Euler Enumeration and Beyond

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Abstract

This paper surveys recent results for flag enumeration of polytopes, Bruhat graphs, balanced digraphs, Whitney stratified spaces and quasi-graded posets.

1 Introduction

In this paper we describe recent developments regarding chain enumeration and the **cd**-index which involve algebra, graph theory and topology. The first is a non-homogeneous **cd**-index for Bruhat graphs due to Billera and Brenti [3]. One motivation for studying the **cd**-index of Bruhat graphs is that the **cd**-index of the interval $[u, v]$ determines the Kazhdan–Lusztig polynomial $P_{u,v}(q)$; see [3, Section 3]. These polynomials arise out of Kazhdan and Lusztig's study of the Springer representations of the Hecke algebra of a Coxeter group [22, 23], and have many applications, including to Verma modules and to the algebraic geometry and topology of Schubert varieties. See section 5 for further discussion.

The second recent development is the theory of balanced graphs, due to Ehrenborg and Readdy [15]. This theory relaxes the graded, poset and Eulerian requirements for chain enumeration in graded posets. Bruhat graphs are a special case of balanced graphs, and the theory simplifies the proof techniques from using quasi-symmetric theory to edge labelings in the graphs. In the case a balanced graph has a linear edge labeling, the authors conjecture the **cd**-index has nonnegative coefficients.

The third development is both a topological and poset theoretic generalization of flag enumeration. Ehrenborg, Goresky and Readdy have extended the theory of face incidence enumeration of polytopes, and more generally, chain enumeration in graded Eulerian posets, to that of Whitney stratified spaces and quasi-graded posets [12]. Whitney stratifications occur naturally for real and complex algebraic sets, analytic sets, semi-analytic sets and for quotients of smooth manifolds by compact group actions. It is important to point out that, unlike the case of polytopes, the coefficients of the **cd**-index of Whitney stratified manifolds can be negative. It is hoped that by applying topological techniques to stratified manifolds, a tractable interpretation of the coefficients of the **cd**-index will emerge. This may ultimately explain Stanley's non-negativity results for spherically shellable posets [34] and Karu's results for Gorenstein* posets [21]. Additionally, the program of determining inequalities for flag vectors of polytopes is expanded in the Whitney stratified setting.

2 Polytopes and face enumeration

A *convex polytope* in n-dimensional Euclidean space $\mathbb{R}^n$ is the convex hull of k points $x_1, \ldots, x_k$ in $\mathbb{R}^n$, that is, the intersection of all convex sets containing these points. Throughout we will assume all of the polytopes we work with are convex. For general references on polytopes, see [9, 18, 38].

One can also define a polytope as the bounded intersection of a finite number of half-spaces in $\mathbb{R}^n$. These two descriptions can be seen to be equivalent by Fourier–Motzkin elimination [38]. A polytope is *n-dimensional* if it is homeomorphic to a closed n-dimensional ball $\mathbb{B}^n = \{(x_1, \ldots, x_r) : x_1^2 + \cdots + x_n^2 \leq 1, x_{n+1} = \cdots = x_r = 0\}$ in $\mathbb{R}^r$.

Given a polytope P in $\mathbb{R}^n$ with supporting hyperplane H, that is, $P \cap H \neq \emptyset$, $P \cap H_+ \neq \emptyset$ and $P \cup H_- = \emptyset$, where H_+ and H_- are the half open regions determined by the hyperplane H, then we say

$P \cap H$ is a *face*. Observe that a face of a polytope is a polytope in its own right.

For an n-dimensional convex polytope the *f-vector* is $(f_0, \dots, f_{n-1})$, where f_i enumerates the number of i-dimensional faces. It satisfies the *Euler–Poincaré relation* [30]:

$$f_0 - f_1 + f_2 - \cdots + (-1)^{n-1} \cdot f_{n-1} = 1 - (-1)^n. \tag{2.1}$$

Equivalently,

$$\sum_{i=-1}^{n} (-1)^i f_i = 0, \tag{2.2}$$

where f_{-1} denotes the number of empty faces ($= 1$) and $f_n = 1$ counts the entire polytope.

Example 2.1. The n-dimensional simplex Δ_n. *The n-dimensional simplex* is the convex hull of any $n+1$ affinely independent points in $\mathbb{R}^n$. Equivalently, it can be described as the convex hull of the $n+1$ points $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ where $\mathbf{e}_i$ is the ith unit vector $(0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^{n+1}$. It is convenient to intersect this polytope with the hyperplane $x_1 + \cdots + x_{n+1} = 1$ so that it lies in $\mathbb{R}^n$. Its f-vector has entries $f_i = \binom{n+1}{i+1}$, for $i = 0, \dots, n-1$. This is an example of a *simplicial polytope*, that is, a polytope where all of its facets ($(n-1)$-dimensional faces) are combinatorially equivalent to the $(n-1)$-dimensional simplex.

