

SIMION'S TYPE B ASSOCIAHEDRON IS A PULLING TRIANGULATION OF THE LEGENDRE POLYTOPE

RICHARD EHRENBORG, GÁBOR HETYEI AND MARGARET READDY

ABSTRACT. We show that the Simion type B associahedron is combinatorially equivalent to a pulling triangulation of the type A root polytope known as the Legendre polytope. Furthermore, we show that every pulling triangulation of the boundary of the Legendre polytope yields a flag complex. Our triangulation refines a decomposition of the boundary of the Legendre polytope given by Cho. We extend Cho's cyclic group action to the triangulation in such a way that it corresponds to rotating centrally symmetric triangulations of a regular $(2n + 2)$ -gon. Finally, we present a bijection between the faces of the Simion type B associahedron and Delannoy paths.

1. INTRODUCTION

Root polytopes arising as convex hulls of roots in a root system have become the subject of intensive interest in recent years [1, 10, 12, 13, 18, 19]. Another important area where geometry meets combinatorics is the study of noncrossing partitions, associahedra and their generalizations. In this context Simion [23] constructed a type B associahedron whose facets correspond to centrally symmetric triangulations of a regular $(2n + 2)$ -gon. Burgiel and Reiner [5] described Simion's construction as providing "the first motivating example for an equivariant generalization of fiber polytopes, that is, polytopal subdivisions which are invariant under symmetry groups". It was recently observed by Cori and Hetyei [11] that the face numbers in this type B associahedron are the same as the face numbers in any pulling triangulation of the boundary of a type A root polytope, called the Legendre polytope in [13].

In this paper we show that the equality of these face numbers is not a mere coincidence. We prove the type B associahedron is combinatorially equivalent to a pulling triangulation of the Legendre polytope P_n . Hence we can embed the Simion type B associahedron in $(n - 1)$ -dimensional space such that each vertex of the associahedron is arbitrarily close to its associated root of the type A root system. The convex hull of the positive roots among the vertices of the Legendre polytope and of the origin is a type A root polytope P_n^+ . Cho [9] has shown that the Legendre polytope P_n may be decomposed into copies of P_n^+ that meet only on their boundaries and that there is a \mathbb{Z}_{n+1} -action on this decomposition. Our triangulation representing the type B associahedron as a triangulation of the boundary of the Legendre polytope refines Cho's decomposition in such a way that extends the \mathbb{Z}_{n+1} -action to the triangulation. The effect of this \mathbb{Z}_{n+1} -action on the centrally symmetric triangulations of the $(2n + 2)$ -gon is rotation.

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Simion observed algebraically that the number of k -dimensional faces of the type B associahedron is given by the number of lattice paths between $(0, 0)$ and $(2n, 0)$ taking k up steps $(1, 1)$, k down steps $(1, -1)$, and $n - k$ horizontal steps $(2, 0)$. Such paths are known as *balanced Delannoy paths*. In Section 8 we give a combinatorial proof by providing a bijection between the faces of the type B associahedron and Delannoy paths. We give two presentations of this bijection, one recursive and one non-recursive.

Our paper is structured as follows. In the preliminaries we discuss the Simion type B associahedron, the Legendre polytope and pulling triangulations. In Section 3 we show that every pulling triangulation of the Legendre polytope is a flag complex. We introduce an arc representation of Simion's type B diagonals in Section 4 and obtain conditions for when a pair of B -diagonals do not cross. A bijection between the set of B -diagonals and the vertex set of the Legendre polytope is obtained by the intermediary of our arc representation in Section 5. We characterize when B -diagonals cross in terms of crossing and nesting conditions on the arrows associated to the vertices of the Legendre polytope. This characterization is used in Section 6 to define a triangulation of the boundary of the Legendre polytope where each face corresponds to a face in the type B associahedron. Since both complexes are flag and have the same minimal non-faces, we conclude that they are the same polytope. We end this section by describing the facets in our triangulation. In Section 7 we prove that our triangulation refines Cho's decomposition and that his \mathbb{Z}_{n+1} -action corresponds to rotating the regular $(2n+2)$ -gon. In Section 8 we present a bijection between faces of the type B associahedron and Delannoy paths.

In the literature there are two schools regarding the associahedron which are dual to each other. One school views the associahedron as a simple polytope, whereas the other school views the associahedron as a simplicial polytope. In this paper we follow Lee [15] and Simion [23] by viewing the associahedron as a simplicial polytope.

We end the paper with comments and future research directions.

2. PRELIMINARIES

2.1. Simion's type B associahedron. Simion [23] introduced a simplicial complex denoted by Γ_n^B on $n(n+1)$ vertices as follows. Consider a centrally symmetric convex $(2n+2)$ -gon, and label its vertices in the clockwise order with $1, 2, \dots, n, n+1, \bar{1}, \bar{2}, \dots, \bar{n}, \overline{n+1}$. The vertices of Γ_n^B are the B -diagonals, which are one of the two following kinds: diagonals joining antipodal pairs of points, and antipodal pairs of noncrossing diagonals. The diagonals joining antipodal points are all pairs of the form $\{i, \bar{i}\}$ satisfying $1 \leq i \leq n+1$. Simion calls such a B -diagonal a *diameter*. The B -diagonals that are antipodal pairs of noncrossing diagonals are either of the form $\{\{i, j\}, \{\bar{i}, \bar{j}\}\}$ satisfying $1 \leq i < i+1 < j \leq n+1$ or of the form $\{\{i, \bar{j}\}, \{\bar{i}, j\}\}$ satisfying $1 \leq j < i \leq n+1$.

The simplicial complex Γ_n^B is the family of sets of pairwise noncrossing B -diagonals. Simion showed the simplicial complex Γ_n^B is the boundary complex of an n -dimensional convex polytope Q_n^B . This polytope is also known as the Bott-Taubes polytope [4] and the cyclohedron [16]. Simion also computed the face numbers and h -vector. See Theorem 1, Proposition 1 and Corollary 1 in [23], respectively. These values turn out to be identical with the face numbers and h -vector of any pulling

triangulation of the Legendre polytope. We will discuss this polytope in the next subsection. We end with a fact that is implicit in the work of Simion [23].

Lemma 2.1 (Simion [23, Section 3.3]). *Each facet Q_n^B of Simion's type B associahedron contains exactly one B -diagonal of the form $\{i, \bar{i}\}$ connecting an antipodal pair of points.*

2.2. The Legendre polytope or “full” type A root polytope. Consider an $(n + 1)$ -dimensional Euclidean space with orthonormal basis $\{e_1, e_2, \dots, e_{n+1}\}$. The convex hull of the vertices $\pm 2e_1, \dots, \pm 2e_{n+1}$ is an $(n + 1)$ -dimensional cross-polytope. The intersection of this cross-polytope with the hyperplane $x_1 + x_2 + \dots + x_{n+1} = 0$ is an n -dimensional centrally symmetric polytope P_n first studied by Cho [9]. It is called the *Legendre polytope* in the work of Heteyi [13], since the polynomial $\sum_{j=0}^n f_{j-1} \cdot ((x-1)/2)^j$ is the n th Legendre polynomial, where f_i is the number of i -dimensional faces in any pulling triangulation of the boundary of P_n . See Lemma 2.9 below. Furthermore, it is called the “full” type A root polytope in the work of Ardila–Beck–Hoşten–Pfeifle–Seashore [1]. It has $n(n + 1)$ vertices consisting of all points of the form $e_i - e_j$ where $i \neq j$.

We use the shorthand notation (i, j) for the vertex $e_j - e_i$ of the Legendre polytope P_n . We may think of these vertices as the set of all directed nonloop edges on the vertex set $\{1, 2, \dots, n + 1\}$. A subset of these edges is contained in some face of P_n exactly when there is no $i \in \{1, 2, \dots, n + 1\}$ that is both the head and the tail of a directed edge. Equivalently, the faces are described as follows.

