2008 Q7:

a. first, WLOG assume all r.v. have mean 0. [otherwise subtract the mean]

The Lindeberg Condition: $\forall\, \varepsilon > 0$

$$(*) \qquad \sum_{k=1}^{n} \frac{\mathbb{E}(X_{nk})^2}{\sigma_n^2} I[\,|X_{nk}| > \varepsilon\,\sigma_n] \longrightarrow 0 \quad \text{as } n \to \infty$$

We need to verify this with the given condition.

Since $|X_{nk}| < M$, $\Rightarrow$ $(X_{nk})^2 \leq M^2$

$$\text{So } (*) \leq \sum_{k=1}^{n} \frac{M^2}{\sigma_n^2} \mathbb{E} I[|X_{nk}| > \varepsilon \sigma_n] = \frac{M^2}{\sigma_n^2} \sum_{k=1}^{n} P(|X_{nk}| > \varepsilon \sigma_n)$$

from here we can either use Chebychev ineq. on the probability (as we did in class)

OR, argue that, since $\sigma_n^2 \to \infty$, therefore σ_n will be larger than M/ε for all sufficient large n, and thus,

"$|X_{nk}| > \varepsilon \sigma_n$" is impossible. [left hand is at most M, right hand is larger than M]

$\Rightarrow$ the prob $= 0$,

$\Rightarrow$ $P(|X_{nk}| > \varepsilon \sigma_n)$ for those large n, is 0. $(\forall k)$

$$\Rightarrow \lim_{n \to \infty} \frac{M^2}{\sigma_n^2} \sum_{k=1}^{n} P(|X_{nk}| > \varepsilon \sigma_n) = 0$$

b. First, rewrite the estimator $\hat{f}_n(x)$ as

$$\hat{f}_n(x) = \frac{1}{2b_n} \cdot \frac{1}{n} \sum_{i=1}^{n} I_{[x-b_n < X_i \le x+b_n]}$$

$$= \sum_{i=1}^{n} \frac{1}{2b_n \cdot n} I_{[x-b_n < X_i \le x+b_n]} = \frac{\sum_{k=1}^{n} Y_{nk}}{2b_n \cdot n} \quad (\text{say})$$

where Y_{nk} are indep. r.v.s, and $I[\quad]$ is a bounded r.v.

(always less than or eq to 1)

Using the result of (a), we have, (i) if the $Var\left(\sum I[x-b_n < X_i \le x+b_n]\right) \to \infty$, then

$$\frac{\sum_{i=1}^{n} I[x-b_n < X_i \le x+b_n] - E(\ )}{\sigma_n} \to N(0,1)$$

where.

$$\sigma_n^2 = Var\left(\sum I[\quad]\right) = \sum_{i=1}^{n} P(x-b_n < X_i \le x+b_n) \cdot [1-P(\quad)]$$

$$= n \cdot P(x-b_n < X_1 \le x+b_n) \cdot [1-P(x-b_n < X_1 \le x+b_n)]$$ by iid-ness, by Var of Bernoulli

$$\doteq n \cdot f(\xi) \cdot 2b_n \cdot [1-f(\xi) \cdot 2b_n]$$ by mean value Th of integral

i.e. $\int_a^b g(t)dt = g(\xi)(b-a)$

As $n \to \infty$,

$n \cdot f(\xi) 2b_n \to \infty$ using the conditions given, $\left[\begin{matrix} nb_n \to \infty \\ f(x) > 0 \end{matrix} \right]$

$[1 - f(\xi) 2b_n] \to 1$, since $b_n \to 0$

Therefore

$$Var\left(\sum_{i=1}^{n} I[x-b_n < X_i \leq x+b_n]\right) \longrightarrow \infty \cdot 1 = \infty, \quad \text{as } n \to \infty$$

This sequence satisfy all conditions needed in ⓐ, so,

$$\frac{\sum_{i=1}^{n} I[x-b_n < X_i \leq x+b] - E(\ )}{\sigma_n} \to N(0,1)$$

i.e.

$$(*) \quad \frac{\sum I[x-b_n < X_i \leq x+b_n] - E(\ )}{\sqrt{n \cdot f(\xi) 2b_n \cdot 1}} \to N(0,1)$$

by Slutsky, and

$f(\xi) \approx f(x)$, because $f(\cdot)$ is Cont. at x.
smooth

and we easily check $(*)$ is same as the statement of (b).

(c) We just need to check

$$(n2b_n)^{1/2}\left[E(\hat{f}(x)) - f(x)\right] \longrightarrow 0, \quad \text{Recall } E(I\ ) = P(\quad)$$

$= P(x-b_n < X_1 \leq x+b_n) = f(\xi) 2b_n$; thus $E\hat{f}(x) = f(\xi)$.

$\Rightarrow \quad E\hat{f}(x) - f(x) = f(\xi) - f(x) = f'(\eta)(\xi - x)$; Assume $f'(\cdot)$ exist therefore

$$|\mathbb{E}\hat{f}(x) - f(x)| \leq |f'(\eta)||\xi - x| \leq |f'(\eta)| \, 2b_n$$

therefore

$$(2nb_n)^{1/2} \cdot |\mathbb{E}\hat{f}(x) - f(x)| \leq (2nb_n)^{1/2} \cdot |f'(\eta)| \cdot 2b_n$$

$$= (nb_n)^{1/2} \cdot b_n \cdot 2 \cdot \sqrt{2} \, |f'(\eta)|$$

Assume $f'(\cdot)$ is finite.

Assume $(nb_n)^{1/2} b_n \to 0$ as $n \to \infty$, then we are O.K.

To summarize Conditions on b_n:

$$\left.\begin{array}{l} b_n \to 0 \\ n b_n \to \infty \\ (nb_n)^{1/2} b_n \to 0 \end{array}\right\}$$

for example $b_n = \frac{1}{\sqrt{n}}$

$b_n =$ [illegible] will work for all 3

Check

(i) $\frac{1}{\sqrt{n}} \to 0$, ✓, (ii) $n \cdot b_n = \sqrt{n} \to \infty$, o.k, ✓

(iii) $(nb_n)^{1/2} b_n = (\sqrt{n})^{1/2} \cdot \frac{1}{\sqrt{n}} = \frac{n^{1/4}}{n^{1/2}} = \frac{1}{n^{1/4}} \to 0$, O.K. ✓