SCATTERING FOR CRITICAL WAVE EQUATIONS WITH VARIABLE COEFFICIENTS

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ABSTRACT. We prove that solutions to the quintic semilinear wave equation with variable coefficients in $\mathbb{R}^{1+3}$ scatter to a solution to the corresponding linear wave equation. The coefficients are small and decay as $|x| \to \infty$, but are allowed to be time dependent. The proof uses local energy decay estimates to establish the decay of the L^6 norm of the solution as $t \to \infty$.

CONTENTS

1. INTRODUCTION

In Minkowski space, solutions of the equation $\Box u = |u|^{p-1}u$ with $\Box = -\partial_t^2 + \Delta$ have a conserved and positive-definite energy

$$E(t) = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{p+1}|u(t,x)|^{p+1}\,dx$$

and the scaling symmetry

$$u(t,x) \mapsto \lambda^{\frac{2}{1-p}} u(\frac{t}{\lambda}, \frac{x}{\lambda}).$$

In three dimensions, the exponent $p = 5$ is called the energy-critical exponent, because solutions of the equation have an energy that is invariant under the scaling symmetry.

For the Cauchy problem with initial data in the energy space $\dot{H}^1 \times L^2$, local well-posedness is proven for $1 < p \leq 5$ by Strichartz estimates. Global existence for small initial data is a straightforward adaptation of the proof of local existence. In addition, there is global existence for large initial data due to the existence of a blowup criterion, which informally says that the energy cannot concentrate at any point in spacetime. Moreover, given any finite-energy initial data there is a unique global solution with finite energy lying in $L^4L^{12}([0,\infty) \times \mathbb{R}^3)$; these solutions are known as strong (or Shatah-Struwe) solutions. See [5],[6],[7],[8],[13],[28],[29],[30],[31],[32],[33],[36] for details and more. The results in [2] and [1] then combine to prove scattering of solutions with finite-energy initial data using a profile decomposition, which describes the failure of a sequence of uniformly bounded solutions to the free wave equation to be compact in the sense of Strichartz estimates. A similar result holds for the focusing equation with energy below that of the ground state: see [14].

This paper considers the equation

$$\begin{cases} Pu(t,x) = u(t,x)^5 \quad (t,x) \in (0,\infty) \times \mathbb{R}^3, \quad P = \partial_\alpha g^{\alpha\beta} \partial_\beta \\ u[0] \in \dot{H}^1 \times L^2 \end{cases} \tag{1.1}$$

Global existence and uniqueness of strong solutions (lying in $C(\mathbb{R}_t, \dot{H}^1) \cap L^5_{loc}L^{10}$) was shown in [9] in the stationary setting. A similar result for classical solutions in the non-stationary setting was shown in [20]. These results require minimal assumptions on the coefficients, as eliminating the blowup scenario only requires local-in-time arguments.

Our main theorem establishes scattering of strong solutions to (1.1) for certain small, asymptotically flat perturbations of the Minkowski metric. To the authors' knowledge, this is the first such result for small perturbations of the Minkowski metric m with variable coefficients.

Definition 1.1 (Scattering in the energy space)**.** We say that the solution u to (1.1) *scatters in the energy space* if there exists $(f, g) \in \dot{H}^1 \times L^2$ such that

$$\lim_{t\to\infty} \|u[t] - S(t,0)(f,g)\|_{\dot{H}^1\times L^2} = 0.$$

Definition 1.2. We define $\underline{L} := \sum_{i=1}^{3} \frac{x^i}{|x|}\partial_{x_i} - \partial_t$.

Theorem 1.3. *Let $g^{\alpha\beta}(t,x)$ be a Lorentzian metric, let $P = \partial_\alpha g^{\alpha\beta}\partial_\beta$, and let $h := g - m$ denote the perturbative terms of the metric g. The unique global strong solution to the Cauchy problem* (1.1) *scatters in the energy space $\dot{H}^1 \times L^2$ provided that*

$$|h| \lesssim \epsilon \frac{\langle t - |x|\rangle^{1/2}}{\langle x\rangle^{\gamma}\langle t + |x|\rangle^{1/2}}, \tag{1.2}$$

$$|h^{\underline{L}\underline{L}}| \lesssim \epsilon \frac{\langle t - |x|\rangle}{\langle x\rangle^{\gamma}\langle t + |x|\rangle}, \tag{1.3}$$

$$|\partial^J h| \lesssim \epsilon \frac{1}{\langle x\rangle^{|J|+\gamma}} \quad \text{for } |J| = 1 \text{ and } |J| = 2 \tag{1.4}$$

where $\gamma > 0$ is an arbitrarily small constant and $\epsilon > 0$ is a sufficiently small constant. In these assumptions, $\partial^J h$ denotes $\partial^J h^{\alpha\beta}$ for all multi-indices α and β, and $h^{\underline{L}\underline{L}} = h^{\alpha\beta}\underline{L}_\alpha\underline{L}_\beta$, where we lower indices with respect to the Minkowski metric.

This says that the unique global solution of the non-linear problem on small, asymptotically flat perturbations of Minkowski space that have appropriate decay at infinity behave, in the asymptotic sense, like the solution to the linear homogeneous problem $Pu = 0$, at least in the energy space.

Remark 1.4. The assumptions (1.2), (1.3), and (1.4) are satisfied by metrics that arise as solutions to Einstein's Vacuum Equations when expressed in harmonic coordinates, see [22].

Remark 1.5. One of the key ingredients in our proof is the fact that Strichartz estimates for the linear problem hold. Assuming

$$|\partial^J h| \lesssim \epsilon \frac{1}{\langle x\rangle^{|J|+\gamma}} \quad \text{for } 0 \le |J| \le 2$$

this was proved by Metcalfe-Tataru [25]. Our assumptions are the same, except that we require more decay of h (but not its derivatives) near the light cone. This is due to the fact that we need to control certain boundary terms that appear when multiplying the equation by $t\partial_t + r\partial_r$. In particular the extra decay requirement (1.3) is needed to control the term $|\underline{L}u|^2$ on the boundary, and geometrically it implies that the light cones of the perturbed metric are comparable to those of the Minkowski metric.