In 1906 Steinitz [36] completely characterized the f-vectors of 3-dimensional polytopes.

Theorem 2.2 (Steinitz)**.** *For a 3-dimensional polytope, the f-vector is uniquely determined by the values f_0 and f_2. The (f_0, f_2)-vector of every 3-dimensional polytope satisfies the following two inequalities:*

$$2(f_0 - 4) \geq f_2 - 4 \text{ and } f_0 - 4 \leq 2(f_2 - 4).$$

Furthermore, every lattice point in this cone has at least one 3-dimensional polytope associated to it.

For polytopes of dimension greater than three the problem of characterizing their f-vectors is still open.

Open question 2.3. *Characterize f-vectors of n-dimensional polytopes where $n \geq 4$.*

The f-vectors of simplicial polytopes have been completely characterized by work of McMullen [27], Billera and Lee [5] and Stanley [33]. This characterization, known as the g-theorem, involved a geometric construction of Billera and Lee [5] for the sufficiency proof, and tools from algebraic geometry for Stanley's necessity proof [33]. In particular, this required the Hard Lefschetz Theorem. We include Björner's reformulation of the g-theorem as stated in [18, section 10.6]:

Theorem 2.4 (The g-theorem)**.** (Billera–Lee; Stanley) *The vector $(1, f_0, \ldots, f_{n-1})$ is the f-vector of an n-dimensional simplicial polytope if and only if it is a vector of the form $\mathbf{g} \cdot M_n$, where M_n is the $([n/2]+1) \times (n+1)$ matrix with nonnegative entries given by*

$$M_n = \left(\binom{n+1-j}{n+1-k} - \binom{j}{n+1-k} \right)_{0 \leq j \leq n, 0 \leq k \leq n}, \tag{2.3}$$

*and $\mathbf{g} = (g_0, \ldots, g_{[n/2]})$ is an M-*sequence*, that is, a nonnegative integer vector with $g_0 = 1$ and $g_{k-1} \geq \partial^k(g_k)$ for $0 < k \leq n/2$. The upper boundary operator ∂^k is given by*

$$\partial^k = \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \cdots + \binom{a_i - 1}{i-2} \tag{2.4}$$

where the unique binomial expansion of a positive integer m is

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_i}{i}, \tag{2.5}$$

with $a_k > a_{k-1} > \cdots a_i \geq i > 0$. For a given polytope P the vector $g = g(P)$ is determined by the f-vector, respectively h-vector, as $g_k = h_k - h_{k-1}$ for $0 < k \leq n/2$ with $g_0 = 1$.

3 Flag vectors

One would like to keep track of not just the number of faces in a polytope, but also the face incidences. We encode this with the

S	f_S	h_S	u_s	$\mathbf{c}^3$	$10 \cdot \mathbf{dc}$	$6 \cdot \mathbf{cd}$
$\emptyset$	1	1	**aaa**	1	0	0
$\{0\}$	12	11	**baa**	1	10	0
$\{1\}$	18	17	**aba**	1	10	6
$\{2\}$	8	7	**aab**	1	0	6
$\{0,1\}$	36	7	**bba**	1	0	6
$\{0,2\}$	36	17	**bab**	1	10	6
$\{1,2\}$	36	11	**abb**	1	10	0
$\{0,1,2\}$	72	1	**bbb**	1	0	0

Table 1: The flag f- and flag h-vectors, **ab**-index and **cd**-index of the hexagonal prism. The sum of the last three columns equals the flag h column, showing the **cd**-index of the hexagonal prism is $\mathbf{c}^3 + 10 \cdot \mathbf{dc} + 6 \cdot \mathbf{cd}$.

flag f-vector (f_S), where $S \subseteq \{0, \ldots, n-1\}$. More formally, for $S = \{s_1 < \cdots < s_k\} \subseteq \{0, \ldots, n-1\}$, define f_S to be the number of flags of faces

$$f_S = \#\{F_1 \subsetneq F_2 \cdots \subsetneq F_k\}$$

where $\dim(F_i) = s_i$. Observe that for an n-dimensional polytope the flag f-vector has 2^n entries. It also contains the f-vector data.

