The Gaussian coefficient revisited

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Abstract

We give new q-$(1+q)$-analogue of the Gaussian coefficient, also know as the q-binomial which, like the original q-binomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$, is symmetric in k and $n-k$. We show this q-$(1+q)$-binomial is more compact than the one discovered by Fu, Reiner, Stanton and Thiem. Underlying our q-$(1+q)$-analogue is a Boolean algebra decomposition of an associated poset. These ideas are extended to the Birkhoff transform of any finite poset. We end with a discussion of higher analogues of the q-binomial.

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1 Introduction

Inspired by work of Fu, Reiner, Stanton and Thiem [2], Cai and Readdy [1] asked the following question. Given a combinatorial q-analogue

$$X(q) = \sum_{w \in X} q^{a(w)},$$

where X is a set of objects and $a(\cdot)$ is a statistic defined on the elements of X, when can one find a smaller set Y and two statistics s and t such that

$$X(q) = \sum_{w \in Y} q^{s(w)} \cdot (1+q)^{t(w)}.$$

Such an interpretation is called an q-$(1+q)$-*analogue*. Examples of q-$(1+q)$-analogues have been determined for the q-binomial by Fu, Reiner, Stanton and Thiem [2], and for the q-Stirling numbers of the first and second kinds by Cai and Readdy [1], who also gave poset and homotopy interpretations of their q-$(1+q)$-analogues.

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In 1916 MacMahon [3, 4, 5] observed that the Gaussian coefficient, also known as the q-binomial coefficient, is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{w \in \Omega_{n,k}} q^{\mathrm{inv}(w)}.$$

Here $\Omega_{n,k} = \mathfrak{S}(0^{n-k}, 1^k)$ denotes all permutations of the multiset $\{0^{n-k}, 1^k\}$, that is, all words $w = w_1 \cdots w_n$ of length n with $n-k$ zeroes and k ones, and $\mathrm{inv}(\cdot)$ denotes the inversion statistic defined by $\mathrm{inv}(w_1 w_2 \cdots w_n) = |\{(i,j) : 1 \leq i < j \leq n, w_i > w_j\}|$. Fu et al. defined a subset $\Omega'_{n,k} \subseteq \Omega_{n,k}$ and two statistics a and b such that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{w \in \Omega'_{n,k}} q^{a(w)} \cdot (1+q)^{b(w)}.$$

In this paper we will return to the original study by Fu et al. of the Gaussian coefficient. We discover a more compact q-$(1+q)$-analogue which, like the original Gaussian coefficients, is also symmetric in the variables k and $n-k$; see Corollary 2.6 and Theorem 3.5. This symmetry was missing in Fu et al.'s original q-$(1+q)$-analogue. We give a Boolean algebra decomposition of the related poset $\Omega_{n,k}$. Since this poset is a distributive lattice, in the last section we extend these ideas to poset decompositions of any distributive lattice and other analogues.

2 A poset interpretation

In this section we consider the poset structure on 0-1-words in $\Omega_{n,k}$. For further poset terminology and background, we refer the reader to [6].

We begin by making the set of elements $\Omega_{n,k}$ into a graded poset by defining the cover relation to be

$$u \circ 01 \circ v \prec u \circ 10 \circ v,$$

where $\circ$ denotes concatenation of words. The word $0^{n-k}1^k$ is the minimal element and the word $1^k 0^{n-k}$ is the maximal element in the poset $\Omega_{n,k}$. Furthermore, this poset is graded by the inversion statistic. This poset is simply the interval $[\widehat{0}, x]$ of Young's lattice, where the minimal element $\widehat{0}$ is the empty Ferrers diagram and x is the Ferrers diagram consisting of $n-k$ columns and k rows.

An alternative description of the poset $\Omega_{n,k}$ is that it is isomorphic to the Birkhoff transform of the Cartesian product of two chains. Let C_m denote the m-element chain. The poset $\Omega_{n,k}$ is isomorphic to the distributive lattice of all lower order ideals of the product $C_{n-k} \times C_k$, usually denoted by $J(C_{n-k} \times C_k)$.

Definition 2.1. *Let $\Omega''_{n,k} \subseteq \Omega_{n,k}$ consist of all 0,1-words $v = v_1 v_2 \cdots v_n$ in $\Omega_{n,k}$ such that*

$$v_1 \leq v_2, \;\; v_3 \leq v_4, \;\; \ldots, \;\; v_{2 \cdot \lfloor n/2 \rfloor - 1} \leq v_{2 \cdot \lfloor n/2 \rfloor}.$$

Observe that when n is odd there is no condition on the last entry w_n. Define two maps ϕ and ψ on $\Omega_{n,k}$ by sending the word $w = w_1 w_2 \cdots w_n$ to

$$\phi(w) = \min(w_1, w_2), \max(w_1, w_2), \;\; \min(w_3, w_4), \max(w_3, w_4), \;\; \ldots,$$
$$\psi(w) = \max(w_1, w_2), \min(w_1, w_2), \;\; \max(w_3, w_4), \min(w_3, w_4), \;\; \ldots.$$

The map ϕ sorts the entries in positions 1 and 2, 3 and 4, and so on. If n is odd, the entry w_n remains in the same position. Similarly, the map ψ sorts in reverse order in each pair of positions. Note that the map ϕ maps $\Omega_{n,k}$ surjectively onto the set $\Omega''_{n,k}$.

