1. (30) Suppose random variables (X, Y) have a joint probability density function given by

$$f(x, y) = c\, e^{-(2x+y)}; \quad \text{for } \ y \geq x \geq 0.$$

(a) find the constant c that make f a density.

(b) find the two marginal p.d.f.s.

(c) find the joint p.d.f. of (U, V) with

$$U = X + Y, \quad V = Y - X.$$

(d) find the marginal distribution or p.d.f. of $V = Y - X$.

(e) find the conditional density of V given U.

2. (25) Let X be a Poisson(λ) random variable. Show that, after proper standardization, as $\lambda \to \infty$ we have X goes in distribution to a normal random variable.

3. (25) State and proof the Chebychev inequality.

4. (8) If $X \sim N(2, 1)$ and $Y \sim N(0, \sigma^2 = 4)$ and they are independent. Please compute $Var(XY) =?$.

5. (12) Suppose $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of exponential ($\lambda = 3$) random variables.

Show that

$$\frac{X_n}{\sqrt{n}} \longrightarrow 0$$

almost surely as $n \to \infty$.

solution: For any $\epsilon > 0$, we compute $P(|X_n/\sqrt{n}| > \epsilon) = P(X_n > \epsilon\sqrt{n}) = e^{-3\epsilon\sqrt{n}}$.

Since $\sum_{n=1}^{\infty} e^{-3\epsilon\sqrt{n}} < \infty$ for any $\epsilon > 0$, we have $P(A_n\ i.o.) = 0$ where $A_n = \{|X_n/\sqrt{n}| > \epsilon\}$. This is the almost sure convergence we want.

One last question to think about: Show that

$$\frac{\max_{1 \leq i \leq n}\{X_i\}}{\sqrt{n}} \longrightarrow 0$$

almost surely as $n \to \infty$.