Remark 1.6. Theorem 1.3 also holds if we replace P by the geometric wave operator

$$\Box_g = \frac{1}{\sqrt{|g|}}\partial_\alpha \sqrt{|g|} g^{\alpha\beta}\partial_\beta, \quad |g| := |\det g^{\alpha\beta}|.$$

Indeed, in the estimates below one integrates with respect to the volume form $\sqrt{|g|}dtdx$, and uses the fact that $\sqrt{|g|}g^{\alpha\beta} \approx g^{\alpha\beta}$. There are extra error terms of the form $\partial\sqrt{|g|}u^6$ arising, which can be absorbed since by (1.4) we have

$$\partial\sqrt{|g|} \lesssim \frac{\epsilon}{\langle x\rangle^{1+\gamma}}$$

Remark 1.7. A key tool in proving scattering on variable-coefficient backgrounds is local energy decay. Such an estimate was proven in [27], [35], and [15] for Minkowski space and in [23], [24] for perturbations of Minkowski space, and became a valuable tool in the study of both linear and nonlinear problems. In particular, they imply Strichartz estimates on certain variable-coefficient backgrounds, see [25]. Our result is one of several showing that local energy decay is fruitful for understanding the long-time behavior and asymptotics of solutions to nonlinear dispersive equations on variable-coefficient backgrounds.

Remark 1.8. For the energy-critical problem on Minkowski space, global a priori estimates were proven in [28], [29], from which scattering for the wave and Klein-Gordon equations were deduced. Analogous results in the exterior of obstacles were obtained in [15], [34], [3], [4]. Scattering on Riemannian manifolds for a class of non-trapping obstacles close to the two-convex framework has been shown as well for the energy-critical non-linear problem ([19]). Profile decompositions akin to [1] have been shown for waves on hyperbolic space in [21]; a similar result was shown for $(\Box + a|x|^{-2})u = u^5$ in [26]. Finally, for the equation $(\Box + V(x))u = u^5$, scattering to steady states was shown in [11], [12].

For the nonlinear Schrödinger equation, the energy-critical problem for the defocusing quintic problem with initial data in the energy space for small, compactly supported perturbations of the Euclidean metric also exhibits scattering to linear solutions for all finite-energy data, as shown in [10]. There are many other known results for the energy-critical energy Schrödinger with potential, and for the exterior of a strictly convex obstacle, see [37], [17], [18] etc.

2. Notation and Preliminaries

We fix the spatial dimension to be $d = 3$ and define $\Box = -\partial_t^2 + \Delta$ and $P = \partial_\alpha g^{\alpha\beta}\partial_\beta$ where $g = g(t,x)$ is a Lorentzian metric. We write either $X \lesssim Y$ or $X = O(Y)$ to indicate that

$$|X| \le CY$$

(rather than $X \le CY$) for some absolute constant C which may vary by line. Similarly, $X \approx Y$ means that there are constants $0 < C_1 < C_2$ so that

$$C_1|X| \le |Y| \le C_2|X|.$$

We let $\langle r\rangle = \langle x\rangle = (1 + |x|^2)^{1/2}$. We write $\nabla = (\partial_t, \nabla_x)$ for the spacetime gradient. Throughout the paper, we use the Einstein summation convention, and we let Greek (resp. Latin) indices denote spacetime (resp. space) indices. We write $u[T] = (u(T,x), \partial_t u(T,x))$.

The energy of the solution u is defined to be

$$E(t) = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{6}u(t,x)^6\,dx.$$

We will also use the notation

$$E_K(t) := \int_K \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{6}u(t,x)^6\,dx$$

for some subset K of $\mathbb{R}^3$.

For any $(f,g) \in \dot{H}^1 \times L^2$, we denote by $S(t,s)(f,g)$ the unique solution $u \in C(\mathbb{R}_t, \dot{H}^1)$ with $\partial_t u \in C(\mathbb{R}_t, L^2)$ to the equation

$$\begin{cases} Pu = 0 & (t,x) \in (s,\infty) \times \mathbb{R}^3 \\ u[s] = (f,g) \end{cases} \tag{2.1}$$

Let

$$X = \left(C(\mathbb{R}_t, \dot{H}^1) \cap L^5_{loc} L^{10} \right) \times C(\mathbb{R}_t, L^2)$$

and for any closed, finite interval

$$X(I) = \left(C(I, \dot{H}^1) \cap L^5(I) L^{10} \right) \times C(I, L^2)$$

Consider the Cauchy problem

$$\begin{cases} Pu(t,x) = u(t,x)^5 & (t,x) \in (0,\infty) \times \mathbb{R}^3 \\ u[0] \in \dot{H}^1 \times L^2 \end{cases} \tag{2.2}$$

By Duhamel's formula, classical solutions to (2.2) satisfy

$$u(t) = S(t,0)u[0] + \int_0^t \frac{1}{g^{00}} S(t,s)(0, u^5(s)) ds \tag{2.3}$$

We can thus define a strong solution to be a solution of (2.3) so that $(u, \partial_t u)$ also lies in X.

The results of [9] and [20] show that, for smooth initial data, there is a unique global classical solution to (2.2) that is also a strong solution. Moreover, this result is extended to initial data in the energy space in [9] for time-independent coefficients, and the same argument can be used to prove it in the time-dependent case. We will be interested in studying the asymptotic properties of the unique strong solution in the energy space, in particular the fact that it approaches a solution to the linear equation in the energy space.

3. Energy conservation and local energy decay in Minkowski space

In order to motivate the discussion that follows, we devote this section toward certain key results for the linear problem in the setting of Minkowski space. Consider the Cauchy problem

$$\begin{cases} \Box u = 0 & (t,x) \in (0,\infty)\times\mathbb{R}^3 \\ u[0] \in \dot{H}^1 \times L^2 \end{cases} \tag{3.1}$$

The energy of the solution u to (3.1) is defined to be

$$E^{lin}(t) = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u(t,x)|^2\, dx$$

and it is conserved: for all $T \geq 0$, $E^{lin}(T) = E^{lin}(0) = E$.