Lemma 2.2. *The faces of the Legendre polytope P_n are of the form $\text{conv}(I \times J) = \text{conv}(\{(i, j) : i \in I, j \in J\})$ where I and J are two non-empty disjoint subsets of the set $\{1, 2, \dots, n + 1\}$. The dimension of a face is given by $|I| + |J| - 2$. A face is a facet if and only if the union of I and J is the set $\{1, 2, \dots, n + 1\}$.*

Especially, when the two sets I and J both have cardinality two, the associated face is geometrically a square. Furthermore, the other two-dimensional faces are equilateral triangles.

Affine independent subsets of vertices of faces of the Legendre polytope are easy to describe. A set $S = \{(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)\}$ is a $(k - 1)$ -dimensional simplex if and only if, disregarding the orientation of the directed edges, the set S contains no cycle, that is, it is a forest [13, Lemma 2.4].

The Legendre polytope P_n contains the polytope P_n^+ , defined as the convex hull of the origin and the set of points $e_i - e_j$, where $i < j$. The polytope P_n^+ was first studied by Gelfand, Graev and Postnikov [12] and later by Postnikov [21]. Some of the results for P_n^+ may be easily generalized to P_n .

2.3. Pulling triangulations. The notion of pulling triangulations is originally due to Hudson [14, Lemma 1.4]. For more modern formulations, see [24, Lemma 1.1] and [2, End of Section 2]. We refer to [13, Section 2.3] for the version presented here.

For a polytopal complex \mathcal{P} and a vertex v of \mathcal{P} , let $\mathcal{P} - v$ be the complex consisting of all faces of \mathcal{P} not containing the vertex v . Also for a facet F let $\mathcal{P}(F)$ be the complex of all faces of \mathcal{P} contained in F .

Definition 2.3 (Hudson [14, Lemma 1.4]). *Let \mathcal{P} be a polytopal complex and let $<$ be a linear order on the set V of its vertices. The pulling triangulation $\Delta(\mathcal{P})$ with respect to $<$ is defined recursively as follows. We set $\Delta(\mathcal{P}) = \mathcal{P}$ if \mathcal{P} consists of a single vertex. Otherwise let v be the least element of V with respect to $<$ and set*

$$\Delta(\mathcal{P}) = \Delta(\mathcal{P} - v) \cup \bigcup_F \{\text{conv}(\{v\} \cup G) : G \in \Delta(\mathcal{P}(F))\},$$

where the union runs over the facets F not containing v of the maximal faces of \mathcal{P} which contain v . The triangulations $\Delta(\mathcal{P} - v)$ and $\Delta(\mathcal{P}(F))$ are with respect to the order $<$ restricted to their respective vertex sets.

Theorem 2.4 (Hudson [14, Lemma 1.4]). *The pulling triangulation $\Delta(\mathcal{P})$ is a triangulation of the polytopal complex \mathcal{P} which does not introduce any new vertices.*

We will need the following two fundamental results on unimodular matrices.

Theorem 2.5 (Stanley [24, Example 2.4 and Corollary 2.7]). *Suppose that one of the vertices of a polytope P is the origin and that the matrix whose rows are the coordinates of the vertices of P is totally unimodular. Let $<$ be any ordering on the vertex set of P such that the origin is the least vertex with respect to $<$. Then the f -vector of the induced triangulation depends only on the original polytope P and not on the pulling order $<$.*

Corollary 2.6. *Suppose the vertices of a polytope P lie in a hyperplane not containing the origin, and that the matrix whose rows are the coordinates of the vertices of P is totally unimodular. Then the f -vector of any pulling triangulation of P is independent of the pulling order.*

This statement follows by applying Theorem 2.5 to the polytope $P' = \text{conv}(P \cup \{0\})$, after noting that adding a row of zeroes does not change the unimodular property. The set of pulling triangulations of P' obtained by pulling 0 first coincides with the set of triangulations obtained by taking an arbitrary pulling triangulation of P and replacing each facet σ in this triangulation by the cone of σ having apex 0.

Theorem 2.7. *The incidence matrix of a directed graph is totally unimodular.*

This is a well-known fact, whose proof may be found in many textbooks. See Veblen and Franklin for a short proof [25]. As noted by Schrijver [22, Section 19.3, Example 2], perhaps the earliest proof goes back to Poincaré [20, page 304].

Corollary 2.8 (Heteyi [13, Corollary 4.11]). *The f -vector of any pulling triangulation of the boundary of the Legendre polytope P_n is independent of the pulling order.*

Proof. Let $F = \text{conv}(I \times J)$ be a proper face of the Legendre polytope P_n ; see Lemma 2.2. The matrix whose rows are the vertices of F is the incidence matrix of the directed graph on the vertex set $\{1, \dots, n\}$, consisting of the edges $\{(i, j) : i \in I, j \in J\}$. By Theorem 2.7 this matrix is unimodular. By Corollary 2.6, any pulling triangulation of F has the same face numbers. Consider now any pulling

order $<$ on the boundary complex ∂P_n of the Legendre polytope P_n . Using the fact the face lattice of any polytope is Eulerian, by inclusion-exclusion we obtain

$$f_i(\Delta_{<}(\partial P_n)) = \sum_F (-1)^{n-1-\dim(F)} \cdot f_i(\Delta_{<}(F)),$$

where F ranges over all non-trivial faces of P_n . Since the right-hand side does not depend on the pulling order $<$, the result follows. \square

An analogous statement to Corollary 2.8 also holds for the positive root polytope P_n^+ of type A .

2.4. Face vectors of pulling triangulations of the Legendre polytope. Among all triangulations of the boundary of the Legendre polytope P_n obtained by pulling the vertices, counting faces is most easily performed for the *lexicographic triangulation* in which we pull (i, j) before (i', j') exactly when $i < i'$ or when $i = i'$ and $j < j'$. Counting faces in this triangulation amounts to counting lattice paths; see [13, Lemma 5.1] and [1, Proposition 17]. From this we obtain the following expression for the face numbers.

Lemma 2.9 (Hetyei [13, Theorem 5.2]). *For any pulling triangulation of the boundary of the Legendre polytope P_n , the number f_{j-1} of $(j-1)$ -dimensional faces is*

$$(2.1) \quad f_{j-1} = \binom{n+j}{j} \binom{n}{j} \quad \text{for } 0 \leq j \leq n.$$

It was first noted in [11] that these face numbers are the same as that of Simion's type B associahedron Q_n^B . Theorem 6.1 will explicitly explain this fact by showing that the boundary of Q_n^B can be realized as a pulling triangulation of the Legendre polytope.

3. THE FLAG PROPERTY

Recall that a simplicial complex is a *flag complex* if every minimal nonface has two elements. The main result of this section is the following.

Theorem 3.1. *Every pulling triangulation of the boundary of the Legendre polytope P_n is a flag simplicial complex.*

To prove this theorem and Theorem 6.1, we need the following observation.

Lemma 3.2. *Let $\{x_1, x_2, y_1, y_2\}$ be a four element subset of the set $\{1, 2, \dots, n+1\}$. Then the set $\{x_1, x_2\} \times \{y_1, y_2\} = \{(x_1, y_1), (x_1, y_2), (x_2, y_2), (x_2, y_1)\}$ is the vertex set of a square face of the Legendre polytope P_n , and the sets $\{(x_1, y_1), (x_2, y_2)\}$ and $\{(x_1, y_2), (x_2, y_1)\}$ are the diagonals of this square. For any pulling triangulation the diagonal containing the vertex that was pulled first is an edge of the triangulation and the other diagonal is not an edge.*

Theorem 3.1 may be rephrased as follows.

Theorem 3.3. *Let $<$ be any order on the vertices of the Legendre polytope P_n and consider the pulling triangulation of the boundary of P_n induced by this order. Suppose we are given a set of vertices $\{(u_1, v_1), (u_2, v_2), \dots, (u_k, v_k)\}$ such that any pair $\{(u_i, v_i), (u_j, v_j)\}$ is an edge in the pulling triangulation. Then this set is a face of the pulling triangulation.*

Proof. Assume that the pulling order on the set of vertices is given by $(u_1, v_1) < (u_2, v_2) < \dots < (u_k, v_k)$. We prove the statement by induction on k . The statement is directly true for $k \leq 2$. Assume from now on $k \geq 3$. Observe first that the sets $\{u_1, \dots, u_k\}$ and $\{v_1, \dots, v_k\}$ must be disjoint. Indeed, if for any $i \neq j$ we have $u_i = v_j$ then the pair of vertices $\{(u_i, v_i), (u_j, v_j)\}$ is not an edge, contradicting our assumption. Thus our set of vertices is contained in the face $\text{conv}(\{u_1, \dots, u_k\} \times \{v_1, \dots, v_k\})$ of the polytope P_n . Note that the lists u_1, \dots, u_k and v_1, \dots, v_k may contain repeated elements.