We have the following Boolean algebra decomposition of the poset $\Omega_{n,k}$.

Theorem 2.2. *The distributive lattice $\Omega_{n,k}$ has the Boolean algebra decomposition*

$$\Omega_{n,k} = \bigcup_{v \in \Omega''_{n,k}} [v, \psi(v)].$$

Proof. Observe that the maps ϕ and ψ satisfy the inequalities $\phi(w) \leq w \leq \psi(w)$. Furthermore, the fiber of the map $\phi : \Omega_{n,k} \longrightarrow \Omega''_{n,k}$ is isomorphic to a Boolean algebra, that is, $\phi^{-1}(v) \cong [v, \psi(v)]$. □

For $v \in \Omega''_{n,k}$ define the statistic

$$\mathrm{asc}_{\mathrm{odd}}(v) = |\{i \,:\, v_i < v_{i+1}, i \text{ odd}\}|,$$

that is, $\mathrm{asc}_{\mathrm{odd}}(\cdot)$ enumerates the number of ascents in odd positions.

Corollary 2.3. *The q-binomial is given by*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{v \in \Omega''_{n,k}} q^{\mathrm{inv}(v)} \cdot (1+q)^{\mathrm{asc}_{\mathrm{odd}}(v)}. \tag{2.1}$$

Proof. It is enough to observe that the sum of the inversion statistic over the elements in the fiber $\phi^{-1}(v) = [v, \psi(v)]$ for $v \in \Omega''_{n,k}$ is given by $q^{\mathrm{inv}(v)} \cdot (1+q)^{\mathrm{asc}_{\mathrm{odd}}(v)}$. □

A geometric way to understand this q-$(1+q)$-interpretation is to consider lattice paths from the origin $(0,0)$ to $(n-k,k)$ which only use east steps $(1,0)$ and north steps $(0,1)$. Color the squares of this $(n-k) \times k$ board as a chessboard, where the square incident to the origin is colored white. The map ϕ in the proof of Theorem 2.2 corresponds to taking a lattice path where every time there is a north step followed by an east step that turns around a white square, we exchange these two steps. The statistic $\mathrm{asc}_{\mathrm{odd}}$ enumerates the number of times an east step is followed by a north step when this pair of steps borders a white square.

Let $\mathrm{er}(n,k)$ denote the cardinality of the set $\Omega''_{n,k}$. Then we have

Proposition 2.4. *The cardinalities $\mathrm{er}(n,k)$ satisfy the recursion*

$$\mathrm{er}(n,k) = \mathrm{er}(n-2,k-2) + \mathrm{er}(n-2,k-1) + \mathrm{er}(n-2,k) \quad \textit{for } n,k \geq 2$$

with $\mathrm{er}(n,n) = 1$ and $\mathrm{er}(n,k) = 0$ whenever $k > n$, $k < 0$ or $n < 0$.

Proof. A word in $\Omega''_{n,k}$ begins with either 00, 01 or 11, yielding the three cases of the recursion. □

Directly we obtain the generating polynomial.

Theorem 2.5. *The generating polynomial for* $\mathrm{er}(n,k)$ *is given by*

$$\sum_{k=0}^{n} \mathrm{er}(n,k) \cdot x^k = (1+x+x^2)^{\lfloor n/2 \rfloor} \cdot (1+x)^{n-2\cdot\lfloor n/2 \rfloor}.$$

We end with a statement concerning the symmetry of the q-$(1+q)$-binomial.

Corollary 2.6. *The set of defining elements for the* q-$(1+q)$*-binomial satisfy the following symmetric relation:*

$$|\Omega''_{n,k}| = |\Omega''_{n,n-k}|.$$

Proof. This follows from the fact that the generating polynomial for $\mathrm{er}(n,k)$ is a product of palindromic polynomials, and thus is itself is a palindromic polynomial. □

3 Analysis of the Fu–Reiner–Stanton–Thiem interpretation

A *weak partition* is a finite non-decreasing sequence of non-negative integers. A weak partition $\lambda = (\lambda_1, \ldots, \lambda_{n-k})$ with $n-k$ parts and each part at most k where $\lambda_1 \leq \cdots \leq \lambda_{n-k}$ corresponds to a Ferrers diagram lying inside an $(n-k) \times k$ rectangle with column i having height λ_i. These weak partitions are in direct correspondence with the set $\Omega_{n,k}$.