The solution u to (3.1) also satisfies the local energy estimate

$$\iint_{[0,T]\times\mathbb{R}^3} \frac{|\nabla u|^2}{\langle r\rangle^{1+\gamma}} + \frac{u^2}{\langle r\rangle^{3+\gamma}}\, dxdt \lesssim E$$

where $\gamma > 0$ is an arbitrarily small fixed constant. This estimate is proven by multiplying both sides of the equation by $a(r)u + b(r)\partial_r u + C\partial_t u$ in the region $[0,T]\times\mathbb{R}^3$, with:

$$b(r) = \sum_{j=0}^{\infty} 2^{-j\gamma}\frac{r}{r+2^j},$$

where for each j, this is a function catered to the region $r \approx 2^j$ with the factor $2^{-j\gamma}$, where the small number $\gamma > 0$ is introduced in order to obtain convergence of the series; $a(r) = b(r)/r$; and $C > 0$ is a constant chosen to be sufficiently large.

More precisely, one obtains

$$\iint_{[0,T]\times\mathbb{R}^3} \Box u b(r)\partial_r u\, dxdt = -\iint b'(\partial_r u)^2 + \frac{b}{r}|\partial_\omega u|^2 + \frac{1}{2}(b' + 2\frac{b}{r})(u_t^2 - |\nabla_x u|^2)\, dxdt$$
$$+ \int_{\mathbb{R}^3} -b\partial_r u\partial_t u|_0^T\, dx,$$

where $|\partial_\omega u|^2 := |\nabla_x u|^2 - |\partial_r u|^2$, and

$$\iint_{[0,T]\times\mathbb{R}^3} \Box u a(r) u\, dxdt = \iint a(u_t^2 - |\nabla_x u|^2) + \frac{1}{2}\Delta a u^2\, dxdt + \frac{1}{2}\int_{\mathbb{R}^3} -au\partial_t u|_0^T\, dx.$$

Since $a = O(\langle r\rangle^{-1})$ and $b = O(1)$, Hardy's inequality $\int_{\mathbb{R}^3} u^2/r^2\, dx \lesssim \int_{\mathbb{R}^3} |\nabla_x u|^2\, dx$ shows that there exists a sufficiently large constant $C > 0$ such that

$$\int_{\{t\}\times\mathbb{R}^3} b\partial_t u \partial_r u + a\partial_t u u + C|\nabla u|^2\, dx \approx E^{lin}(t)$$

for all $t \geq 0$. We obtain

$$E^{lin}(T) + \iint_{[0,T]\times\mathbb{R}^3} \frac{1}{2}b'(u_r^2 + u_t^2) - (\frac{1}{2}b' - \frac{b}{r})|\partial_\omega u|^2 - \frac{\Delta a}{2}u^2\, dxdt \lesssim E^{lin}(0)$$

One can check directly that

$$b' \gtrsim \langle r\rangle^{-1-\gamma}, \quad b/r - \frac{1}{2}b' \gtrsim \langle r\rangle^{-1-\gamma}, \quad -\Delta a \gtrsim \langle r\rangle^{-3-\gamma}$$

and thus

$$\frac{1}{2}b'(u_r^2 + u_t^2) + (-\frac{1}{2}b' + \frac{b}{r})|\partial_\omega u|^2 - \frac{\Delta a}{2}u^2 \gtrsim \frac{|\nabla u|^2}{\langle r\rangle^{1+\gamma}} + \frac{u^2}{\langle r\rangle^{3+\gamma}} \tag{3.2}$$

which finishes the proof.

4. Uniform energy bounds and local energy decay for the nonlinear problem on perturbations

We now come to certain key tools, analogous to the results presented in the previous section, that will be used in the proof of scattering for the non-linear problem on certain perturbations of Minkowski space that have appropriate decay at infinity.

If $g(t,x)$ is a non-stationary metric that satisfies certain decay conditions, and u is a solution of the Cauchy problem (2.2), then we have uniform energy bounds: with the energy now defined to be

$$E(t) := \int_{\{t\}\times\mathbb{R}^3} \frac{1}{2}|\nabla u|^2 + \frac{1}{6}|u|^6\, dx,$$

if g and its derivatives satisfy certain decay conditions, then

$$E(T) \lesssim E := E(0)$$

for some implicit constant that is independent of T. In fact, we may prove local energy decay and uniform energy bounds in one fell swoop for (2.2), as the following proposition shows.

Theorem 4.1 (Integrated local energy decay for the nonlinear Cauchy problem)**.** *Let u be a solution of (2.2) and let $|J| \leq 1$ be a multi-index. If $\partial^J h^{\alpha\beta} \lesssim \epsilon \langle r \rangle^{-|J|-\gamma}$ where $\gamma > 0$ is an arbitrarily small constant and $\epsilon > 0$ is a sufficiently small constant then*

$$\|u\|^2_{LE^1[T_1,T_2]} + E(T_2) \lesssim E(T_1) \tag{4.1}$$

for some implicit constant that is independent of T_1 and T_2, where

$$\|u\|^2_{LE^1[T_1,T_2]} = \iint_{[T_1,T_2]\times\mathbb{R}^3} \frac{|\nabla u|^2}{\langle r \rangle^{1+\gamma}} + \frac{u^2}{\langle r \rangle^{3+\gamma}} + \frac{|u|^6}{\langle r \rangle} \, dxdt$$

Let us first assume that u is a classical solution to the equation

$$Pu = u^5 + F$$

Following [24] and the discussion in the previous section, we multiply the equation by $a(r)u + b(r)\partial_r u + C\partial_t u$, with

$$b(r) = \sum_{j=0}^{\infty} 2^{-j\gamma} \frac{r}{r+2^j}, \quad a(r) = b(r)/r.$$

Upon integrating by parts, we obtain

$$E(T_2) + \iint_{[T_1,T_2]\times\mathbb{R}^3} \frac{1}{2} b'(u_r^2 + u_t^2) - (\frac{1}{2}b' - \frac{b}{r})|\partial_\omega u|^2 - \frac{\Delta a}{2} u^2 + Err$$
$$+ (\frac{2}{3}a(r) - \frac{1}{6}b'(r))u^6 \, dxdt \lesssim E(T_1) + \iint_{[T_1,T_2]\times\mathbb{R}^3} |F|(|\nabla u| + \frac{|u|}{\langle r \rangle}) \, dxdt.$$

where the error satisfies

$$Err \lesssim (\frac{|h|}{\langle x \rangle} + |\nabla h|)(|\nabla u|^2 + |\nabla u| \frac{|u|}{\langle x \rangle})$$

Since $\partial^J h^{\alpha\beta} \lesssim \epsilon \langle r \rangle^{-|J|-\gamma}$, we can estimate by Cauchy-Schwarz

$$Err \lesssim \epsilon \Big(\frac{|\nabla u|^2}{\langle r \rangle^{1+\gamma}} + \frac{u^2}{\langle r \rangle^{3+\gamma}} \Big)$$

