If $X_1 \sim Poi(\lambda)$ and $X_2 \sim Poi(3\lambda)$ and indep. then

$$f(x_1, x_2) = e^{-\lambda} \frac{\lambda^{x_1}}{x_1!} \cdot e^{-3\lambda} \frac{(3\lambda)^{x_2}}{x_2!}, \quad \text{for } x_1 = 0, 1, 2, \cdots \quad x_2 = 0, 1, 2, \cdots$$

$$= e^{-4\lambda} \underbrace{\frac{1}{x_1!\, x_2!} 3^{x_2}}_{h(x_1 x_2)} \cdot \underbrace{\lambda^{x_1 + x_2}}_{g(T(x), \theta)}$$

$$= c(\theta) \cdot h(x_1 x_2) \cdot g(T(x), \theta)$$

by factorization Lemma, $T = x_1 + x_2$ is suff for λ.

Suppose $X_1 X_2 \cdots X_n \sim$ iid $exp(\lambda)$, $\mathbb{E} X_1 = \frac{1}{\lambda}$, $Var(X_1) = \frac{1}{\lambda^2}$

By WLLN for iid sum,

$$\bar{X} \xrightarrow{P} \mathbb{E} X_1 = \frac{1}{\lambda}.$$

Also by CLT for iid sum

$$\sqrt{n}\left(\bar{X} - \frac{1}{\lambda}\right) \xrightarrow{\mathcal{D}} N\left(0, Var = \frac{1}{\lambda^2}\right).$$

Using $f(x) = \frac{1}{x}$ and Delta method, we have

$$\sqrt{n}\left(f(\bar{x}) - f\left(\tfrac{1}{\lambda}\right)\right) \xrightarrow{\mathcal{D}} N\left(0, Var = \left[f'\left(\tfrac{1}{\lambda}\right)\right]^2 \cdot \frac{1}{\lambda^2}\right)$$

that is $\sqrt{n}\left(1/\bar{X} - \lambda\right) \xrightarrow{\mathcal{D}} N\left(0, \left[\left(\tfrac{1}{\lambda}\right)^{-1}\right]^4 \cdot \frac{1}{\lambda^2}\right) = N(0, \lambda^2)$.

$$\sqrt{n}\left[\frac{1}{\bar{X}} - \lambda\right] \Big/ ? \xrightarrow{\mathcal{D}} N(0, 1)$$

According to Slutsky Th. ? can be taken as anything that Converges to (in prob) λ will be suffice.

So, we may take (*)

$$? = 1/\bar{X} \quad , \quad \text{then by Slutsky}$$

$$\sqrt{n}\left[\frac{1}{\bar{X}} - \lambda\right] \Big/ (\bar{X})^{-1} \xrightarrow{\mathcal{D}} N(0, \lambda^2) \Big/ \left(\frac{1}{\lambda}\right)^{-1} = N(0, 1)$$

$\boxed{(*) \text{ By Cont. mapping, } \frac{1}{\bar{X}} \xrightarrow{P} \lambda .}$

So,

$$\sqrt{n}\left(\frac{1}{\bar{X}} - \lambda\right)\bar{X} \xrightarrow{\mathcal{D}} N(0, 1) \quad \text{as } n \to \infty$$

HW6

We compute

$$\mathbb{E}\min(X_i) = \theta + \frac{1}{n+1}$$

$$\mathbb{E}\max(X_i) = \theta + \frac{n}{n+1}$$

thus, define

$$g(\min(X_i), \max(X_i)) = \max(X_i) - \min(X_i) - \frac{n-1}{n+1}$$

We have

$$\mathbb{E}\, g(\min(X_i), \max(X_i)) = 0 \quad \text{for all } \theta$$

but clearly $g(\cdot, \cdot)$ is not always 0, [it is a random variable with non-degenerate dist]