If $f(a)$ is a Countinuous function for $a \in R^1$, and its derivative always exists except on points $\{c_1, c_2 \cdots c_K\}$;

then any (local) min point of $f(a)$ must satisfy: either

① derivative $= 0$;

OR ② derivative do not exist, (i.e. one of $\{c_1 \cdots c_K\}$ points)

but the derivative to the immediate left must be negative, and derivative to the immediate right must be positive.

Let us compute the derivative (wrt a) of the expectation.

$$\frac{\partial}{\partial a}\left[\sum_{i=1}^{K}\frac{1}{K}|C_i-a|\right] = \sum \frac{1}{K}|C_i-a|'$$

$$= \sum_{i=1}^{K}\frac{1}{K}\left(I[C_i<a]-I[C_i>a]\right)$$

(except those points
$C_1\ C_2 \cdots C_K$ that derivative do not exist)

notice for small a, most $C_i > a$, and the derivative is negative. For large a, most $C_i < a$ and the derivative is positive.

So, this function is decreasing for small a, but increasing for large a. Thus, in the middle, when it change from decreasing to increasing, that will be the minimum.
[When derivative change from (<0) to (>0).

[At the point of change, the derivative may not exist.]
and it may or may not attain the value of zero when change from <0 to >0.]

[In fact, we can show this is a convex function]

[try to think what if prob $= p_i$ instead of $\frac{1}{K}$.]

#2

$P(X=0) = 1-P$

$P(X=1) = P/2$

$P(X=-1) = P/2$

So, if we observe $X=0$, the likelihood is $f(p|x=0)=1-P$

So to make it large, p needs to be zero.

If we observe $x=1$ the likelihood is $f(p|x=1) = P/2$.

To make it large, p needs to be 1.

Ditto for $X=-1$

So the MLE of p is $\hat{p} = \begin{cases} 0 & \text{if } x=0 \\ 1 & \text{if } x=1, \text{ or } -1 \end{cases}$

(b) $\mathbb{E}\, T(x) = 2 \cdot \mathbb{E}\, I[X=1] = 2 \times P(x=1) = 2 \times P/2 = P$

$Var\ T(x) = 4\ Var\ I[X=1] = 4 \times [\frac{P}{2}(1-\frac{P}{2})]$

(c) $\mathbb{E}\, K(x) = \mathbb{E}\, I[X=-1 \text{ or } +1] = P(X=-1 \text{ or } +1) = P/2 + P/2 = p$

$Var\ K(x) = p(1-p)$

easy to see $Var\ K(x) \leq Var\ T(x)$ and "<" is strick

unless $p=0$, in which case both $Var = 0$.