Moreover,

$$\frac{2}{3}a(r) - \frac{1}{6}b'(r) = \sum_{j=0}^{\infty} 2^{-j\gamma} \left(\frac{1}{2} \frac{1}{r+2^j} + \frac{1}{6} \frac{r}{(r+2^j)^2} \right) \gtrsim \frac{1}{\langle r \rangle}$$

Taking (3.2) into account, and applying Hölder and Hardy to control the inhomogeneity, we get

$$\|u\|^2_{LE^1[T_1,T_2]} + E(T_2) \lesssim E(T_1) + \|F\|_{L^1[T_1,T_2]L^2}\|\nabla u\|_{L^\infty[T_1,T_2]L^2} \tag{4.2}$$

Consider now a strong solution u, and a sequence of classical solutions u_n so that $u_n(T_1) \to u(T_1)$ in the energy norm. After dividing the interval $I = [T_1, T_2]$ into finitely many intervals so that the L^5L^{10} norm of u is suitably small on each interval, a contraction argument shows that $u_n \to u$ in $X(I)$. In particular this implies that u_n^5 is a Cauchy sequence in $L^1[T_1,T_2]L^2$, and thus by (4.2) we must have $u_n \to u$ in $LE^1[T_1,T_2]$. The desired conclusion (4.1) now follows.

5. L^6 norm decay of solutions in Minkowski space

In order to motivate the the next section, which contains the main result and its proof, in this section we shall present the highlights of the proof of L^6 norm decay in Minkowski space for the non-linear problem, as done in [2].

5.1. **More notation.** First, we fix some notation which will be used for the rest of the paper. Let

$$\Gamma = \{(t,x) : |x| - c < t, t > 0\}$$

be a forward solid light cone with $c \geq 0$ to be determined and let $\Gamma(I) = \Gamma \cap (I \times \mathbb{R}^3)$ where $I \subset [0,\infty)$ is a time interval. Let $D(T) = \{(t,x) : t = T, |x| - c < t\}$ denote its $t = T$ slices and let $L(I) = \{(t,x) : t \in I, |x| - c = t\}$ denote the lateral boundary of $\Gamma(I)$ with

$$L_c(I) := L(I)$$

also used for emphasis, but usually we shall simply write $L(I)$.

Consider next the Cauchy problem

$$\begin{cases} \Box u = u^5 & (t,x) \in (0,\infty) \times \mathbb{R}^3 \\ u[0] \in \dot{H}^1 \times L^2 \end{cases} \tag{5.1}$$

We now sketch a proof of the L^6 norm decay in the energy space for solutions to (5.1) (see [33], [1] for more details). We will adapt this proof to the variable-coefficient case in the next section.

Given $\delta > 0$, pick c sufficiently large so that the energy in the exterior region $|x| > c$ is less than $\delta/2$. Now given any $c \geq 0$, the flux on the time interval I is defined to be the integral on the lateral boundary arising from multiplying the equation $\Box u = u^5$ by $\partial_t u$, namely

$$\text{flux}(I) = \int_{L(I)} \frac{1}{2}|\partial_\omega u|^2 + \frac{1}{2}|\partial_t u + \partial_r u|^2 + \frac{1}{6}u^6 \frac{d\sigma}{\sqrt{2}}$$

It is clear that the flux is non-negative. As the upper and lower limits of I approach infinity, the flux decays, as an application of the divergence theorem in the interior of the $\Gamma(I)$ region shows. More precisely, if $I = [T_1, T_2]$, then in Minkowski space one obtains for the arbitrary number $c \geq 0$ chosen

$$E_{|x|<c+T_2}(T_2) = E_{|x|<c+T_1}(T_1) + \text{flux}([T_1, T_2]).$$

Thus $E_{|x|<c+t}(t)$ is monotone non-decreasing; moreover, it is bounded; therefore it converges to a limit as $t \to \infty$, as claimed. In particular, for all T_2 such that $T_2 > T_1$, and any $c \geq 0$,

$$\lim_{T_1 \to \infty} \lim_{T_2 \to \infty} \text{flux}([T_1, T_2]) = 0.$$

We now multiply (5.1) by

$$Xu := (t + c)\partial_t u + x^i \partial_i u + u = Su + c\partial_t u + u$$

where $S := t\partial_t + \sum_{i=1}^3 x^i \partial_i u$ and apply the divergence theorem in $\Gamma(I)$. We obtain

$$P(T_2) + \iint_{\Gamma(I)} \frac{u^6}{3}\, dxdt = P(T_1) + \int_{L(I)} (t + c) \left(\frac{Xu}{t + c}\right)^2 \frac{d\sigma}{\sqrt{2}} \tag{5.2}$$

where

$$P(T) := \int_{D(T)} \frac{t + c}{2} \left[\left(\frac{Xu}{t + c}\right)^2 + \left(|\nabla_x u|^2 - (\frac{x}{t + c} \cdot \nabla_x u)^2 \right) \right] + \frac{u^2}{t + c} + \frac{t + c}{6} u^6\, dx$$

Recall that on $L(I)$ we have $r = t + c$, enabling us to write $\frac{Xu}{t+c} = \partial_t u + \partial_r u + \frac{u}{t+c}$. By Cauchy-Schwarz and Hölder we obtain

$$\int_{L(I)} (t+c)\left(\frac{Xu}{t+c}\right)^2 \frac{d\sigma}{\sqrt{2}} \lesssim (T_2+c)\int_{L(I)} (\partial_t u + \partial_r u)^2 \, d\sigma + \int \frac{u^2}{t+c}\, d\sigma$$
$$\lesssim (T_2+c)(\|(\partial_t+\partial_r)u\|^2_{L^2(L(I))} + \|u\|^2_{L^6(L(I))})$$

In summary,

$$T_2 \int_{D(T_2)} u^6 \, dx \lesssim P(T_2) + \iint_{\Gamma(I)} u^6 \, dxdt \lesssim P(T_1) + (T_2+c)G(\text{flux}([T_1,T_2]))$$
$$\lesssim (T_1+c)E_{|x|<T_1+c}(T_1) + (T_2+c)G(\text{flux}([T_1,T_2]))$$

and

$$G(\theta) := \theta + \theta^{1/3}$$

is a function which decays to zero as its argument decays to zero. Take $T_1 = \delta T_2$ to see that, since δ was arbitrary and the flux decays,

$$\limsup_{t\to\infty} \|u(t,\cdot)\|_{L^6(\mathbb{R}^3)} = 0.$$

6. L^6 norm decay and scattering of solutions on small asymptotically flat perturbations of Minkowski space

We now come to the main result and its proof.

