \emptyset$. Suppose $\sigma^{-1} \neq \tau$, since $r(D(\sigma^{-1}), D(\tau))$ can not be true, by the assumption, there is an element $\theta \in \overline{\Sigma}''$ such that $\tau = \sigma^{-1}\theta$. Because $D(\tau)$ is nondegenerate, so is $D(\theta)$. Then $\sigma\tau = \sigma\sigma^{-1}\theta = \theta|_{D(\sigma)}$. Because $D(\tau) \neq \emptyset$, $D(\theta) \cap D(\sigma) \neq \emptyset$, then we deduce as before that σ and θ are f-factors of each other, so they have the same domain, therefore $\sigma\tau = \theta$ is an element of $\overline{\Sigma}''$. If $\sigma = \tau^{-1}$, then $\sigma\tau$ is the identity map on $D(\sigma)$. From the above discussion, we can prove (a) by induction on the word length of w.

(b) Suppose there is an element $\sigma \in \overline{\Sigma}''$ such that $D(\sigma) \cap R(\sigma) \neq \emptyset$, if $D(\sigma)$ consists of one point, then $D(\sigma) = R(\sigma)$, so σ fixes the only point in its domain, this is impossible. Assume that $D(\sigma)$ is nondegenerate, then $D(\sigma)$ and $R(\sigma)$ do not have the relation r, by the conditions of this example, $\sigma = \sigma^{-1}$, therefore σ^2 is the identity map on $D(\sigma)$. Because the lift of σ^2 is not trivial, this is impossible by Lemma 3.6 (a). So, $D(\sigma) \cap R(\sigma) = \emptyset$ and consequently, $D(\sigma^2) = \emptyset$.

(c) The condition implies that $D(\sigma^{-1}) \cap D(\tau^{-1}) \neq \emptyset$. By the assumption of this example, either $r(D(\sigma^{-1}), D(\tau^{-1}))$, i.e. $r(R(\sigma), R(\tau))$ or σ^{-1} and τ^{-1} are f-factors of each other, then σ and τ are t-factors of each other.

(d) According to (c), the ranges of elements of Σ_k satisfy the same conditions for the domains. (d) is proved similar to (a). ◊

Lemma 4.7: *Any minimal generating subset Σ_0 of $\overline{\Sigma}''$ is a reduced generating subset.*

Proof: Assume we have a minimal generating subset Σ_0 of $\overline{\Sigma}''$ which is not a reduced generating subset. Then we have a reduced trivial word $w = \sigma_1\sigma_2 \cdots \sigma_k$ in element of Σ_0.

Because w is trivial, σ_w fixes a point in its domain. According to Lemma 3.1, $D(\sigma_i)$ is nondegenerate for $i \leq k$. We claim that $\sigma_i \neq \sigma_j$ if $i \neq j$.

Proof of the claim: Suppose there are integers $i < j$ such that $\sigma_i = \sigma_j$. We may assume there is no integer l between i and j with $\sigma_l = \sigma_i = \sigma_j$. If $j = i + 1$, then $D(w) = \emptyset$ since according to Lemma 4.6 (b), $D(\sigma_i^2) = \emptyset$, this contradicts the assumption. Assume $j > i + 1$, by Lemma 4.6 (a), if $w' = \sigma_{i+1} \cdot \sigma_{i+2} \cdots \sigma_{j-1}$, then $D(w') = D(\sigma_{i+1})$ is nondegenerate and $\tau = \sigma_{w'}$ is an element of $\overline{\Sigma}''$ or it is the identity map on $D(\sigma_{i+1})$.

Assume $\sigma_{w'} \in \overline{\Sigma}''$, since $w = \cdots \sigma_i \cdot w' \cdot \sigma_j \cdots$ has nonempty domain, $D(\sigma_i) \cap D(\tau^{-1}) = D(\sigma_j) \cap D(\tau^{-1}) \neq \emptyset$, and $R(\sigma_i) \cap R(\tau^{-1}) \neq \emptyset$, therefore $\sigma_i = \tau^{-1} = \sigma_{j-1}^{-1} \cdots \sigma_{i+1}^{-1}$, but we assumed that Σ_0 is a minimal generating subset of $\overline{\Sigma}''$, this is impossible. Suppose τ is the identity map on $D(\sigma_{i+1})$, then $D(\sigma_i\tau\sigma_j) \subset D(\sigma_i^2) = \emptyset$, this is impossible. So the claim is true.