The *flag h-vector* $(h_S)_{S \subseteq \{0,\ldots,n-1\}}$ is defined by the invertible relation

$$h_S = \sum_{T \subseteq \{0,\ldots,n-1\}} (-1)^{|S-T|} f_T. \tag{3.1}$$

Equivalently, by the Möbius Inversion Theorem

$$f_S = \sum_{T \subseteq \{0,\ldots,n-1\}} h_T. \tag{3.2}$$

See Table 1 for the computation of the flag f- and flag h-vectors of the hexagonal prism. Observe that the symmetry of the flag h-vector reduces the number of entries we have to keep track of by half. This is true in general.

Theorem 3.1 (Stanley). *For an n-dimensional polytope, and more generally, an Eulerian poset of rank n,*

$$h_S = h_{\overline{S}},$$

where $\overline{S}$ denotes the complement of S with respect to $\{0, 1, \ldots, n-1\}$.

Recall a *partially ordered set* P, or *poset* for short, consists of a finite number of elements with a partial order $\leq$ which is reflexive ($x \leq x$ for all elements $x \in P$), antisymmetric (if $x \leq y$ and $y \leq x$ then $x = y$), and transitive ($x \leq y$ and $y \leq z$ implies $x \leq z$). Unless stated otherwise the posets we will work with have unique minimal and maximal elements, denoted by $\hat{0}$ and $\hat{1}$ respectively. A poset P with unique minimal and maximal elements is *graded* if any saturated chain of elements from $\hat{0}$ to x, that is, $c = \{\hat{0} = x_0 \prec x_1 \prec \cdots \prec x_k = x\}$ has the same length for a fixed element $x \in P$. We call this length the *rank* of x, denoted $\rho(x)$ and the rank of a graded poset is $\rho(\hat{1})$.

In important subclass of graded posets are the *Eulerian posets*. These satisfy the condition that $\mu(x, y) = (-1)^{\rho(x,y)}$, where $\rho(x, y) = \rho(y) - \rho(x)$ and the *Möbius function* is defined by $\mu(x, x) = 1$ and $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$. Equivalently, in every non-trivial interval the number of elements of even rank equals the number of elements of odd rank. Important families of Eulerian posets include the face lattice of a convex polytope, the face poset of a regular cell decomposition of a homology sphere and the (strong) Bruhat order on a Coxeter group.

4 The ab-index and cd-index

In this section we describe the **cd**-index, a compact encoding of the flag vector data of a polytope.

The **ab**-*index* of an n-dimensional polytope P is defined by

$$\Psi(P) = \sum_{S} h_S \cdot u_S,$$

where the sum is taken over all subsets $S \subseteq \{0, \ldots, n-1\}$ and $u_S = u_0 u_1 \ldots u_{n-1}$ is the non-commutative monomial encoding the subset S by

$$u_i = \begin{cases} \mathbf{a} & \text{if } i \notin S, \\ \mathbf{b} & \text{if } i \in S. \end{cases}$$

Observe the resulting **ab**-index is a noncommutative polynomial of degree n in the noncommutative variables **a** and **b**.

We now introduce another change of basis. Let $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$ be two noncommutative variables of degree 1 and 2, respectively. The following result was conjectured by J. Fine and proven by Bayer–Klapper for polytopes, and Stanley for Eulerian posets [2, 34].

Theorem 4.1 (Bayer–Klapper, Stanley)**.** *For the face lattice of a polytope, and more generally, an Eulerian poset, the* **ab***-index* $\Psi(P)$ *can be written uniquely in terms of the noncommutative variables* $\mathbf{c} = \mathbf{a} + \mathbf{b}$ *and* $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$*, that is,* $\Psi(P) \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$*.*

The resulting noncommutative polynomial is called the **cd**-*index*.

Prior to the introduction of the **cd**-index, Bayer and Billera determined all the linear relations which hold among the flag f-vector entries [1], known as the the *generalized Dehn–Sommerville relations*:

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \cup \{j\}} = (1 - (-1)^{k-i-1}) \cdot f_S. \tag{4.1}$$

Here $i \leq k-2$, the elements i and k are elements of $S \cup \{-1, n\}$, and the subset S contains no integer between i and k. Observe that the Euler-Poincaré relation follows if we take $S = \emptyset$, $i = -1$ and $k = n$. Besides showing the existence of the **cd**-index for Eulerian posets, Bayer and Klapper proved that the **cd**-index removes *all* of the linear redundancies holding among the flag vector entries [2]. Hence the **cd**-monomials form a natural basis for the vector space of **ab**-indexes of polytopes.

The **cd**-index did not generate very much excitement in the mathematical community until Stanley's proof of the nonnegativity of its coefficients [34].