Theorem 6.1 (Main Theorem)**.** *Let u be the unique global strong solution of* (2.2)*.*

(1) *We make the following assumptions on the perturbation h:*

$$|\partial h| \lesssim \epsilon \langle x\rangle^{-1-\gamma} \tag{6.1}$$

$$|h| \lesssim \epsilon \frac{\langle t-r\rangle^{1/2}}{\langle x\rangle^{\gamma}\langle t+r\rangle^{1/2}} \tag{6.2}$$

$$|h^{\underline{LL}}| \lesssim \epsilon \frac{\langle t-r\rangle}{\langle x\rangle^{\gamma}\langle t+r\rangle} \tag{6.3}$$

where $\gamma > 0$ is an arbitrarily small constant and $\epsilon > 0$ is a sufficiently small constant.

Then

$$\limsup_{t\to\infty} \|u(t,\cdot)\|_{L^6(\mathbb{R}^3)} = 0. \tag{6.4}$$

(2) *If in addition*

$$|\partial^J h| \lesssim \epsilon\langle x\rangle^{-2-\gamma}, \quad |J| = 2, \tag{6.5}$$

then u scatters in the energy space.

Recall that we define the normal derivative to the cone

$$\underline{L} = \frac{x^i}{|x|}\partial_i - \partial_0 = \frac{x^i}{|x|}\partial_i - \partial_t$$

and

$$h^{\underline{L}\underline{L}} = h^{\alpha\beta}\underline{L}_\alpha\underline{L}_\beta = h^{00} - 2\sum_i h^{0i}\frac{x^i}{|x|} + \sum_{i,j} h^{ij}\frac{x^i x^j}{|x|^2}.$$

We also remark that the decay rates on h and ∂h are consistent with the ones required for local energy decay Theorem 4.1 except near the cone $t \approx |x|$, where we need better decay rates to close the argument.

We now sketch the proof our the main theorem. Let us first assume that u is a classical solution to the equation

$$Pu = u^5 + F \tag{6.6}$$

The main estimate of the paper is the following:

Proposition 6.2. If u solves (6.6), and for any R, T_1 and T_2 so that $R \geq 0$, $1 < T_1$ [1], and $T_1 + 2R < T_2$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} u^6(T_2,x)dx \lesssim \frac{T_1+2R}{T_2}E_{\{|x|<T_1+2R\}}(T_1) + \frac{E}{T_2^\gamma}\\ + G\Big(E_{\{|x|>T_1+R\}}(T_1) + \|u\|^2_{LE^1[T_1,T_2]} + \|F\|_{L^1[T_1,T_2]L^2}\|\nabla u\|_{L^\infty[T_1,T_2]L^2}\Big)\end{aligned} \tag{6.7}$$

where $E := E(0)$.

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[1] We shall only be interested in certain sufficiently large values of T_1 and R.

We now make the key observation that, unlike in the case of Minkowski, flux(I) is not necessarily nonnegative on arbitrary light cones. Instead, by averaging we obtain that there is $c \in [R, 2R]$ so that

$$\int_{L_c([T_1,T_2])} \frac{|\nabla u|^2}{\langle x\rangle^{1+\gamma}}\, d\sigma \lesssim R^{-1}\|u\|^2_{LE^1[T_1,T_2]}. \tag{6.8}$$

For the rest of this proof, fix c as above. Note that the hypothesis $T_1 + 2R < T_2$ implies that $T_2 \approx T_2 + c$. Moreover, in this case c depends on T_1, so we make sure that we carefully track the dependence on c in our estimates. In fact, the implicit constants do not depend on c, T_1, T_2, or R.

Proposition 6.2 follows from the results of Sections 6.1 and 6.2. We then finish the proof of Theorem 6.1 in Section 6.3.

6.1. L^6 **norm decay of solutions on spacelike slices exterior to the cone.** The next lemma shows that we can control both the outside energy and the flux through L_c. Note that, unlike in the Minkowski case, it is not clear that this can be done for all c.

Lemma 6.3. *Let u solve (6.6). Then*

$$\begin{aligned} E_{\{|x|>T_2+c\}}(T_2) + \mathit{flux}([T_1,T_2]) &\lesssim E_{\{|x|>T_1+c\}}(T_1) + \|u\|^2_{LE^1[T_1,T_2]} \\ &\quad + \|F\|_{L^1[T_1,T_2]L^2}\|\nabla u\|_{L^\infty[T_1,T_2]L^2}. \end{aligned} \tag{6.9}$$

Here,

$$\mathit{flux}([T_1,T_2]) := \int_{L_c([T_1,T_2])} \frac{1}{2}|\bar\partial u|^2 + \frac{1}{6}u^6 \,\frac{d\sigma}{\sqrt{2}}$$

and

$$\bar\partial u := \{Lu, (r\sin\phi)^{-1}\partial_\theta u, r^{-1}\partial_\phi u\}, \quad L = \frac{x^i}{|x|}\partial_i + \partial_t$$

denote the tangential derivatives of u to the light cone.