Set $w_0 = \sigma_1 \cdot \sigma_2 \cdots \sigma_{k-1}$. Since w is a trivial word, the lift of w_0 can not be trivial, then by Lemma 3.6 (a), σ_{w_0} can not be an identity map. According to Lemma 4.6 (a), σ_{w_0} is an element of $\overline{\Sigma}''$ and $D(w_0) = D(\sigma_1)$. Since w fixes a point of its domain, we have $D(\sigma_k^{-1}) \cap D(w_0) = R(\sigma_k) \cap D(\sigma_1) \neq \emptyset$. By Lemma 4.6 (d), $R(w_0) = R(\sigma_{k-1})$, then $R(w_0) \cap R(\sigma_k^{-1}) = R(\sigma_{k-1}) \cap D(\sigma_k) \neq \emptyset$. Because all the sets involved are nondegenerate, we must have $\sigma_k^{-1} = \sigma_{w_0}$. Then σ_k is generated by $\sigma_1, \sigma_2, \ldots, \sigma_{k-1}$, contradiction the minimal generating property of Σ_0. This proves that there is no trivial word in elements of Σ_0, therefore Σ_0 is a reduced generating subset of $\overline{\Sigma}''$. ◊

Assume $\sigma, \tau \in \overline{\Sigma}''$, $(D(\sigma))^\circ \cap (D(\tau))^\circ \neq \emptyset$, then $D(\sigma)$ and $D(\tau)$ are nondegenerate, so they do not have relation r, therefore, σ, τ are f-factors of each other.

Lemma 4.8: *Assume that $\tau_1, \tau_2, \ldots, \tau_k$ and σ are elements of $\overline{\Sigma}''$, then there is an element $\sigma' \in \overline{\Sigma}''$ satisfying the following properties:*

(a) $\sigma = \sigma_w \sigma' \sigma_{w'}$, where w, w' are words in elements of the set $T = \{\tau_i | i = 1, 2, \ldots, k\}$, they could be the empty word.

(b) $(D(\sigma'))^\circ \cap (D(\tau_i))^\circ = \emptyset$ and $(R(\sigma'))^\circ \cap (R(\tau_i))^\circ = \emptyset$ for every $i \leq k$.

Proof: If $D(\sigma)$ is a one point set, then we take $\sigma' = \sigma$, w, w' be the empty word. Assume $D(\sigma)$ is nondegenerate. If there is a τ_i such that $(D(\sigma))^\circ \cap (D(\tau_i))^\circ \neq \emptyset$, then $\sigma = \tau_i \sigma_1$ for some element $\sigma_1 \in \overline{\Sigma}''$. Again, if $(D(\sigma_1)^\circ \cap (D(\tau_j))^\circ \neq \emptyset$ for some $\tau_j \in T$, then $\sigma_1 = \tau_j \sigma_2$, for some $\sigma_2 \in \Sigma$. In this way we get $\sigma_1, \sigma_2 \ldots$. We claim that there is a nonnegative number m such that no element of T is a f-factor of σ_m (if $m = 0$, $\sigma_m = \sigma$), which is equivalent to the fact that $(D(\sigma_m))^\circ \cap (D(\tau_i))^\circ = \emptyset$ for every $\tau_i \in T$. This claim is clear by the claim of Lemma 4.7, note that T is a finite set. We have $\sigma = \sigma_w \sigma_m$ where w is a word in elements of T.

From Lemma 4.6 (c), we deduce similarly to the above that there is a word w' in elements of T and an element σ' of $\overline{\Sigma}''$ such that $\sigma_m = \sigma' \sigma_{w'}$ and $(R(\sigma'))^\circ \cap (R(\tau_i))^\circ = \emptyset$, for every $\tau_i \in T$. Since no element of T can be a f-factor of σ', $(D(\sigma'))^\circ \cap (D(\tau_i))^\circ = \emptyset$ for every i. ◊

Given any minimal generating subset Σ_0 of $\overline{\Sigma}''$, we now construct another one in the following way:

Assume $\Sigma_0 = \{\sigma_j | j = 1, 2, \ldots, m\}$. Suppose that for some $i \leq m$, we have chosen a set $T_{i-1} = \{\sigma_j' | j \leq i-1\}$ of elements of $\overline{\Sigma}''$ such that for every $j < i$, there are words w_j, w_j' in elements of the set $T_{j-1} = \{\sigma_l' | l \leq j-1\}$ such that $\sigma_j = \sigma_{w_j} \sigma_j' \sigma_{w_j'}$ and the interiors of domains (ranges resp.) of elements in the set T_{i-1} are disjoint for each other. By Lemma 4.8, there exists an element σ_i' of $\overline{\Sigma}''$ and there are words w_i, w_i' in elements of T_{i-1} such that $\sigma_i = \sigma_{w_i} \sigma_i' \sigma_{w'}$ and elements in the set $T_i = \{\sigma_j' | j \leq i\}$ have disjoint interiors of domains and disjoint interiors of ranges. In this way, we choose σ_i'.