Theorem 4.2 (Stanley)**.** *The* **cd***-index of the face lattice of a polytope, more generally, the augmented face poset of any spherically-shellable regular CW-sphere, has nonnegative coefficients*

Stanley's result opened the door to the following question.

Open question 4.3. *Give a combinatorial interpretation of the coefficients of the* **cd***-index.*

In his dissertation, Purtill's gave an interpretation of the **cd**-index coefficients for the n-dimensional simplex and the n-dimensional cube respectively in terms of simsun and signed simsun permutations [32]. See also [19]. Ehrenborg and Readdy introduced coalgebraic techniques to describe how the **cd**-index changes under geometric operations applied to a polytope [14]. These techniques have lead to new results regarding flag vector inequalities. See the concluding remarks for details. More recently for each **cd**-monomial Karu gave a sequence of operators on sheaves of vector spaces to show the non-negativity of the coefficients of the **cd**-index for Gorenstein* posets [21].

5 Bruhat graphs

Another family of Eulerian posets is formed by taking the (strong) Bruhat order on a Coxeter group [37]. Hence any interval has a **cd**-index which is homogeneous of degree one more than the length of the interval. By removing the adjacent rank assumption on the cover relation of the Bruhat order, a directed graph known as the Bruhat graph is obtained which in effect allows algebraic "short cuts" between elements.

More formally, let (W, S) be a Coxeter system, where W denotes a (finite or infinite) Coxeter group with generators S and $\ell(u)$ denotes the length of a group element u. Let T be the set of reflections, that is, $T = \{w \cdot s \cdot w^{-1} \;:\; s \in S, w \in W\}$. The *Bruhat graph* has the group W as its vertex set and its set of labels Λ is the set of reflections T. The edges and their labeling are defined as follows. There is a directed edge from u to v labeled t if $u \cdot t = v$ and $\ell(u) < \ell(v)$. The underlying poset of the Bruhat graph is called the *(strong) Bruhat order*. It is important to note that every interval of the Bruhat order is Eulerian, that is, every interval $[x, y]$ has Möbius function given by $\mu(x, y) = (-1)^{\rho(y)-\rho(x)}$, where ρ denotes the rank function. For a more complete description of Coxeter systems, see Björner and Brenti's text [7].

Using the fact that the generalized Dehn–Sommerville relations hold for coefficients of polynomials arising in Kazhdan–Lusztig polynomials [8, Theorem 8.4] and quasisymmetric functions, Billera and Brenti show that the Bruhat graph has a non-homogeneous **cd**-index [3].

Theorem 5.1 (Billera–Brenti)**.** *For an interval $[u, v]$ in the Bruhat order, where $u < v$, the following three conditions hold:*

(i) The interval $[u, v]$ in the Bruhat graph has a **cd***-index $\Psi([u, v])$.*

(ii) Restricting the **cd***-index $\Psi([u, v])$ to those terms of degree $\ell(v) - \ell(u) - 1$ equals the* **cd***-index of the graded poset $[u, v]$.*

(iii) The degree of a term in the **cd***-index $\Psi([u, v])$ is less than or equal to $\ell(v) - \ell(u) - 1$ and has the same parity as $\ell(v) - \ell(u) - 1$.*

For an alternate proof using labelings of balanced graphs, see [15].

Billera and Brenti also show the important connection with Kazhdan–Lusztig polynomials [3, Theorem 3.3].

Theorem 5.2 (Billera–Brenti)**.** *For an interval $[u, v]$ in the Bruhat order, the Kazhdan–Lusztig polynomial can be explicitly computed from the* **cd***-index $\Psi([u, v])$ of the interval $[u, v]$ in the Bruhat graph.*

See also Brenti's work on computing the Kazhdan–Lusztig polynomial using lattice paths, as well as Morel's follow-up paper [8, 29].

6 Bruhat and balanced graphs

In this section we describe a graph theoretic framework for flag enumeration. The notion of a labeled acyclic digraph was introduced in [15] in order to model poset structure in this more general setting.

Let $G = (V, E)$ be a directed, acyclic and locally finite graph with multiple edges allowed. Recall that an *acyclic graph* does not have any directed cycles and the property of a graph being *locally finite* requires that there are a finite number of paths between any two vertices. Each directed edge e has a tail and a head, denoted respectively by $\text{tail}(e)$ and $\text{head}(e)$. View each directed edge as an arrow from its tail to its head. A directed path p of length k from a vertex x to a vertex y is a list of k directed edges $(e_1, e_2, \ldots, e_k)$ such that $\text{tail}(e_1) = x$, $\text{head}(e_k) = y$ and $\text{head}(e_i) = \text{tail}(e_{i+1})$ for $i = 1, \ldots, k-1$. We denote the length of a path p by $\ell(p)$.