We next show that (u_1, v_1) is the least element with respect of the order $<$ in the set $\{u_1, \dots, u_k\} \times \{v_1, \dots, v_k\}$. To prove this, suppose (u_i, v_j) is the least element. If $i = j$ then either $i = 1$ or we may use that we were given $(u_1, v_1) < (u_i, v_i)$ for all $i > 1$. If $i \neq j$ but $u_i = u_j$ then we have $(u_i, v_j) = (u_j, v_j)$ and again we are done since $(u_j, v_j) > (u_1, v_1)$ if $j > 1$. Similarly, if $i \neq j$ but $v_i = v_j$ then we may use $(u_i, v_j) = (u_i, v_i)$. Finally if $i \neq j$, $u_i \neq u_j$ and $v_i \neq v_j$ hold, then apply Lemma 3.2 to the square with vertex set (u_i, v_i) , (u_i, v_j) , (u_j, v_j) and (u_j, v_i) . Since (u_i, v_j) is the least with respect to the pulling order, the diagonal $\{(u_i, v_i), (u_j, v_j)\}$ is not an edge of the triangulation, contradicting our assumption. Hence we conclude (u_1, v_1) is the least vertex in the set $\{u_1, \dots, u_k\} \times \{v_1, \dots, v_k\}$.

We claim the smallest face of the boundary of P_n containing the vertex set $\{(u_2, v_2), \dots, (u_k, v_k)\}$ does not contain the vertex (u_1, v_1) . Assume, by way of contradiction, that $\{u_1, u_2, \dots, u_k\} = \{u_2, \dots, u_k\}$ and $\{v_1, v_2, \dots, v_k\} = \{v_2, \dots, v_k\}$ hold. Then there is an index $i > 1$ and an index $j > 1$ such that $u_1 = u_i$ and $v_1 = v_j$ hold. Note that we must also have $v_1 \neq v_i$ and $u_1 \neq u_j$. Consider the square face with the following four vertices: (u_1, v_1) , $(u_1, v_i) = (u_i, v_i)$, (u_j, v_i) and $(u_j, v_j) = (u_j, v_1)$. The first vertex that was pulled is (u_1, v_1) . By Lemma 3.2 the edge $\{(u_i, v_i), (u_j, v_j)\}$ is not an edge in the pulling triangulation, contradicting our assumptions.

By the induction hypothesis the set $\{(u_2, v_2), \dots, (u_k, v_k)\}$ is a face in the polytopal complex $\partial P_n - \{w : w \leq (u_1, v_1)\}$. By definition, we obtain that $\{(u_1, v_1)\} \cup \{(u_2, v_2), \dots, (u_k, v_k)\}$ is a face of the pulling triangulation. \square

Since the Cartesian product of an m -dimensional simplex and an n -dimensional simplex is a face of the Legendre polytope of dimension $m + n + 1$, we obtain the following corollary.

Corollary 3.4. *Every pulling triangulation of the Cartesian product of two simplices is a flag complex.*

4. THE ARC REPRESENTATION OF Γ_n^B

In this section we describe a representation of Γ_n^B as a simplicial complex whose vertices are centrally symmetric pairs of “arcs” on a circle. This representation has a natural circular symmetry.

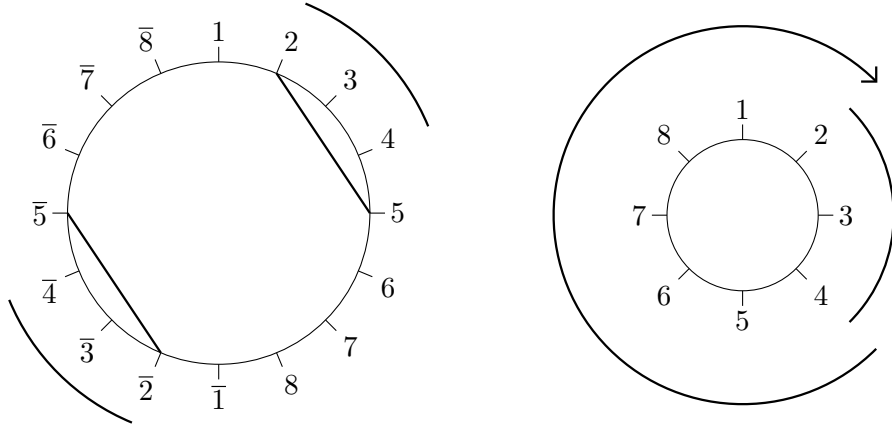


FIGURE 1. The arc representation of the B -diagonal consisting of the two diagonals $\{2, 5\}$ and $\{\bar{2}, \bar{5}\} = \{10, 13\}$ is the two arcs $[2, 4]$ and $[\bar{2}, \bar{4}] = [10, 12]$. By considering the arcs modulo $n + 1 = 8$ (see the second circle) we obtain that this B -diagonal is represented by the arrow $(4, 2)$.

Consider a regular $(2n + 2)$ -gon whose vertices are labeled $1, 2, \dots, n + 1, \bar{1}, \bar{2}, \dots, \overline{n + 1}$ in the clockwise order. Identify each vertex \bar{i} with $n + 1 + i$ for $i = 1, 2, \dots, n + 1$. Subject to this identification, each B -diagonal, that is, a pair of diagonals, may be represented as an unordered pair of diagonals of the form $\{\{u, v\}, \{u + n + 1, v + n + 1\}\}$ for some $\{u, v\} \subseteq \{1, 2, \dots, 2n + 2\}$, where addition is modulo $2n + 2$. For B -diagonals $\{k, \bar{k}\}$ joining antipodal points, the unordered pair $\{\{k, k + n + 1\}, \{k + n + 1, k + 2n + 2\}\}$ contains two copies of the same two-element set.

For any two points x and y on the circle $\mathbb{R}/(2n + 2)\mathbb{Z}$ which are not antipodal, let $[x, y]$ denote the shortest arc from x to y .

Definition 4.1. We define the arc-representation on the vertices of Γ_n^B as follows. Subject to the above identifications, represent the B -diagonal $\{\{u, v\}, \{u + n + 1, v + n + 1\}\}$ with the centrally symmetric pair of arcs $\{[u, v - 1], [u + n + 1, v + n]\}$ on the circle $\mathbb{R}/(2n + 2)\mathbb{Z}$.

The representation above is well-defined: for any pair of vertices u and v , the arc $[u, v - 1]$ and the arc $[u + n + 1, v + n]$ form a centrally symmetric pair, so the definition is independent of the selection of the element of $\{\{u, v\}, \{u + n + 1, v + n + 1\}\}$. Note that for B -diagonals of the form $\{\{k, n + 1 + k\}, \{k, n + 1 + k\}\}$ corresponding to antipodal pairs of points, the union of the arcs $[k, k + n]$ and $[k + n + 1, k - 1]$ is not the full circle.

See Figure 1 for an example where $n = 7$ with the B -diagonal $\{\{2, 5\}, \{\bar{2}, \bar{5}\}\}$.

Lemma 4.2. The arc-representation of the vertices of Γ_n^B is one-to-one: distinct B -diagonals are mapped to distinct centrally symmetric pairs of arcs.

Indeed, the pair $\{[u, v - 1], [u + n + 1, v + n]\}$ can only be the image of the unordered pair $\{\{u, v\}, \{u + n + 1, v + n + 1\}\}$.

The following theorem plays an important role in connecting the type B associahedron with the Legendre polytope.