Fu, Reiner, Stanton and Thiem used a pairing algorithm to determine a subset $\Omega'_{n,k} \subseteq \Omega_{n,k}$ of 0-1-sequences to define their q-$(1+q)$-analogue of the q-binomial; see [2, Proposition 6.1]. This translates into the following statement. The set $\Omega'_{n,k}$ is in bijection with weak partitions into $n-k$ parts with each part at most k such that

(a) if k is even, each odd part has even multiplicity,

(b) if k is odd, each even part (including 0) has even multiplicity.

Definition 3.1. *Let* $\mathrm{frst}(n,k)$ *be the cardinality of the set* $\Omega'_{n,k}$.

Lemma 3.2. *The quantity* $\mathrm{frst}(n,k)$ *counts the number of weak partitions into* $n-k$ *parts where each part is at most* k *and each odd part has even multiplicity.*

Proof. When k is even there is nothing to prove. When k is odd, by considering the complement of weak partitions with respect to the rectangle of size $(n-k) \times k$, we obtain a bijective proof. The same complement proof also shows the case when k is even holds. □

Theorem 3.3. *The* frst*-coefficients satisfy the recursion*

$$\begin{aligned} \mathrm{frst}(n,k) &= \mathrm{frst}(n-1,k-1) + \mathrm{frst}(n-1,k) && \text{for } k \text{ even,} \\ \mathrm{frst}(n,k) &= \mathrm{frst}(n-2,k-2) + \mathrm{frst}(n-2,k-1) + \mathrm{frst}(n-2,k) && \text{for } k \text{ odd,} \end{aligned}$$

where $1 \leq k \leq n-1$.

Proof. We use the characterization in Lemma 3.2. When k is even there are two cases. If the last part is k, remove it to obtain a weak partition counted by $\operatorname{frst}(n-1,k)$. If the last part is less than k, then the weak partition is counted by $\operatorname{frst}(n-1,k-1)$.

When k is odd there are three cases. If the last two parts are equal to k, then removing these two parts yields a weak partition counted by $\operatorname{frst}(n-2,k)$. Note that we cannot have the last part equal to k and the next to last part less than k since k is odd. If the last part is equal to $k-1$, we can remove it to obtain a weak partition counted by $\operatorname{frst}(n-2,k-1)$. Finally, if the last part is less than or equal to $k-2$, the weak partition is counted by $\operatorname{frst}(n-2,k-2)$. □

Lemma 3.4. *The inequality* $\operatorname{frst}(n,k) \leq \operatorname{frst}(n+1,k+1)$ *holds.*

Proof. The weak partitions which lie inside the rectangle $(n-k) \times k$ and satisfy the conditions of Lemma 3.2 are included among the weak partitions which lie inside the larger rectangle $(n-k) \times (k+1)$ and satisfy the same conditions. □

Theorem 3.5. *For all* $0 \leq k \leq n$ *the inequality* $|\Omega''_{n,k}| = \operatorname{er}(n,k) \leq \operatorname{frst}(n,k) = |\Omega'_{n,k}|$ *holds.*

Proof. We proceed by induction on n. The induction base is $n \leq 3$. Furthermore, the inequality holds when k is 0, 1, $n-1$ and n. When k is odd we have that

$$\begin{aligned} \operatorname{er}(n,k) &= \operatorname{er}(n-2,k-2) + \operatorname{er}(n-2,k-1) + \operatorname{er}(n-2,k) \\ &\leq \operatorname{frst}(n-2,k-2) + \operatorname{frst}(n-2,k-1) + \operatorname{frst}(n-2,k) \\ &= \operatorname{frst}(n,k). \end{aligned}$$

Similarly, when k is even we have

$$\begin{aligned} \operatorname{er}(n,k) &= \operatorname{er}(n-2,k-2) + \operatorname{er}(n-2,k-1) + \operatorname{er}(n-2,k) \\ &\leq \operatorname{frst}(n-2,k-2) + \operatorname{frst}(n-2,k-1) + \operatorname{frst}(n-2,k) \\ &\leq \operatorname{frst}(n-1,k-1) + \operatorname{frst}(n-2,k-1) + \operatorname{frst}(n-2,k) \\ &= \operatorname{frst}(n-1,k-1) + \operatorname{frst}(n-1,k) \\ &= \operatorname{frst}(n,k), \end{aligned}$$

where the second inequality follows from Lemma 3.4. These two cases complete the induction hypothesis. □

See Table 1 to compare the values of $\operatorname{frst}(n,k)$ and $\operatorname{er}(n,k)$ for $n \leq 10$.