Since the graph is acyclic, it does not have any loops. Furthermore, the acyclicity condition implies there is a natural partial order on the vertices of G by defining the order relation $x \leq y$ if there is a directed path from the vertex x to the vertex y. It is straightforward to verify that this relation is reflexive, antisymmetric and transitive. It allows us to define the *interval* $[x, y]$ to be the set of all vertices z in $V(G)$ such that there is a directed path from x to z and a directed path from z to y. We view the interval $[x, y]$ as the vertex-induced subgraph of the digraph G, where the edges have the same labels as in the digraph G. The locally finite condition is now equivalent to that every interval $[x, y]$ in the graph has finite cardinality.

We next relax the notions of R-labeling and the **ab**-index of a poset. Let Λ be a set with a relation $\sim$, that is, there is a subset $R \subseteq \Lambda \times \Lambda$ such that for $i, j \in \Lambda$ we have $i \sim j$ if and only if $(i, j) \in R$. A *labeling of* G is a function λ from the set of edges of G to the set Λ. Let $\mathbf{a}$ and $\mathbf{b}$ be two non-commutative variables each of degree one. For a path $p = (e_1, \ldots, e_k)$ of length k, where $k \geq 1$, we define the *descent word* $u(p)$ to be the **ab**-monomial $u(p) = u_1 u_2 \cdots u_{k-1}$, where

$$u_i = \begin{cases} \mathbf{a} & \text{if } \lambda(e_i) \sim \lambda(e_{i+1}), \\ \mathbf{b} & \text{if } \lambda(e_i) \not\sim \lambda(e_{i+1}). \end{cases}$$

Observe that the descent word $u(p)$ has degree $k-1$, that is, one less than the length of the path p. The **ab**-*index* of an interval $[x, y]$ is defined to be

$$\Psi([x, y]) = \sum_{p} u(p), \tag{6.1}$$

where the sum is over all directed paths p from x to y.

An analogue of the coalgebraic groundwork for flag enumeration in posets holds for labeled acyclic digraphs [15, Corollary 3.4].

Theorem 6.1 (Ehrenborg–Readdy)**.** *The* **ab**-*index of a labeled acyclic digraph is a coalgebra homomorphism from the linear span of bounded labeled acyclic digraphs to the polynomial ring* $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$.

The following result gives three equivalent statements which imply the (non-homogeneous) **ab**-index of an acyclic digraph can be written as a (non-homogeneous) **cd**-index [15, Theorem 4.7].

Theorem 6.2 (Ehrenborg–Readdy)**.** *For a labeled acyclic digraph G, the following three statements are equivalent:*

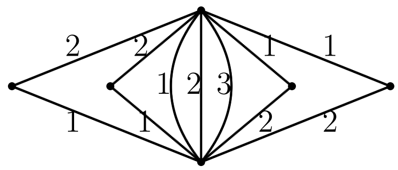
(i) For every interval $[x, y]$ in the digraph G and for every non-negative integer k, the number of rising paths from x to y of length k is equal to the number of falling paths from x to y of length k.

(ii) For every interval $[x, y]$ in the digraph G and for every even *positive integer k, the number of rising paths from x to y of length k is equal to the number of falling paths from x to y of length k.*

(iii) The **ab**-*index of every interval $[x, y]$ in the digraph G, where $x < y$, is a polynomial in* $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$.

Definition 6.3. *A labeled acyclic digraph G is said to be* balanced *if it satisfies condition (i) in Theorem 6.2. Such a labeling is called a* balanced labeling *or* B-labeling *for short.*

An edge labeling *linear* if the underlying relation $(\Lambda, \sim)$ is that of a linear order.



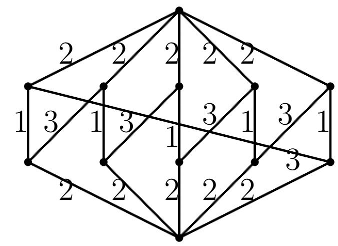


Figure 1: Two balanced directed graphs where the relation on the labeled set $\Lambda = \{1, 2, 3\}$ is the natural linear order. Their respective **cd**-indexes are $2 \cdot \mathbf{c} + 3$ and $5 \cdot \mathbf{d}$. These two examples show that the **cd**-index of a graph is not necessarily homogeneous and that the coefficient of the **c**-power term is not necessarily 1.

Theorem 6.4 (Ehrenborg–Readdy)**.** *Let u be a non-zero* **cd***-polynomial with non-negative coefficients. Then there exists a bounded balanced labeled acyclic digraph G where the relation on the set of labels is a linear order and which satisfies $\Psi(G) = w$.*

Theorem 6.4 (see [15, Theorem 8.1]) motivates the following conjecture.