Theorem 4.3. *The B -diagonal represented by the pair of arcs $\{[u_1, v_1 - 1], [u_1 + n + 1, v_1 + n]\}$ and the B -diagonal represented by the pair of arcs $\{[u_2, v_2 - 1], [u_2 + n + 1, v_2 + n]\}$ are noncrossing if and only if for either arc $I \in \{[u_1, v_1 - 1], [u_1 + n + 1, v_1 + n]\}$ and for either arc $J \in \{[u_2, v_2 - 1], [u_2 + n + 1, v_2 + n]\}$, the arcs I and J are either nested or disjoint.*

Proof. Assume first that the two B -diagonals cross. Since we may replace $\{u_i, v_i\}$ with $\{u_i + n + 1, v_i + n + 1\}$ if necessary, without loss of generality we may assume that the diagonal $\{u_1, v_1\}$ crosses the diagonal $\{u_2, v_2\}$. Exactly one of the endpoints of the diagonal $\{u_2, v_2\}$ must then belong to the arc $[u_1 + 1, v_1 - 1]$ and the other one does not belong even to the larger arc $[u_1, v_1]$. If $u_2 \in [u_1 + 1, v_1 - 1]$ and $v_2 \notin [u_1, v_1]$ then we have

$$[u_1, v_1 - 1] \cap [u_2, v_2 - 1] = [u_2, v_1 - 1].$$

This arc contains u_2 , but does not contain u_1 (since $u_2 \in [u_1 + 1, v_1 - 1]$), nor does it contain $v_2 - 1 \notin [u_1, v_1 - 1]$. The arcs $[u_1, v_1 - 1]$ and $[u_2, v_2 - 1]$ do not contain each other and they are not disjoint.

If $v_2 \in [u_1 + 1, v_1 - 1]$ and $u_2 \notin [u_1, v_1]$ then we have

$$[u_1, v_1 - 1] \cap [u_2, v_2 - 1] = [u_1, v_2 - 1].$$

Similar to the previous case, we obtain that the arcs $[u_1, v_1 - 1]$ and $[u_2, v_2 - 1]$ do not contain each other and they are not disjoint.

Assume next that the two diagonals do not cross. If $u_2 \in [u_1 + 1, v_1 - 1]$ then we must have $v_2 \in [u_2 + 1, v_1]$, implying

$$[u_1, v_1 - 1] \cap [u_2, v_2 - 1] = [u_2, v_2 - 1].$$

Similarly, if $v_2 \in [u_1 + 1, v_1 - 1]$ then we must have $u_2 \in [u_1, v_2]$, implying

$$[u_1, v_1 - 1] \cap [u_2, v_2 - 1] = [u_2, v_2 - 1].$$

Finally, if neither u_2 nor v_2 belongs to $[u_1 + 1, v_1 - 1]$ then either the arc $[u_2, v_2 - 1]$ contains the arc $[u_1, v_1 - 1]$ or it is disjoint from it. The above argument remains valid if we replace either (or both) of $[u_1, v_1 - 1]$ and $[u_2, v_2 - 1]$ with the arc $[u_1 + n + 1, v_1 + n]$ or $[u_2 + n + 1, v_2 + n]$, respectively. \square

Corollary 4.4. *The B -diagonal represented by the pair of arcs $\{[u_1, v_1 - 1], [u_1 + n + 1, v_1 + n]\}$ and the B -diagonal represented by the pair of arcs $\{[u_2, v_2 - 1], [u_2 + n + 1, v_2 + n]\}$ are noncrossing if and only if the set $[u_1, v_1 - 1] \cup [u_1 + n + 1, v_1 + n]$ and the set $[u_2, v_2 - 1] \cup [u_2 + n + 1, v_2 + n]$ are nested or disjoint.*

5. EMBEDDING Γ_n^B AS A FAMILY OF SIMPLICES ON ∂P_n

In this section we describe a way to represent the boundary complex Γ_n^B of the Simion type B associahedron Q_n^B as a family of simplices on the vertex set of the Legendre polytope P_n . We do this so that each simplex is contained in a face of the boundary ∂P_n of P_n . In Section 6 we will prove

that our map represents the boundary complex of the type B associahedron as a pulling triangulation of ∂P_n .

We begin by defining a bijection between the vertex set of Γ_n^B and that of P_n . Recall that we use the shorthand notation (i, j) for the vertex $e_j - e_i$ of P_n . We refer to (i, j) as *the arrow from i to j* . Using the term “arrow” as opposed to “directed edge” will eliminate the confusion that $e_j - e_i$ is a vertex of P_n . Instead we think of it as an arrow from i to j in the complete directed graph on the vertex set $\{1, 2, \dots, n+1\}$ having no loops.

Definition 5.1. *Let $\{\{i, j\}, \{\bar{i}, \bar{j}\}\}$ be the arc representation of a B -diagonal in Γ_n^B , where $1 \leq i \leq n+1$ and $i < j$. Define the arrow representation of this B -diagonal in P_n to be the arrow (j, i) .*

In other words, the arrow encodes the complement of the image of the arcs in the circle $\mathbb{R}/(n+1)\mathbb{Z}$. We refer to the second circle in Figure 1 for the continuation of the example of the B -diagonal $\{\{2, 5\}, \{\bar{2}, \bar{5}\}\}$.

When making this definition explicit for a B -diagonal $\{\{i, j\}, \{\bar{i}, \bar{j}\}\}$, we obtain several cases:

- (1) For each i satisfying $2 \leq i \leq n+1$, represent the B -diagonal $\{i, \bar{i}\}$ connecting two antipodal points with the arrow $(i-1, i)$. Represent the B -diagonal $\{1, \bar{1}\}$ with the arrow $(n+1, 1)$.
- (2) For each i and j satisfying $1 \leq i < i+1 < j \leq n+1$, represent the B -diagonal $\{\{i, j\}, \{\bar{i}, \bar{j}\}\}$ with the arrow $(j-1, i)$.
- (3) For each i and j satisfying $2 \leq j < i \leq n+1$, represent the B -diagonal $\{\{i, \bar{j}\}, \{\bar{i}, j\}\}$ with the arrow $(j-1, i)$. For each i satisfying $2 \leq i \leq n+1$, represent the B -diagonal $\{\{1, \bar{i}\}, \{\bar{1}, i\}\}$ with the arrow $(n+1, i)$.

This representation yields a bijection between B -diagonals and arrows. The inverse map is given as follows:

- (a) For i satisfying $1 \leq i \leq n$, the arrow $(i, i+1)$ represents the B -diagonal $\{i+1, \overline{i+1}\}$ and the arrow $(n+1, 1)$ represents the B -diagonal $\{1, \bar{1}\}$.
- (b) For each i and j satisfying $1 \leq i < j \leq n$, the arrow (j, i) represents the B -diagonal $\{\{i, j+1\}, \{\bar{i}, \overline{j+1}\}\}$.
- (c) For each i and j satisfying $1 \leq j < i-1 < i \leq n+1$, the arrow (j, i) represents the B -diagonal $\{\{i, \overline{j+1}\}, \{\bar{i}, j+1\}\}$, and for $2 \leq i \leq n+1$, the arrow $(n+1, i)$ represents the B -diagonal $\{\{1, \bar{i}\}, \{\bar{1}, i\}\}$.

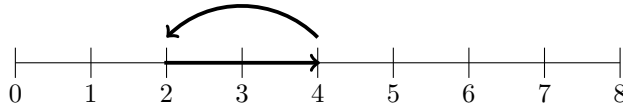
The B -diagonals in item (1) of Definition 5.1 may be thought of as a “degenerate case” of the B -diagonals in item (3). In the case when $i = j$, the set $\{\{\bar{i}, j\}, \{i, \bar{j}\}\}$ becomes a singleton, and the rules in item (1) may be obtained by extending the rules in item (3) in an obvious way. On the other hand, Definition 5.1 has a simpler form in terms of the arc-representation described in Section 4 and in terms of the following continuous map.