Proof. Let $I = [T_1, T_2]$. Multiplying both sides of the equation in (6.6) by $\partial_t u$, we obtain the identity

$$\partial_\alpha(g^{\alpha\beta}\partial_\beta u\partial_t u) - \frac{1}{2}\partial_t(g^{\alpha\beta}\partial_\beta u\partial_\alpha u) + \frac{1}{2}\partial_t g^{\alpha\beta}\partial_\beta u\partial_\alpha u = \frac{1}{6}\partial_t(u^6) + F\partial_t u.$$

Define

$$\Gamma^{\text{ext}}(I) := \{T_1 \le t \le T_2, |x| > t + c\}, \quad D(T)^c := \Gamma^{\text{ext}}(I) \cap \{t = T\}$$

Applying the divergence theorem within the region $\Gamma^{\text{ext}}(I)$ leads to

$$\begin{aligned}\iint_{\Gamma^{\text{ext}}(I)} F\partial_t u\,dxdt &= \iint_{\Gamma^{\text{ext}}(I)} \frac{1}{2}\partial_t g^{\alpha\beta}\partial_\beta u\partial_\alpha u\,dxdt+ \\ &\int_{\partial\Gamma^{\text{ext}}(I)} \nu_\alpha g^{\alpha\beta}\partial_\beta u\partial_t u - \frac{1}{2}\nu_0 g^{\alpha\beta}\partial_\beta u\partial_\alpha u - \frac{1}{6}\nu_0 u^6\,d\sigma.\end{aligned} \tag{6.10}$$

Next, let BD_h denote the part of the energy density on the boundary of $\Gamma^{\text{ext}}(I)$ arising from h

$$\mathrm{BD}_h := \nu_\alpha h^{\alpha\beta}\partial_\beta u\partial_t u - \frac{1}{2}\nu_0 h^{\alpha\beta}\partial_\beta u\partial_\alpha u.$$

Note that BD_h depends on the domain of integration. Expanding (6.10), we have

(6.11)

$$\begin{aligned}&E_{\{|x|>T_2+c\}}(T_2) + \int_{D(T_2)^c} \mathrm{BD}_h\,dx + \mathrm{flux}([T_1,T_2]) + \iint_{\Gamma^{\text{ext}}(I)} \frac{1}{2}\partial_t h^{\alpha\beta}\partial_\alpha u\partial_\beta u\,dxdt \\ &= \iint_{\Gamma^{\text{ext}}(I)} F\partial_t u\,dxdt + E_{\{|x|>T_1+c\}}(T_1) + \int_{D(T_1)^c}\mathrm{BD}_h\,dx + \int_{L_c(I)}\mathrm{BD}_h\,d\sigma\end{aligned}$$

The space-time term is easy to estimate by (6.1)

$$\iint_{\Gamma^{\text{ext}}(I)} \frac{1}{2}\partial_t h^{\alpha\beta}\partial_\alpha u\partial_\beta u\,dxdt \lesssim \iint_{\Gamma^{\text{ext}}(I)} \frac{|\nabla u|^2}{\langle x\rangle^{1+\gamma}}\,dxdt \le \|u\|^2_{LE^1[T_1,T_2]}. \tag{6.12}$$

Similarly, using that $|h| \lesssim \epsilon$, we obtain

$$\int_{D(T_j)^c} \mathrm{BD}_h\,dx \lesssim \epsilon E_{\{|x|>T_j+c\}}(T_j), \quad j=1,2. \tag{6.13}$$

which can be absorbed in $E_{\{|x|>T_j+c\}}(T_j)$ for small enough ϵ.

Finally, we need to estimate the perturbative error term on the lateral boundary; this is where we will use (6.2) and (6.3). We write

$$\begin{aligned}\nu_\alpha h^{\alpha\beta}\partial_\beta &= -\frac{1}{2}h^{\underline{LL}}\underline{L} + O(h)\bar\partial \\ \partial_t &= \frac{1}{2}(L-\underline{L}) \\ h^{\alpha\beta}\partial_\alpha u\partial_\beta u &= \frac{1}{4}h^{\underline{LL}}(\underline{L}u)^2 + O(h)\bar\partial u\partial u\end{aligned}$$

Note that, due to (6.3) and (6.2) we have that on $L(I)$

$$h^{\underline{LL}} \lesssim \frac{R}{\langle x\rangle^{1+\gamma}}, \quad h \lesssim \epsilon\frac{R^{1/2}}{\langle x\rangle^{1/2+\gamma}} \tag{6.14}$$

and thus by Cauchy-Schwarz

$$\int_{L_c(I)} \mathrm{BD}_h \, d\sigma \lesssim \int_{L_c(I)} |h^{\underline{LL}}|(\underline{L}u)^2+|h||\bar{\partial}u||\partial u| d\sigma \lesssim \epsilon \mathrm{flux}([T_1,T_2])+R\int_{L_c(I)} \frac{|\nabla u|^2}{\langle x\rangle^{1+\gamma}}\, d\sigma \tag{6.15}$$

The conclusion of the lemma now follows from (6.8), (6.11), (6.12), (6.13), and (6.15). □

6.2. L^6 **norm decay of solutions on interior spacelike slices of the cone.** The objective within this section is to show that solutions to (6.6) satisfy the following estimate.

Lemma 6.4. *Let u solve (6.6). Then*

$$\begin{aligned}\int_{D(T_2)} u^6(T_2,x)dx \lesssim \frac{T_1+2R}{T_2} E_{\{|x|<T_1+c\}}(T_1) + \frac{E}{T_2^{\gamma}} + G(\mathit{flux}([T_1,T_2]))+ \\ \|u\|^2_{LE^1[T_1,T_2]} + \|F\|_{L^1[T_1,T_2]L^2}\|\nabla u\|_{L^\infty[T_1,T_2]L^2}\end{aligned} \tag{6.16}$$

We remark that Proposition 6.2 easily follows from Lemma 6.3 and Lemma 6.4.

Proof. To prove (6.16), we multiply both sides of (6.6) by Xu and obtain

$$\begin{aligned}\partial_\alpha(g^{\alpha\beta}\partial_\beta u Xu) - \frac12\partial_t((t+c)g^{\alpha\beta}\partial_\beta u\partial_\alpha u) - \frac12\partial_i(x^i g^{\alpha\beta}\partial_\beta u\partial_\alpha u) \\ +\frac12((X-1)g^{\alpha\beta})\partial_\alpha u\partial_\beta u = \partial_t((t+c)\frac{u^6}{6}) + \partial_i(x^i\frac{u^6}{6}) - \frac{u^6}{3} + FXu.\end{aligned} \tag{6.17}$$