Conjecture 6.5 (Ehrenborg–Readdy)**.** *The* **cd***-index of a bounded labeled acyclic digraph G with a balanced linear edge labeling is non-negative.*

7 Euler flag enumeration

Polytopes are examples of regular decompositions of the n-dimensional sphere. In this section we extend the idea of flag enumeration of polytopes to decompositions of more general manifolds. We will see the face poset of such a manifold is an instance of the more general quasi-graded posets. In all cases, we can extend the flag enumeration theory.

We begin with a modest example.

Example 7.1. Consider the non-regular CW-complex Ω consisting of one vertex v, one edge e and one 2-dimensional cell c such that the boundary of c is the union $v \cup e$, that is, boundary of the complex Ω is a one-gon. Its face poset is the four element chain $\mathscr{F}(\Omega) = \{\hat{0} < v < e < c\}$. This is not an Eulerian poset. The **ab**-index of Ω is $\mathbf{a}^2$. Note that $\mathbf{a}^2$ cannot be written in terms of $\mathbf{c}$ and $\mathbf{d}$.

Observe that the edge e is attached to the vertex v twice. Hence it is natural to change the value of f_{01} to 2. The **ab**-index becomes $\Psi(\Omega) = \mathbf{a}^2 + \mathbf{b}^2$ and hence its **cd**-index is $\Psi(\Omega) = \mathbf{c}^2 - \mathbf{d}$.

The motivation for the value 2 in Example 7.1 is best expressed in terms of the Euler characteristic of the link. The link of the vertex v in the edge e is two points whose Euler characteristic is 2. In order to view this example in the right topological setting, we review the notion of a Whitney stratification. For more details, see [10, 16, 17, 24].

Definition 7.2. *Let W be a closed subset of a smooth manifold M which has been decomposed into a finite union of locally closed subsets*

$$W = \bigcup_{X \in \mathcal{P}} X.$$

Furthermore suppose this decomposition satisfies the condition of the frontier*:*

$$X \cap \overline{Y} \neq \emptyset \Longleftrightarrow X \subseteq \overline{Y}.$$

This implies the closure of each stratum is a union of strata, and it provides a partial ordering for the index set $\mathcal{P}$:

$$X \subseteq \overline{Y} \Longleftrightarrow X \leq_{\mathcal{P}} Y.$$

This decomposition of W is a Whitney stratification *if*

1. *Each $X \in \mathcal{P}$ is a (locally closed, not necessarily connected) smooth submanifold of M.*

2. *If $X <_{\mathcal{P}} Y$ then Whitney's conditions (A) and (B) hold: Suppose $y_i \in Y$ is a sequence of points converging to some $x \in X$ and that $x_i \in X$ converges to x. Also assume that (with respect to some local coordinate system on the manifold M) the*

secant lines $\ell_i = \overline{x_i y_i}$ converge to some limiting line ℓ and the tangent planes $T_{y_i}Y$ converge to some limiting plane τ. Then the following inclusions hold:

$$\text{(A) } T_x X \subseteq \tau \qquad \text{and} \qquad \text{(B) } \ell \subseteq \tau.$$

Whitney's conditions A and B ensure there is no fractal behavior and no infinite wiggling. A crucial result is that the links are well-defined in a Whitney stratification. See [12].

Recall the *incidence algebra* of a poset is the set of all functions $f : I(P) \to \mathbb{C}$ where $I(P)$ denotes the set of intervals in the poset. The multiplication is given by $(f \cdot g)(x, y) = \sum_{x \leq z \leq y} f(x, z) \cdot g(z, y)$ and the identity is given by the delta function $\delta(x, y) = \delta_{x,y}$, where the second delta is the usual Kronecker delta function $\delta_{x,y} = 1$ if $x = y$ and zero otherwise. For other poset terminology, we refer the reader to Stanley's text [35].

We introduce the notion of a quasi-graded poset. This extends the notion of a ranked poset.

Definition 7.3. *A quasi-graded poset $(P, \rho, \bar{\zeta})$ consists of*

(i) a finite poset P (not necessarily ranked),

(ii) a strictly order-preserving function ρ from P to $\mathbb{N}$, that is, $x < y$ implies $\rho(x) < \rho(y)$ and

(iii) a function $\bar{\zeta}$ in the incidence algebra $I(P)$ of the poset P, called the weighted zeta function*, such that $\bar{\zeta}(x, x) = 1$ for all elements x in the poset P.*

Observe that we do not require the poset to have a minimal element or a maximal element. Since $\bar{\zeta}(x, x) \neq 0$ for all $x \in P$, the function $\bar{\zeta}$ is invertible in the incidence algebra $I(P)$ and we denote its inverse by $\bar{\mu}$.