Definition 5.2. Define the map $\pi : \mathbb{R}/(2n+2)\mathbb{Z} \rightarrow \mathbb{R}/(n+1)\mathbb{Z}$ to be the modulo $n+1$ map. Furthermore, identify the circle $\mathbb{R}/(n+1)\mathbb{Z}$ with the half-open interval $(0, n+1]$. Thus π sends each $x \in (0, n+1]$ to x and each $x \in (n+1, 2n+2]$ to $x - n - 1$.

Observe that the map π depends on n . However, we suppress this dependency by not writing π_n . Also observe that the map π is a two-to-one mapping: for each $y \in \mathbb{R}/(n+1)\mathbb{Z}$ we have $|\pi^{-1}(y)| = 2$.

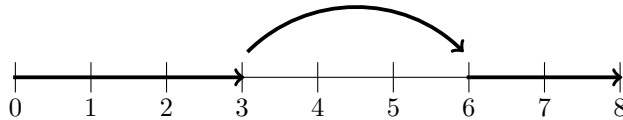
Remark 5.3. For any pair of arcs $\{[u, v-1], [u+n+1, v+n]\}$ there is a unique way to select u to be an element of the set $\{1, 2, \dots, n+1\}$, that is, $\pi(u) = u$. We may distinguish two cases depending upon whether the arc $[u, v-1]$ is a subset of the arc $[u, n+1]$ or not.

- (i) If $[u, v-1] \subseteq [u, n+1]$ then visualize the set $\pi([u, v-1]) = \pi([u+n+1, v+n])$ as the subinterval $[u, v-1]$ of $(0, n+1]$. The direction of both arcs $[u, v-1]$ and $[u+n+1, v+n]$ corresponds to parsing the interval $[u, v-1]$ in increasing order. Adding the associated backward arrow $(v-1, u)$ closes a directed cycle with this directed interval.



As an example, when $n = 7$ then the B -diagonal $\{\{2, 5\}, \{\bar{2}, \bar{5}\}\}$ is represented by the backward arrow $(4, 2)$ as drawn above on the interval $(0, 8]$.

- (ii) If the arc $[u, v-1]$ is not contained in the arc $[u, n+1]$ then $n+1 < v-1 < u+n+1$. The integer $\pi(v-1) = v-1 - (n+1)$ is congruent to $v-1$ modulo $(n+1)$ and satisfies $1 \leq \pi(v-1) < u$. The image of the arc $[u, v-1]$ under π , that is, $\pi([u, v-1]) = \pi([u+n+1, v+n])$ is then the subset $(0, \pi(v)-1] \cup [u, n+1]$ of the interval $(0, n+1]$. We may consider $(0, \pi(v)-1] \cup [u, n+1]$ as a “wraparound interval” modulo $n+1$ from u to $\pi(v-1)$. The direction of both pieces corresponds to listing the elements of this “wraparound interval” in increasing order modulo $n+1$. Adding the associated forward arrow $(\pi(v-1), u)$ closes a directed cycle with the directed wraparound interval.



For instance, when $n = 7$ the B -diagonal $\{\{4, \bar{6}\}, \{\bar{4}, 6\}\}$ yields the forward arrow $(3, 6)$.

Proposition 5.4. The B -diagonal represented by the arrow $(\pi(v_1-1), \pi(u_1))$ and the B -diagonal represented by the arrow $(\pi(v_2-1), \pi(u_2))$ are noncrossing if and only if the images $\pi([u_1, v_1-1])$ and $\pi([u_2, v_2-1])$ are disjoint or contain each other.

Proof. The statement is an easy consequence of the following observation. Both arcs of the arc representation $\{[u, v-1], [u+n+1, v+n]\}$ of a B -diagonal are mapped onto the same arc $[\pi(u), \pi(v-1)]$ by π and $[u, v-1] \cup [u+n+1, v+n] = \pi^{-1}([\pi(u), \pi(v-1)])$. \square

Next we translate the noncrossing conditions for B -diagonals into conditions for the arrows representing them.

Proposition 5.5. *Suppose a pair of B -diagonals is represented by a pair of arrows as defined in Definition 5.1. These B -diagonals cross if and only if one of the following conditions is satisfied:*

- (1) *Both arrows are backward and they cross.*
- (2) *Both arrows are forward and they do not nest.*
- (3) *One arrow is forward, the other one is backward, and the backward arrow nests or crosses the forward arrow.*
- (4) *The head of one arrow is the tail of the other arrow.*

Proof. Suppose the arc representations of the two B -diagonals are $\{[u_1, v_1 - 1], [u_1 + n + 1, v_1 + n]\}$ and $\{[u_2, v_2 - 1], [u_2 + n + 1, v_2 + n]\}$, respectively. By Proposition 5.4, the represented B -diagonals are crossing if and only if the images $\pi([u_1, v_1 - 1])$ and $\pi([u_2, v_2 - 1])$ are not disjoint and do not contain each other.

We will consider three cases, depending on the direction of the two arrows $(\pi(v_1 - 1), \pi(u_1))$ and $(\pi(v_2 - 1), \pi(u_2))$. These arrows are either both forward, both backward or have opposite directions.

If both arrows are backward then neither the image $\pi([u_1, v_1 - 1])$ nor the image $\pi([u_2, v_2 - 1])$ contain the point $n + 1$. Two such intervals intersect nontrivially in an interval of positive length exactly when the corresponding arrows cross. They intersect in a single point exactly when there is a vertex that is the tail of one of the arrows and the head of the other arrow.

If both arrows are forward then both images $\pi([u_1, v_1 - 1])$ and $\pi([u_2, v_2 - 1])$ contain the point $n + 1$, so they cannot be disjoint. They do not contain each other exactly when the corresponding arrows are not nested. Note that a pair of forward arrows such that the head of one arrow is the same as the tail of the other is particular example of a pair of nonnested arrows.

Assume finally that one of the arrows, say $(\pi(v_1 - 1), \pi(u_1))$, is a backward arrow and the other one, say $(\pi(v_2 - 1), \pi(u_2))$, is a forward arrow. The image $\pi([u_2, v_2 - 1])$ cannot be a subset of $\pi([u_1, v_1 - 1])$ as the first image contains the point $n + 1$ whereas the second does not. The image $\pi([u_1, v_1 - 1])$ is the interval $[\pi(u_1), \pi(v_1 - 1)]$, whereas the image $\pi([u_2, v_2 - 1])$ is the union $(0, \pi(v_2 - 1)] \cup [\pi(u_2), n + 1]$. The intersection of these two sets has either zero, one or two connected components. If one component is a single point then this point is the head of one arrow and the tail of the other. If both components are non-trivial intervals then the backward arrow nests the forward arrow. If they intersect in one interval but the image $\pi([u_1, v_1 - 1])$ does not contain the image $\pi([u_2, v_2 - 1])$ then the two arrows cross. Finally, if the image $\pi([u_1, v_1 - 1])$ does contain the image $\pi([u_2, v_2 - 1])$ then the arrows neither cross nor nest. \square

An immediate consequence of Proposition 5.5 and Lemma 2.2 is the following corollary.

Corollary 5.6. *Noncrossing sets of B -diagonals correspond to subsets of vertices contained in a facet of the Legendre polytope P_n .*

Order of nodes	Arrow first pulled	Edge	Not an edge
$x_1 < x_2 < y_1 < y_2$	(x_2, y_1)		
$x_1 < y_1 < x_2 < y_2$	(x_2, y_1)		
$x_1 < y_1 < y_2 < x_2$	(x_2, y_2)		
$y_1 < x_1 < x_2 < y_2$	(x_1, y_1)		
$y_1 < x_1 < y_2 < x_2$	(x_1, y_1) or (x_2, y_2)		
$y_1 < y_2 < x_1 < x_2$	(x_1, y_2)		

TABLE 1. Pairs of arrows that are edges or minimal nonfaces

6. THE TYPE B ASSOCIAHEDRON REPRESENTED AS A PULLING TRIANGULATION

In this section we show that the arc representation given in Definition 5.1 constitutes the boundary of Simion's type B associahedron Q_n^B as a pulling triangulation of the boundary of the Legendre polytope P_n . Our main result is the following.