Indeed, (6.17) follows by the following computations and the symmetry of $g^{\alpha\beta}$:

$$\partial_\alpha(g^{\alpha\beta}\partial_\beta u\partial_t u) - \frac12\partial_t(g^{\alpha\beta}\partial_\beta u\partial_\alpha u) + \frac12\partial_t g^{\alpha\beta}\partial_\alpha u\partial_\beta u = (Pu)\partial_t u;$$

similarly, we have

$$\begin{aligned}\partial_\alpha(g^{\alpha\beta}\partial_\beta u t\partial_t u) - \frac12\partial_t(tg^{\alpha\beta}\partial_\beta u\partial_\alpha u) - g^{0\beta}\partial_\beta u\partial_t u + \frac12 g^{\alpha\beta}\partial_\alpha u\partial_\beta u \\ +\frac12 t\partial_t g^{\alpha\beta}\partial_\beta u\partial_\alpha u = (Pu)t\partial_t u,\end{aligned}$$

and

$$\begin{aligned}\partial_\alpha(g^{\alpha\beta}\partial_\beta u x^j\partial_j u) - g^{j\beta}\partial_\beta u\partial_j u - \frac12\partial_j(g^{\alpha\beta}\partial_\beta u x^j\partial_\alpha u) + \frac12 x^j\partial_j g^{\alpha\beta}\partial_\beta u\partial_\alpha u \\ +\frac32 g^{\alpha\beta}\partial_\beta u\partial_\alpha u = (Pu)(x^j\partial_j u)\end{aligned}$$

as well as

$$\partial_\alpha(g^{\alpha\beta}u\partial_\beta u) - g^{\alpha\beta}\partial_\beta u\partial_\alpha u = (Pu)u.$$

The nonlinear term follows in a similar manner. Upon summing these terms we obtain (6.17).

We now integrate (6.17) on $\Gamma(I)$ and apply the divergence theorem. We obtain

$$\iint_{\Gamma(I)} \frac{u^6}{3} + \frac{1}{2}((X-1)g^{\alpha\beta})\partial_\alpha u\partial_\beta u - FXu\,dxdt = -\int_{\partial\Gamma(I)} \nu_\alpha g^{\alpha\beta}\partial_\beta u Xu - \frac{1}{2}\nu\cdot(t+c,x)g^{\alpha\beta}\partial_\beta u\partial_\alpha u - \frac{1}{2}\nu\cdot(t+c,x)\frac{u^6}{6}\,d\sigma$$

Recall that on $L(I)$ the outward unit normal vector ν to $L(I)$ is $(-1, x/|x|)/\sqrt{2}$, and thus $\nu\cdot(t+c,x) = 0$ on $L(I)$. The boundary term can now be written more explicitly as

$$-P(T_2) + P(T_1) + \text{flux}([T_1,T_2]) - BDR_h$$

where the first three terms come from the Minkowski case, and BDR_h denotes the part of the energy density on the boundary arising from h:

$$BDR_h := \int_{D(T_2)} h^{0\beta}\partial_\beta u Xu - \frac{1}{2}(t+c)h^{\alpha\beta}\partial_\beta u\partial_\alpha u\,dx - \int_{D(T_1)} h^{0\beta}\partial_\beta u Xu - \frac{1}{2}(t+c)h^{\alpha\beta}\partial_\beta u\partial_\alpha u\,dx + \int_{L(I)} \nu_\alpha h^{\alpha\beta}\partial_\beta u Xu\,d\sigma$$

As explained in Section 5, we know that

$$P(T_2) \gtrsim T_2\int_{D(T_2)} u^6(T_2,x)dx$$

$$P(T_1) \lesssim (T_1+c)E_{\{|x|<T_1+c\}}(T_1)$$

We can also make the trivial estimate

$$\iint_{\Gamma(I)} FXu\,dxdt \lesssim T_2\|F\|_{L^1[T_1,T_2]L^2}\|\nabla u\|_{L^\infty[T_1,T_2]L^2}$$

Moreover, our assumptions on $h^{\alpha\beta}$ immediately imply that

$$(X-1)h^{\alpha\beta} \lesssim t\langle x\rangle^{-1-\gamma}$$

and thus

$$\iint_{\Gamma(I)} |(X-1)g^{\alpha\beta}\partial_\alpha u\partial_\beta u|\,dxdt \lesssim T_2\iint_{\Gamma(I)} \frac{|\nabla u|^2}{\langle x\rangle^{1+\gamma}}\,dxdt \le T_2\|u\|^2_{LE^1[T_1,T_2]} \tag{6.18}$$

The conclusion (6.16) will follow if we show that

$$BDR_h \lesssim \epsilon(P(T_2) + P(T_1)) + T_2^{1-\gamma} E + T_2 \left(G(\text{flux}([T_1, T_2])) + \|u\|^2_{LE^1[T_1,T_2]} \right)$$

Let us write

$$D(T_2) = D_{int}(T_2) \cup D_{ext}(T_2)$$

where

$$D_{int}(T_2) = D(T_2) \cap \{|x| \leq \frac{T_2 + c}{2}\}, \quad D_{ext}(T_2) = D(T_2) \cap \{|x| \geq \frac{T_2 + c}{2}\},$$

Since $|h| \lesssim \epsilon$ in D_{int}, we have that

$$\int_{D_{int}(T_2)} h^{0\beta} \partial_\beta u X u - \frac{1}{2}(t + c) h^{\alpha\beta} \partial_\beta u \partial_\alpha u \, dx \lesssim \epsilon P(T_2)$$

On the other hand, $|h| \lesssim \frac{1}{T_2^\gamma}$ in D_{ext}, and thus by the boundedness of energy

$$\int_{D_{ext}(T_2)} h^{0\beta} \partial_\beta u X u - \frac{1}{2}(t + c) h^{\alpha\beta} \partial_\beta u \partial_\alpha u \, dx \lesssim T_2^{1-\gamma} E(T_2) \lesssim T_2^{1-\gamma} E$$

Adding the last two inequalities we obtain

$$\int_{D(T_2)} h^{0\beta} \partial_\beta u X u - \frac{1}{2}(t + c) h^{\alpha\beta} \partial_\beta u \partial_\alpha u \, dx \lesssim \epsilon P(T_2) + T_2^{1-\gamma} E$$

Similarly we can show that

$$\int_{D(T_1)} h^{0\beta} \partial_\beta u X u - \frac{1}{2}(t + c) h^{\alpha\beta} \partial_\beta u \partial_\alpha u \, dx \lesssim \epsilon P(T_1) + T_1^{1-\gamma} E$$

We are left with dealing with the lateral terms. We will show that

$$\int_{L(I)} \nu_\alpha h^{\alpha\beta} \partial_\beta u X u \, d\sigma \lesssim T_2 \left(G(\text{flux}([T_1, T_2])) + \|u\|^2_{LE^1[T_1,T_2]} \right) \tag{6.19}$$