For a chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ in the face poset of a Whitney stratified space, define

$$\bar{\zeta}(c) = \chi(c_1) \cdot \chi(\mathrm{link}_{x_2}(x_1)) \cdots \chi(\mathrm{link}_{x_{k-1}}(x_k)),$$

where χ denotes the Euler characteristic.

The usual approach for the **ab**-index of polytopes and Eulerian posets is via the flag f- and flag h-vectors. We extend this route by introducing the flag $\bar{f}$- and flag $\bar{h}$-vectors. Let $(P, \rho, \bar{\zeta})$ be a quasi-graded poset of rank $n+1$ having a $\hat{0}$ and $\hat{1}$ such that $\rho(\hat{0}) = 0$. For $S = \{s_1 < s_2 < \cdots < s_k\}$ a subset of $\{1, \ldots, n\}$, define the *flag $\bar{f}$-vector* by

$$\bar{f}_S = \sum_c \bar{\zeta}(c), \tag{7.1}$$

where the sum is over all chains $c = \{\hat{0} = x_0 < x_1 < \cdots < x_{k+1} = \hat{1}\}$ in P such that $\rho(x_i) = s_i$ for all $1 \leq i \leq k$. The *flag $\bar{h}$-vector* is defined by the relation (and by inclusion–exclusion, we also display its inverse relation)

$$\bar{h}_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot \bar{f}_T \quad \text{and} \quad \bar{f}_S = \sum_{T \subseteq S} \bar{h}_T. \tag{7.2}$$

The **ab**-index of the quasi-graded poset $(P, \rho, \bar{\zeta})$ is then given by

$$\Psi(P, \rho, \bar{\zeta}) = \sum_S \bar{h}_S \cdot u_S,$$

where the sum ranges over all subsets S. Again, in the case when we take the weighted zeta function to be the usual zeta function ζ, the flag $\bar{f}$ and flag $\bar{h}$-vectors correspond to the usual flag f- and flag h-vectors.

Definition 7.4. *A quasi-graded poset is said to be* Eulerian *if for all pairs of elements $x \leq z$ we have that*

$$\sum_{x \leq y \leq z} (-1)^{\rho(x,y)} \cdot \bar{\zeta}(x, y) \cdot \bar{\zeta}(y, z) = \delta_{x,z}. \tag{7.3}$$

In other words, the function $\bar{\mu}(x, y) = (-1)^{\rho(x,y)} \cdot \bar{\zeta}(x, y)$ is the inverse of $\bar{\zeta}(x, y)$ in the incidence algebra. In the case $\bar{\zeta}(x, y) = \zeta(x, y)$, we refer to relation (7.3) *as the* classical Eulerian relation*.*

Generalizing the classical result of Bayer and Klapper for graded Eulerian posets, we have the analogue for quasi-graded posets [12, Theorem 4.2].

S	$\bar{f}_S$	$\bar{h}_S$	$3\mathbf{dc}$	$-2\mathbf{cd}$
$\emptyset$	0	0	0	0
$\{0\}$	3	3	3	0
$\{1\}$	1	1	3	-2
$\{2\}$	-2	-2	0	-2
$\{0,1\}$	2	-2	0	-2
$\{0,2\}$	2	1	3	-2
$\{1,2\}$	2	3	3	0
$\{0,1,2\}$	4	0	0	0

Table 2: The flag $\bar{f}$- and flag $\bar{h}$-vectors, **ab**-index and **cd**-index of the 2-dimensional torus with an edge and one isolated vertex on it. The sum of the last two columns equals the flag h column, showing the **cd**-index is $3\mathbf{baa} + \mathbf{aba} - 2\mathbf{aab} - 2\mathbf{bba} + \mathbf{bba} + 3\mathbf{abb} = 3\mathbf{dc} - 2\mathbf{cd}$.

Theorem 7.5 (Ehrenborg–Goresky–Readdy)**.** *For an Eulerian quasi-graded poset $(P, \rho, \bar{\zeta})$ its* **ab***-index $\Psi(P, \rho, \bar{\zeta})$ can be written uniquely as a polynomial in the non-commutative variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$.*

Example 7.6. Consider the 2-torus with one edge with two incident vertices on it and an isolated vertex. See Table 2 for the **cd**-index computation.