Theorem 6.1. *Let $<$ be any linear order on the vertex set of the Legendre polytope P_n subject to the following conditions:*

- (1) $(x_1, y_1) < (x_2, y_2)$ whenever $x_1 - y_1 > 0 > x_2 - y_2$.
- (2) On the subset of vertices (x, y) satisfying $x < y$, the order relation $(x_1, y_1) < (x_2, y_2)$ holds whenever the interval $[x_1, y_1] = \{x_1, x_1 + 1, \dots, y_1\}$ is contained in the interval $[x_2, y_2]$.
- (3) On the subset of vertices (x, y) satisfying $x > y$, the order relation $(x_1, y_1) < (x_2, y_2)$ holds whenever the interval $[y_1, x_1]$ is contained in the interval $[y_2, x_2]$.

Then the arc representation of Γ_n^B given in Definition 5.1 is a pulling triangulation of the boundary of the Legendre polytope P_n with respect to $<$.

Proof. Fix any pulling order $<$ satisfying the above conditions. The first condition requires all backward arrows to precede all forward arrows. The next two conditions require that for a pair of nested arrows of the same direction the nested arrow should precede the nesting arrow.

Recall that Γ_n^B is a flag complex and its minimal nonfaces are the pairs of crossing B -diagonals. By Theorem 3.1 the pulling triangulation we defined is also a flag complex. It suffices to show that the minimal nonfaces are in bijection. Equivalently, for any pair of arrows $\{(x_1, y_1), (x_2, y_2)\}$ that form an edge in the pulling triangulation of P_n , these arrows correspond to a pair of noncrossing B -diagonals in Γ_n^B . By Proposition 5.5 this amounts to showing the following: backward arrows cannot cross, forward arrows must nest, and for a pair of arrows of opposite direction the backward arrow cannot cross or nest the forward arrow.

Given a four-element subset $\{x_1, x_2, y_1, y_2\}$ of $\{1, 2, \dots, n+1\}$, where $x_1 < x_2$ and $y_1 < y_2$, consider the four arrows in the set $\{x_1, x_2\} \times \{y_1, y_2\}$. As seen in Lemma 3.2, these four arrows form the vertex set of a square face of the Legendre polytope P_n , and only the diagonal which contains the first vertex to be pulled is an edge of the pulling triangulation. Table 1 lists all six possible orderings of this four element set.

In each of the six cases we note which vertex is pulled first, which diagonal of the square becomes an edge in the triangulation and which diagonal does not become an edge. Note that in the fifth row of Table 1 we have two possibilities for selecting the vertex to be pulled first. However, these two vertices belong to the same diagonal. In every case we obtain that none of the pairs of arrows with distinct heads and tails that is an edge corresponds to a pair of crossing B -diagonals. \square

As a corollary we obtain Simion's polytopal result.

Corollary 6.2 (Simion [23, Theorem 1]). *Simion's simplicial complex Γ_n^B is the boundary complex of an n -dimensional simplicial polytope.*

Following Simion, let Γ_n^A be the simplicial complex whose vertex set is the $\binom{n+2}{2} - (n+2)$ diagonals of a convex $(n+2)$ -gon. A set of diagonals forms a face if they do not intersect in the interior of the $(n+2)$ -gon. This simplicial complex is known as the associahedron. We now obtain the following classical result; see the work of Haiman, Lee [15] and Stasheff. For a brief history, see the introduction of [6].

Corollary 6.3. *The associahedron Γ_n^A is the boundary complex of an $(n-1)$ -dimensional simplicial polytope.*

Proof. The associahedron Γ_n^A is the link of a vertex of the form $\{i, \bar{i}\}$ in the Simion type B associahedron Q_n^B . Hence a polytopal realization of the associahedron is obtained by the vertex figure of this vertex in the Simion type B associahedron Q_n^B of dimension n . \square

We end this section by describing the structure of all facets of Simion's type B associahedron in terms of arrows.

Theorem 6.4. *A set of arrows $S = \{(x_1, y_1), \dots, (x_n, y_n)\}$ represents a facet of Simion's type B associahedron Q_n^B if and only if the following conditions are satisfied:*

- (1) *There is exactly one k satisfying $1 \leq k \leq n + 1$ such that $(k - 1, k)$ (or $(n + 1, 1)$ if $k = 1$) belongs to the set S . We call this k the type of the facet.*
- (2) *Backward arrows do not nest any forward arrow, in particular, they cannot nest $(k - 1, k)$ if $k > 1$.*
- (3) *If $k = 1$ then there is no forward arrow in the set S .*
- (4) *Forward arrows must nest. In particular, if $k > 1$ then for each forward arrow $(x, y) \in S$ must satisfy $x \leq k - 1$ and $y \geq k$. (Forward arrows must nest $(k - 1, k)$.)*
- (5) *No head of an arrow in the set S is also the tail of another arrow in S .*
- (6) *No two arrows cross.*

Proof. Condition (1) is equivalent to Lemma 2.1. Except for Condition (3), the remaining conditions are stated for all faces in Proposition 5.5. To prove Condition (3), observe that $k = 1$ implies that the backward arrow $(n + 1, 1)$ belongs to S . This arrow would nest any forward arrow, contradicting Condition (3) in Proposition 5.5. \square

7. TRIANGULATING CHO'S DECOMPOSITION

The type A root polytope P_n^+ is the convex hull of the origin and the set of points $\{e_i - e_j : 1 \leq i < j \leq n + 1\}$. Cho [9] gave a decomposition of the Legendre polytope P_n into $n + 1$ copies of P_n^+ as follows. The symmetric group \mathfrak{S}_{n+1} acts on the Euclidean space \mathbb{R}^{n+1} by permuting the coordinates, that is, the permutation $\sigma \in \mathfrak{S}_{n+1}$ sends the basis vector e_i into $e_{\sigma(i)}$. Hence the permutation σ acts on the Legendre polytope P_n by sending each $e_i - e_j$ into $e_{\sigma(i)} - e_{\sigma(j)}$. Cho's main result is the following decomposition.

Theorem 7.1 (Cho [9, Theorem 16]). *The Legendre polytope P_n has the decomposition*

$$P_n = \bigcup_{k=0}^n \zeta^k(P_n^+)$$

where ζ is the cycle $(1, 2, \dots, n + 1)$. Furthermore, for $0 \leq k < r \leq n$ the polytopes $\zeta^k(P_n^+)$ and $\zeta^r(P_n^+)$ have disjoint interiors.

In this section we show that each copy $\zeta^k(P_n^+)$ of P_n^+ is the union of simplices of the triangulation given in Definition 5.1, representing the boundary complex Γ_n^B of Simion's type B associahedron Q_n^B .

Theorem 7.2. *Every facet F of the arc representation of Γ_n^B given in Definition 5.1 is contained in $\zeta^{k-1}(P_n^+)$ where k is the unique arrow of the form $(k - 1, k)$ in F or $(n + 1, 1)$ if $k = 1$. Equivalently, the facet F is contained in $\zeta^k(P_n^+)$ exactly when it represents a facet of Γ_n^B that contains the diagonal $\{k, \bar{k}\}$.*

Proof. The polytope P_n^+ is the convex hull of the origin and of all vertices that are represented by backward arrows. The facets of type 1 (as defined in Theorem 6.4) form a pulling triangulation of the part of the boundary of P_n^+ that does not contain the origin. In fact, the restriction of the pulling order to the backward arrows may be taken in the *revlex order*, as defined in [13, Definition 4.5], giving rise to the *standard triangulation* described in [12]. A facet in our triangulation of P_n belongs to the

standard triangulation of the boundary of the part of P_n^+ not containing the origin exactly when it has type 1.

