A proof of Central Limit Theorem using continuous functions

0. For sequences of independent random variables X_{ni}, we want to proof that $Ef(\sum_{i=1}^n X_{ni})$ converges to $Ef(N(0,1))$ for *any given* smooth, C^3 function f. (as $n \to \infty$)

It then takes a little work/imagination to show this imply the conventional definition of convergence in distribution.

The moment generating functions (indexed by t), the characteristic functions are nice families of continuous functions.

The conditions we shall be imposing/refering to are

$X_{ni}, i = 1, \cdots, n$ are independent. $EX_{ni} = 0$, $\sum_{i=1}^n EX_{ni}^2 = 1$. (these conditions are WLOG)

Lindeberg: $\sum_i EX_{ni}^2 I_{[|X_{ni}|>\epsilon]} \to 0$ for any $\epsilon > 0$ as $n \to \infty$.

Liaponov: $\sum_i E|X_{ni}|^3 \to 0$, as $n \to \infty$.

1. Suppose a given function $f()$ is three times continuous differentiable. Also suppose X, Y, W are three r.v.s such that $EY = EW = 0$, $EY^2 = EW^2$ and X indep Y; X indep W.

Then $Ef(X+W)$ and $Ef(X+Y)$ do not differ by much. We have

$$|Ef(X+W) - Ef(X+Y)| \le C(E|Y|^3 + E|W|^3)$$

where the constant C can taken to be...(depends on f). (prove this by using Taylor's expansion).

2. If Y is normal then $E|Y|^3 \le KE|W|^3$ where the constant K can taken to be $E|N(0,1)|^3$. (In fact, this is true for any power larger than 2). [Jensen's inequality].

2.5. If Y_{ni} $i = 1, 2, ...n$ are normal, independent r.v.s such that $EY_{ni} = 0$ and $\sum_{i=1}^n EY_{ni}^2 = 1$. Under what conditions we have $\sum \sigma_{ni}^3 \to 0$? ($\sigma_{ni}^2 = EY_{ni}^2$) Give examples that it converges to 0 and not converges to 0. (under Lindeberg condition it does. Ironically the Lindeberg condition is for the W's. but Y and W have same variances.)

3. Now suppose X_{ni} $i = 1, 2, \cdots n$ are independent random variables with mean zero and variances σ_{ni}^2 that sum to one: ($\sum_i \sigma_{ni}^2 = 1$).

Notice $N(0,1)$ can be written as the sum of a sequence of independent normal random variables: $N(0,1) = \sum_{i=1}^n N(0, \sigma_{ni}^2)$. [at least in distribution]

We want to show the expectation of $f(\sum X_{ni})$ is very close to the expectation of $f(\sum N(0, \sigma_{ni}^2)) = f(N(0,1))$.

Using the inequality in (1) above repeatedly (one term at a time) we can get

$$|Ef(\sum X_{ni}) - Ef(N(0,1))| \le C(\sum E|X_{ni}|^3 + E|N(0, \sigma_{ni}^2)|^3)$$

Using the inequality in 2 we have

$$|Ef(\sum X_{ni}) - Ef(N(0,1))| \le K(\sum E|X_{ni}|^3)$$

Under Liaponov condition the convergence is proved.

4. We have shown $Ef(\sum X_{ni})$ converge to $Ef(N(0,1)$ for *any* C^3 function. Must be true for any $sin()$, $cos()$ functions, therefore must be true for characteristic functions or moment generating functions (if the expectation exist).

5. Finally, it can be shown that Liaponov condition can be replaced by Lindberg condition. It is believed the Lindberg condition is almost necessary (cannot be relaxed).