We first remark that $Xu = (rL + 1)u$ on $L(I)$, and we again write

$$\nu_\alpha h^{\alpha\beta} \partial_\beta u = -\frac{1}{2} h^{\underline{L}\underline{L}} \underline{L} u + O(h) \bar{\partial} u \tag{6.20}$$

Note that (6.14) in particular imply the weaker estimates

$$h^{\underline{L}\underline{L}} \lesssim \frac{R^{1/2}}{\langle x \rangle^{1/2+\gamma}}, \quad h \lesssim 1$$

We can now estimate by Cauchy-Schwarz, (6.14) and the fact that $r \le T_2 + c \lesssim T_2$:

$$\begin{aligned}\int_{L(I)} |h^{\underline{LL}}\underline{L}u(rLu)|\, d\sigma &\lesssim T_2 \int_{L(I)} |\frac{R^{1/2}}{\langle x\rangle^{1/2+\gamma}}\underline{L}uLu|\, d\sigma \\ &\le T_2 \left(R\int_{L(I)} \frac{|\nabla u|^2}{\langle x\rangle^{1+\gamma}}\, d\sigma + \int_{L(I)} |Lu|^2\, d\sigma \right) \\ &\lesssim T_2(\|u\|^2_{LE^1[T_1,T_2]} + \text{flux}([T_1,T_2]))\end{aligned}$$

where in the last inequality we used (6.8).

Similarly,

$$\begin{aligned}\int_{L(I)} |h\bar{\partial}u(rLu)|\, d\sigma &\lesssim T_2 \int_{L(I)} |\bar{\partial}uLu|\, d\sigma \\ &\le T_2 \int_{L(I)} |\bar{\partial}u|^2\, d\sigma \lesssim T_2\text{flux}([T_1,T_2])\end{aligned}$$

Thirdly, by an application of (6.14), Cauchy-Schwarz and then Hölder's inequality,

$$\begin{aligned}\int_{L(I)} |h^{\underline{LL}}\underline{L}uu|\, d\sigma &\lesssim \int_{L(I)} \frac{R^{1/2}}{\langle x\rangle^{1/2+\gamma}}|\underline{L}u||u|\, d\sigma \\ &\lesssim \left(\int_{L(I)} R\frac{|\nabla u|^2}{\langle x\rangle^{1+\gamma}}\, d\sigma\right)^{1/2} \left(\int_{L(I)} u^2\, d\sigma\right)^{1/2} \\ &\lesssim \left(\int_{L(I)} R\frac{|\nabla u|^2}{\langle x\rangle^{1+\gamma}}\, d\sigma\right)^{1/2} \|u\|_{L^6(L(I))}T_2 \\ &\lesssim T_2\left(\|u\|^2_{LE^1[T_1,T_2]} + \text{flux}([T_1,T_2])^{1/3}\right)\end{aligned}$$

where again we used (6.8).

Finally,

$$\begin{aligned}\int_{L(I)} |h\bar{\partial}uu|\, d\sigma &\lesssim (\int_{L(I)} |\bar{\partial}u|^2\, d\sigma)^{1/2}(\int_{L(I)} u^2\, d\sigma)^{1/2} \\ &\lesssim (\int_{L(I)} |\bar{\partial}u|^2\, d\sigma)^{1/2}(\int_{L(I)} |u|^6\, d\sigma)^{1/6}T_2 \\ &\lesssim T_2G(\text{flux}[T_1,T_2])\end{aligned}$$

The last four estimates now imply (6.19), which finishes the proof of (6.16). □

6.3. **Proof of Theorem 6.1.**

Proof. Assume now that Proposition 6.2 holds. A similar argument as the one in Section 4 allows us to pass to the limit and deduce the following:

Proposition 6.5. Let u be the strong solution to (2.2). For any $R \geq 1$, $1 < T_1$, and $T_1 + 2R < T_2$, we have

$$\int_{\mathbb{R}^3} u^6(T_2, x)dx \lesssim \frac{T_1 + 2R}{T_2} E_{\{|x|<T_1+2R\}}(T_1) + \frac{E}{T_2^\gamma} + G\Big(E_{\{|x|>T_1+R\}}(T_1) + \|u\|^2_{LE^1[T_1,T_2]}\Big) \tag{6.21}$$

Let us now prove (6.4). Pick any $\tilde{\epsilon} > 0$, and let T_1 be large enough such that

$$\|u\|^2_{LE^1[T_1,\infty)} < \tilde{\epsilon};$$

such a number may be found because of the local energy estimate (4.1). Next pick R large enough so that

$$E_{\{|x|>T_1+R\}}(T_1) < \tilde{\epsilon}$$

Now let $T_2 \to \infty$ in (6.21). We obtain

$$\limsup_{T_2\to\infty} \int_{\mathbb{R}^3} u^6(T_2, x)dx \lesssim G(\tilde{\epsilon})$$

and (6.4) follows by letting $\tilde{\epsilon} \to 0$.

To obtain part (2) of Theorem 6.1, note that if $\partial^J h \lesssim \epsilon\langle x\rangle^{-|J|-\gamma}$ where $|J| \leq 2$ (which are implied by our assumptions in the main theorem), then global Strichartz estimates are implied by a refinement of the local energy decay estimates (see Theorem 2 in [25][2]). Then, for any $\eta > 0$, by choosing a sufficiently large number $T > 0$, we obtain $\|w\|_{L^5L^{10}([T,\infty)\times\mathbb{R}^3)} \leq \eta$ where w solves (1.1). For any $\tilde{w}$ with $\|\tilde{w}\|_{L^5L^{10}([T,\infty)\times\mathbb{R}^3)} \leq \eta$, let W be the solution to

$$PW = (w + \tilde{w})^5$$

with

$$\lim_{t\to\infty} \|\nabla W(t, \cdot)\|_{L^2(\mathbb{R}^3)} = 0.$$

[2]In the current arXiv version of this paper, see Theorem 6 instead.

As $\eta > 0$ was arbitrary, we may select η sufficiently small so that the map $\tilde{w} \mapsto W$ is a contraction mapping, so that for any finite energy solution w of (1.1), there exists a unique solution to (2.1) such that their difference vanishes in the $\dot{H}^1 \times L^2$ norm as $t \to \infty$, and we conclude that the solution scatters in the energy space. □

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