Observe next that the effect of ζ on the arrows (considered as vertices of P_n) is adding 1 modulo $n + 1$ to the head and to the tail of each arrow. Taking into account Definition 5.1 and Remark 5.3, it is not difficult to see that the induced effect on the arc representation is adding 1 modulo $n + 1$, that is, a rotation. It is worth noting that to rotate each vertex of a regular $(2n + 2)$ -gon into itself requires increasing each index by one $2n + 2$ times. However, a centrally symmetric pair of arcs $\{[u, v - 1], [u + n + 1, v + n]\}$ is taken into itself already by $n + 1$ such elementary rotations. In the arc representation the facets containing only backward arrows corresponding to facets containing the pair of arcs $\{[1, n + 1], [n + 2, 2n + 2]\}$. The induced action of ζ^{k-1} takes this pair into $\{[k, n + k], [n + k + 1, 2n + 1 + k]\}$. The conditions stated in Theorem 4.3 are rotation-invariant, so the induced action of ζ^{k-1} takes the family of all facets containing $\{[1, n + 1], [n + 2, 2n + 2]\}$ into the family of all facets containing $\{[k, n + k], [n + k + 1, 2n + 1 + k]\}$. \square

8. A BIJECTION BETWEEN THE FACES OF Γ_n^B AND DELANNOY PATHS

In Simion's original paper, the nontrivial computation of the f -vector of the type B associahedron Q_n^B used the recursive structure of the B -diagonals. Simion asked if there is a direct way to obtain this f -vector. Due to the unimodularity of the Legendre polytope P_n , all pulling triangulations of the boundary give the same f -vector; see Corollary 2.8. A direct enumeration in the case of the lexicographic pulling order, also known as the anti-standard triangulation, is straightforward and well-known; see [1, 13].

In this section we establish a bijection between the faces of Γ_n^B and balanced Delannoy paths of length $2n$ in such a way that $(k - 1)$ -dimensional faces correspond to Delannoy paths with k up steps, thus answering Simion's question.

Definition 8.1. *A balanced Delannoy path of length $2n$ is a lattice path starting at $(0, 0)$, ending at $(2n, 0)$ and using only steps of the following three types: up steps $(1, 1)$, down steps $(1, -1)$ and horizontal steps $(2, 0)$. A Schröder path is a balanced Delannoy path that never goes below the horizontal axis.*

More generally, a Delannoy path is a path taking the steps $(1, 1)$, $(1, -1)$ and $(2, 0)$ with no ending condition. In this paper we will only work with balanced Delannoy paths.

Denote the up, down and horizontal steps by U , D and H , respectively. Denote the *length* of the letters U and D each to be 1, whereas the length of the letter H is 2. A Delannoy path is uniquely encoded by a word in these three letters. We call the associated word a Delannoy word. A balanced Delannoy path of length $2n$ corresponds to a word in which the number k of the occurrences of U is the same as the number of occurrences of D , and the number of occurrences of H is $n - k$. We call such a word *balanced*. Hence the number of Delannoy paths of length n with k up steps is given by the multinomial coefficient $\binom{n+k}{k, k, n-k} = \binom{n+k}{k} \binom{n}{k}$ which is the same as the number of $(k - 1)$ -dimensional faces in any pulling triangulation of the boundary of P_n ; see Lemma 2.9.

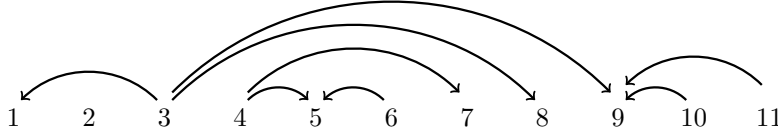


FIGURE 2. A valid digraph on 11 nodes.

We represent the faces of Γ_n^B as a digraph on the set of nodes $\{1, 2, \dots, n+1\}$. A contrapositive formulation of Proposition 5.5 is the following.

Proposition 8.2. *A digraph represents a proper face of Simion's type B associahedron Q_n^B exactly when the following conditions are satisfied:*

- (1) *There are no crossings between arrows.*
- (2) *Forward arrows nest.*
- (3) *A backward arrow cannot nest a forward arrow.*
- (4) *No head of an arrow is the tail of another arrow.*

See also Table 1 occurring in the proof of Theorem 6.1. Call such a set of arrows a *valid* digraph. An example of a valid digraph is shown in Figure 2.

For W a subset of V let A_W denote the induced subgraph on W . Let $\alpha \cdot \beta$ denote the concatenation of the two words α and β .

Let A be a valid digraph of backward arrows on a non-empty finite set of nodes $V \subseteq \mathbb{P}$ and let v be the minimal element of V . We define the Schröder word $\text{SP}(A)$ by the following recursive definition.

- (i) If the set V only consists of one node, that is, $V = \{v\}$, let $\text{SP}(A)$ be the empty word ϵ .
- (ii) If the minimal node v is an isolated node of A , let $\text{SP}(A)$ be the concatenation $H \cdot \text{SP}(A_{V-\{v\}})$.
- (iii) Lastly, when the node v is not isolated then since v is the least element, there is necessarily a backward arrow (x, v) going into v . Let w the smallest node with an arrow to v , that is, $w = \min(\{x \in V : (x, v) \in A\})$. Define $\text{SP}(A)$ by

$$\text{SP}(A) = U \cdot \text{SP}(A_{V \cap (v, w]}) \cdot D \cdot \text{SP}(A_{V - (v, w]}).$$

Lemma 8.3. *Let A be a valid digraph on a set of nodes of size $n+1$ consisting of exactly k backward arrows. The Schröder word $\text{SP}(A)$ then has length $2 \cdot n$ and has exactly k copies of the letters U and k copies of the letters D .*

Proof. Both statements follow by induction on n . The first statement follows from the fact that the two sets of nodes $V \cap (v, w]$ and $V - (v, w]$ are complements of each other. The second statement follows by observing the noncrossing property ensures each arrow in A is either the arrow (w, v) , an arrow in $A_{V \cap (v, w]}$, or an arrow in $A_{V - (v, w]}$. \square

Proposition 8.4. *The map SP is a bijection between the set of all valid digraphs consisting only of backward arrows on the set of $n + 1$ nodes and all Schröder words of length $2n$.*

Proof. Given a Schröder word α of length $2n$ and a node set $V = \{v_1 < v_2 < \dots < v_{n+1}\}$, the inverse map is computed recursively as follows. If α is the empty word ϵ then the inverse image is the isolated node v_1 . If the word α begins with H , that is, $\alpha = H \cdot \beta$ where β has length $2n - 2$, compute the inverse image of β on the nodes $\{v_2, v_3, \dots, v_{n+1}\}$ and add the node v_1 as an isolated node. Otherwise factor α uniquely as $U \cdot \beta \cdot D \cdot \gamma$, where β and γ are Schröder words of lengths $2p$ and $2n - 2p - 2$, respectively. Compute the inverse image of β on the nodes $\{v_2, v_3, \dots, v_{p+2}\}$. Similarly, compute the inverse image of γ on the nodes $\{v_1, v_{p+3}, v_{p+4}, \dots, v_{n+1}\}$. Take the union of these two digraphs and add the backward arrow (v_{p+2}, v_1) . \square

We now extend the bijection SP to all valid digraphs. In order to do this we introduce a *twisting operation* tw on each digraph A that has a forward arrow from the least element $v \in V$ to the largest element $w \in V$. The *twisted digraph* $\text{tw}(A)$ is a digraph on the node set $V - \{v\}$ with arrow set

$$\text{tw}(A) = A_{V-\{v\}} \cup \{(w, z) : (v, z) \in A\}.$$

In other words, we remove the least node v and replace each forward arrow (v, z) starting at v with a backward arrow (w, z) . Note that there is no backward arrow (z, v) in A as v is the tail of the forward arrow (v, w) .

Lemma 8.5. *Let V be a node set of smallest node v and largest node w . The twisting map is a bijection from the set of valid digraphs on the set V containing the forward arrow (v, w) to the set of valid digraphs on the set $V - \{v\}$.*

Proof. We claim that the twisted digraph $\text{tw}(A)$ is a valid digraph. Begin to note that the restriction $A_{V-\{v\}}$ is a valid digraph. If there are no forward arrows in A of the form (v, z) where $z < w$, the equality $\text{tw}(A) = A_{V-\{v\}}$ holds and the claim is true. Hence assume that there are forward arrows of the form (v, z) . We need to show that adding the backward arrow (w, z) the digraph is still valid. We verify conditions (1) through (4) of Proposition 8.2 in order. If the arrow (w, z) crosses an arrow (x, y) then the arrow (v, z) already crossed this arrow in A , verifying condition (1). Since we did not introduce any new forward arrows, condition (2) holds vacuously. Assume that the backward arrow (w, z) nests a forward arrow (x, y) . Then the two forward arrows (v, z) and (x, y) did not nest in A , verifying (3). The last condition (4) holds directly, proving the claim.