4 Concluding remarks

Is it possible to find a q-$(1+q)$-analogue of the Gaussian coefficient which has the smallest possible index set? We believe that our analogue is the smallest, but cannot offer a proof of a minimality. Perhaps a more tractable question is to prove that the Boolean algebra decomposition of $\Omega_{n,k}$ is minimal.

1										
1	1									
1	1	1								
1	2	2	1							
1	2	4	2	1						
1	3	6	5	3	1					
1	3	9	8	8	3	1				
1	4	12	14	16	9	4	1			
1	4	16	20	30	19	13	4	1		
1	5	20	30	50	39	32	14	5	1	
1	5	25	40	80	69	71	36	19	5	1

1										
1	1									
1	1	1								
1	2	2	1							
1	2	3	2	1						
1	3	5	5	3	1					
1	3	6	7	6	3	1				
1	4	9	13	13	9	4	1			
1	4	10	16	19	16	10	4	1		
1	5	14	26	35	35	26	14	5	1	
1	5	15	30	45	51	45	30	15	5	1

Table 1: The frst- and er-triangles for $n \leq 10$.

We can extend these ideas involving of a Boolean algebra decomposition to any distributive lattice. Let P be a finite poset and let A be an antichain of P such that there is no cover relation in A, that is, there is no pair of elements $u, v \in A$ such that $u \prec v$. We obtain a Boolean algebra decomposition of the Birkhoff transform $J(P)$ by defining

$$J''(P) = \{I \in J(P) \ : \ \text{the ideal } I \text{ has no maximal elements in the antichain } A\}.$$

The two maps ϕ and ψ are now defined as

$$\begin{aligned} \phi(I) &= I - \{a \in A \ : \ \text{the element } a \text{ is maximal in } I\}, \\ \psi(I) &= I \cup \{a \in A \ : \ I \cup \{a\} \in J(P)\}. \end{aligned}$$

Then we have the following decomposition theorem

Theorem 4.1. *For P any finite poset the distributive lattice $J(P)$ has the Boolean algebra decomposition*

$$J(P) = \bigcup_{I \in J''(P)} [I, \psi(I)].$$

Yet again, how can we select the antichain A such that the above decomposition A has the fewest possible terms? Furthermore, would this give the smallest Boolean algebra decomposition?

Another way to extend the ideas of Theorem 2.2 is as follows. Define $\Omega^r_{n,k}$ to be the set of all words $v \in \Omega_{n,k}$ satisfying the inequalities

$$v_1 \leq v_2 \leq \cdots \leq v_r, \ \ v_{r+1} \leq v_{r+2} \leq \cdots \leq v_{2r}, \ \ \ldots, \ \ v_{r \cdot \lfloor n/r \rfloor - r+1} \leq v_{r \cdot \lfloor n/r \rfloor - r+2} \leq \cdots \leq v_{r \cdot \lfloor n/r \rfloor}.$$

For $1 \leq i \leq \lfloor r/2 \rfloor$ define the statistics $b_i(v)$ for $v \in \Omega^r_{n,k}$ to be

$$b_i(v) = |\{j \in [\lfloor n/r \rfloor] \ : \ v_{rj-r+1} + v_{rj-r+2} + \cdots + v_{rj} \in \{i, r-i\}\}|.$$

Theorem 4.2. *The distributive lattice $\Omega_{n,k}$ has the decomposition*

$$\Omega_{n,k} = \bigcup_{v \in \Omega^r_{n,k}} \Omega_{r,1}^{b_1(v)} \times \Omega_{r,2}^{b_2(v)} \times \cdots \times \Omega_{r,\lfloor r/2 \rfloor}^{b_{\lfloor r/2 \rfloor}(v)}.$$

Corollary 4.3. *The q-binomial is given by*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{v \in \Omega^r_{n,k}} q^{\mathrm{inv}(v)} \cdot \begin{bmatrix} r \\ 1 \end{bmatrix}_q^{b_1(v)} \cdot \begin{bmatrix} r \\ 2 \end{bmatrix}_q^{b_2(v)} \cdots \begin{bmatrix} r \\ \lfloor r/2 \rfloor \end{bmatrix}_q^{b_{\lfloor r/2 \rfloor}(v)}.$$

The least complicated case is when $r = 3$, where only one term appears in the above poset product. This term is $\Omega_{3,1}$ which is the three element chain C_3. The associated Gaussian coefficient is $1 + q + q^2$. Thus Corollary 4.3 could be called a q-$(1 + q + q^2)$-*analogue*. As an example we have

$$\begin{bmatrix} 6 \\ 3 \end{bmatrix}_q = 1 + q \cdot (1 + q + q^2)^2 + q^4 \cdot (1 + q + q^2)^2 + q^9.$$

On a poset level this a decomposition of $J(C_3 \times C_3)$ into two one-element posets of rank 0 and rank 9, and two copies of $C_3 \times C_3$, where one has its minimal element of rank 1 and the other of rank 4.

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