Elements of $T_m = \{\sigma_j' | j \leq m\}$ obviously generate the pseudo-group P''. We choose a minimal generating subset Σ_0' of T_m, then Σ_0' is a reduced generating subset of $\overline{\Sigma}''$ satisfying the conditions of Example 3.3, hence (DF) is true for the action $T \times G \to T$. ◇

Remark: the following three conditions are equivalent to each other:

(a) For any $\sigma, \tau \in \overline{\Sigma}''$, $\sigma \neq \tau$, if $D(\sigma) \cap D(\tau) \neq \emptyset$, then $r(D(\sigma), D(\tau))$, or σ and τ are f-factors of each other.

(b) For any $\sigma, \tau \in \overline{\Sigma}''$, $\sigma \neq \tau$, if $R(\sigma) \cap R(\tau) \neq \emptyset$, then $r(R(\sigma), R(\tau))$, or σ and τ are t-factors of each other.

(c) For any $\sigma, \tau \in \overline{\Sigma}''$, $\sigma \neq \tau^{-1}$, if $D(\sigma\tau) \neq \emptyset$, then $r(D(\sigma\tau), D(\sigma))$ and $r(R(\sigma\tau), R(\tau))$, or $\sigma\tau$ is an element of $\overline{\Sigma}''$ and $D(\sigma\tau) = D(\sigma)$.

Therefore in Example 4.5, we may replace the condition (a) by (b) or (c).

Assume that S_1, S_2 are subsets of T_0, if $S_1 \cdot z^m \cap S_2 \neq \emptyset$ for some integer m, then we write $S_1 \sim_z S_2$.

Example 4.9: Assume for any $\sigma_g, \sigma_h \in \Sigma'$, the following fact is true: if $D_g \sim_z D_h$, then $D_g = D_h$ or $r(D_g, D_h)$. Then the action $T \times G \to T$ satisfies the Property (DF).

Proof: Assume the action $T \times G \to T$ is free. Suppose $\overline{\sigma}_g, \overline{\sigma}_h \in \overline{\Sigma}''$ be such that $h \neq g$ and $(\overline{D}_g)^\circ \cap (\overline{D}_h)^\circ \neq \emptyset$, then $D_g \sim_z D_h$, so $D_g = D_h$ since, $r(D_g, D_h)$ can not be true, therefore, $\overline{D}_g = \overline{D}_h$. we have

$$\sigma_{g^{-1}}\sigma_h = \sigma_{g^{-1}h}|_{(D_g \cap D_h)\sigma_g} = \sigma_{g^{-1}h}|_{R_g}$$

and then

$$\overline{\sigma}_{g^{-1}}\overline{\sigma}_h = \overline{\sigma}_{g^{-1}h}|_{\overline{R}_g}$$

We have

$$\overline{\sigma}_h = \overline{\sigma}_h|_{\overline{D}_g} = \overline{\sigma}_g\overline{\sigma}_{g^{-1}}\overline{\sigma}_h = \overline{\sigma}_g\overline{\sigma}_{g^{-1}h}|_{\overline{R}_g} = \overline{\sigma}_g\overline{\sigma}_{g^{-1}h}$$

So $\overline{\sigma}_g$ is a f-factor of $\overline{\sigma}_h$, symmetrically, $\overline{\sigma}_h$ is a f-factor of $\overline{\sigma}_g$.

Next, assume that $(\overline{D}_g)^\circ \cap (\overline{D}_h)^\circ \neq \emptyset$ and $(\overline{R}_g)^\circ \cap (\overline{R}_h)^\circ \neq \emptyset$. Then $(\overline{D}_{g^{-1}})^\circ \cap (\overline{D}_{h^{-1}})^\circ \neq \emptyset$, as before, we have $\overline{D}_{g^{-1}} = \overline{D}_{h^{-1}}$, i.e. $\overline{R}_g = \overline{R}_h$. Then $\overline{\sigma}_{g^{-1}}\overline{\sigma}_h$ maps $\overline{R}_g = \overline{R}_h$ to its self, there must be a fixed point for this partial isometry. If $\overline{\sigma}_g \neq \overline{\sigma}_h$, then $g \neq h$, so $\overline{\sigma}_{g^{-1}}\overline{\sigma}_h$ has nontrivial lift, this contradicts Lemma 3.6 (a), impossible. Hence $\overline{\sigma}_g = \overline{\sigma}_h$.

Now we see that this example is a direct consequence of Example 4.5. ◇

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