The next theorem implies the existence of the **cd**-index for any manifold with Whitney stratified boundary [12, Theorem 6.10]. The proof required properties of the Euler characteristic and returning to Mather's idea of "tube systems and control data. See [12, Sections 7–9].

Theorem 7.7 (Ehrenborg–Goresky–Readdy)**.** *Let M be a manifold with a Whitney stratified boundary. Then the face poset is quasi-graded and Eulerian, with*

$$\rho(x) = \dim(x) + 1$$

and

$$\bar{\zeta}(x, y) = \chi(\mathrm{link}_y(x)).$$

8 Concluding remarks

Knowing inequalities for the **cd**-index implies inequalities for the flag h-vector and the flag f-vector. This follows from expanding the **cd**-index back into the **ab**-index ($\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$ are each non-negative linear combinations of monomials in **a** and **b**), then expanding the **ab**-index back into the flag f-vector via equation (3.2) (another non-negative linear combination).

Recall that Stanley proved the nonnegativity of the **cd**-index for polytopes, and more generally, for spherically-shellable regular CW-spheres. See Theorem 4.2. Stanley conjectured that for n-dimensional polytopes, more generally, Gorenstein* lattices, the **cd**-index was minimized on the simplex of the same dimension, respectively Boolean algebra of the same rank. Both of these conjectures were shown to be true. See [4, 13].

Theorem 8.1 (Billera–Ehrenborg)**.** *The* **cd***-index of a convex n-dimensional polytope is coefficient-wise greater than or equal to the* **cd***-index of the n-simplex.*

Theorem 8.2 (Ehrenborg–Karu)**.** *The* **cd***-index of a Gorenstein* lattice of rank n is coefficient-wise greater than or equal to the* **cd***-index of the Boolean algebra B_n.*

Open question 8.3. *Find the linear inequalities that hold among the entries of the* **cd***-index of a Whitney stratified manifold.*

This expands the program of determining linear inequalities for flag vectors of polytopes. Since the coefficients may be negative, one must ask what should the new minimization inequalities be. Observe that Kalai's convolution [20] still holds. More precisely, let M and N be two linear functionals defined on the **cd**-coefficients of any m-dimensional, respectively, n-dimensional manifold. If both M and N are non-negative then their convolution is non-negative on any $(m + n + 1)$-dimensional manifold.

Define an inner product on $\mathbf{k}\langle \mathbf{c}, \mathbf{d} \rangle$ by

$$\langle u | v \rangle = \delta_{u,v}$$

where u and v are **cd**-monomials and extend by linearity. We can use this notation to encode inequalities easily. For example,

$$\left\langle \mathbf{d} - \mathbf{c}^2 | \Psi(P) \right\rangle \geq 0$$

says the for a 2-dimensional polytope the coefficient of $\mathbf{d}$ minus the coefficient of $\mathbf{c}^2$ is nonnegative. (True, as $(n-2) - 1 \geq 0$ for $n \geq 3$.)

We can now state Ehrenborg's lifting technique [11, Theorem 3.1].

Theorem 8.4 (Ehrenborg)**.** *Let u and v be two* **cd***-monomials. Suppose u does not end in* $\mathbf{c}$ *and v does not begin with* $\mathbf{c}$*. Then the inequality*

$$\langle H | \Psi(P) \rangle \geq 0 \textit{ implies } \langle u \cdot H \cdot v | \Psi(P) \rangle \geq 0.$$

where H is a **cd***-polynomial such that the inequality $\langle H | \Psi(P) \rangle \geq 0$ holds for all polytopes P.*

Corollary 8.5 (Ehrenborg)**.** *For two* **cd***-monomials u and v the following inequality holds for all polytopes P:*

$$\langle u \cdot \mathbf{d} \cdot v | \Psi(P) \rangle \geq \left\langle u \cdot \mathbf{c}^2 \cdot v | \Psi(P) \right\rangle.$$

This corollary says the coefficient of a **cd**-monomial increases when replacing a $\mathbf{c}^2$ with a $\mathbf{d}$.

It is natural to ask the following inequality questions:

Open question 8.6. *Can Ehrenborg's lifting technique [11] be extended to stratified manifolds?*

Open question 8.7. *What non-linear inequalities hold among the* **cd***-coefficients?*

Returning to Karu's approach to show nonnegativity of the coefficients of **cd**-index for Gorenstein* posets, we now ask the following question.

Open question 8.8. *Is there a signed analogue of Karu's construction to explain the negative coefficients occurring in the* **cd***-index of quasi-graded posets?*

9 Acknowledgements

The author would like to thank the Princeton University Mathematics Department where this article was written. This work was partially supported by a grant from the Simons Foundation (#206001 to Margaret Readdy).

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