The twisting map is one-to-one. Its inverse is given by

$$\text{tw}^{-1}(B) = \{(v, w)\} \cup B_{V \cap (v, w)} \cup \{(v, z) : (w, z) \in B\}.$$

Note that in case the node w was a ‘tail node’ the inverse map switches it back to being a ‘head node’. \square

We now extend the bijection SP to a map DP which applies to all valid digraphs.

- (i) If the valid digraph A has no forward arrows, let $\text{DP}(A) = \text{SP}(A)$.

- (ii) If the valid digraph A has a forward arrow, let (x, y) be the forward arrow which nests the other forward arrows. Let v and w be the minimal, respectively, the maximal node of the node set V , that is, $v = \min(V)$ and $w = \max(V)$. Observe that $v \leq x < y \leq w$, that is, the two digraphs $A_{V \cap [v, x]}$ and $A_{V \cap [y, w]}$ have no forward arrows. Define $\text{DP}(A)$ to be the concatenation

$$\text{DP}(A) = \text{SP}(A_{V \cap [v, x]}) \cdot D \cdot \text{DP}(\text{tw}(A_{V \cap [x, y]})) \cdot U \cdot \text{SP}(A_{V \cap [y, w]}).$$

As an extension of Lemma 8.3 we have the following lemma.

Lemma 8.6. *Let A be a valid digraph on a set V of $n + 1$ nodes and assume that A consists of k arrows. Then the balanced Delannoy word $\text{DP}(A)$ has length $2n$ and has exactly k copies of the letters U and D , respectively.*

Proof. We only have to check the second case of the map DP . Assume that $V \cap [v, x]$ and $V \cap [y, w]$ have cardinalities a , respectively b . Then the middle part $V \cap [x, y]$ has size $n - a - b + 3$. Thus $\text{tw}(A_{V \cap [x, y]})$ has $n - a - b + 2$ nodes. Hence the total length is $2 \cdot (a - 1) + 2 \cdot (n - a - b + 1) + 2 \cdot (b - 1) + 2 = 2n$. For the second statement it is again enough to check the second case. Assume that the restricted digraphs $A_{V \cap [v, x]}$ and $A_{V \cap [y, w]}$ have c , respectively d arrows. Since there is no arrow nesting the forward arrow (x, y) , the middle digraph has $k - c - d$ arrows. Thus the twisted digraph has one less arrow, namely $k - c - d - 1$. Hence the total number of U letters is $c + (k - c - d - 1) + d + 1 = k$, proving the second claim. \square

We now extend Proposition 8.4 to all valid digraphs.

Theorem 8.7. *The map DP is a bijection between the set of all valid digraphs on a set of $n + 1$ nodes and the set of all balanced Delannoy words of length $2n$.*

Proof. The inverse is computed as follows. Let α be a balanced Delannoy word of length $2n$ and V be a node set $\{v_1 < v_2 < \dots < v_{n+1}\}$. If the Delannoy word α is a Schröder word, apply the inverse map of Proposition 8.4. Otherwise, we can factor the Delannoy word α uniquely as $\beta \cdot D \cdot \gamma \cdot U \cdot \delta$, where γ is a balanced Delannoy word, and β and δ are Schröder words. Note that the factor $\beta \cdot D$ is uniquely determined by the fact it is the shortest initial segment in which the number of D letters exceeds the number of U letters, in other words, the encoded path ends with the first down step going below the horizontal axis. By a symmetric argument, $U \cdot \delta$ is the shortest final segment with one more U than the number of D 's. Assume that β , γ and δ have lengths $2p$, $2q$ and $2n - 2p - 2q - 2$, respectively. Apply the inverse map from the proof of Proposition 8.4 to the words β and δ to obtain digraphs on $\{v_1, v_2, \dots, v_{p-1}\}$, respectively $\{v_{p+q+2}, v_{p+q+3}, \dots, v_{n+1}\}$.

By recursion, apply the inverse of DP to γ to obtain a valid digraph on the node set $\{v_{p+2}, v_{p+3}, \dots, v_{p+q+2}\}$. Then apply the inverse of the twisting map to the digraph thus obtained. Finally, take the union of these three digraphs. \square

We now give non-recursive description of the bijection DP . First we encode a valid digraph with a multiset of indexed letters. Recall that a *weakly connected component* of a digraph is a connected

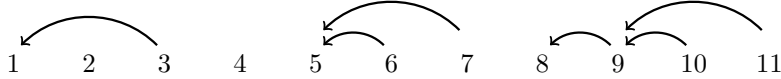


FIGURE 3. The modified set of arrows obtained from the set in Figure 2.

component in the graph obtained by disregarding the direction of the arrows. In our valid digraphs, the weakly connected components are trees where each node is either a source or a sink.

Definition 8.8. *Given a valid digraph A on the node set V , define the associated multiset $M(A)$ with elements from the set of indexed letters $\{D_x, U_x, H_x : x \in \mathbb{P}\}$ by the following three steps:*

- (1) *For each $x < \max(V)$ which is a maximum element in a weakly connected component we add the letter H_x to the multiset $M(A)$.*
- (2) *For each tail x of a forward arrow, consider the set $\text{Head}(x) = \{y > x : (x, y) \in A\}$, that is, the set of heads of forward arrows with tail x . Add a copy of the letter D_x to $M(A)$, also add a copy of the letter U_w for $w = \max(\text{Head}(x))$. Remove the forward arrow (x, w) . For each remaining $y \in \text{Head}(x)$ that is less than w , replace the forward arrow (x, y) with the backward arrow (w, y) . The resulting set of arrows has only backward arrows.*
- (3) *For each head y of a backward arrow, consider the set $\text{Tail}(y) = \{x > y : (x, y) \in A\}$, that is, the set of tails of backward arrows with head y . Add a copy of U_y to $M(A)$. Also add a copy of D_x for $x \in \text{Tail}(y)$ and add a copy of U_x for all but the maximum element of $\text{Tail}(y)$.*

Example 8.9. For the valid set of arrows A shown in Figure 2, in the first step we add the letters H_2 and H_7 to $M(A)$. In the second step we add the letters D_3, U_9, D_4 , and U_7 to $M(A)$, and we remove the forward arrows $(3, 9)$ and $(4, 7)$. We also replace $(3, 8)$ with $(9, 8)$ and $(4, 5)$ with $(7, 5)$. We obtain the set of backward arrows shown in Figure 3. Note that this set of arrows does not need to be valid anymore: in our example, 9 is the head of $(10, 9)$ and $(11, 9)$ and it is also the tail of $(9, 8)$. Finally, in step three we add the letters $U_1, D_3, U_5, D_6, U_6, D_7, U_8, D_9, U_9, D_{10}, U_{10}$, and D_{11} to $M(A)$. We end up with the multiset

$$M(A) = \{U_1, H_2, D_3, D_3, D_4, U_5, D_6, U_6, D_7, U_7, H_7, U_8, D_9, U_9, U_9, D_{10}, U_{10}, D_{11}\}.$$

Define a linear order on the indexed letters by the inequalities $D_x < U_x < H_x < D_{x+1}$ for all positive integers x . We obtain the lattice path as follows.

Proposition 8.10. *The balanced Delannoy word $\text{DP}(A)$ is obtained from the multiset $M(A)$ by reading the indexed letters in order and then omitting the subscripts.*

For the set of arrows shown in Figure 2, we obtain the word $UHDDDDUDUDUHUDUUDUD$. The lattice path encoded by this word is shown in Figure 4.

A quick outline of a proof of Proposition 8.10 is as follows. First prove the statement when there are no forward arrows. In this case, each nonmaximal element y of $\text{Tail}(x)$ contributes two consecutive letters $D_y U_y$. Next, observe that a digraph and its twisted digraph have the same weakly connected components. Furthermore, the arrows that moved under the twisted operation still have the same set

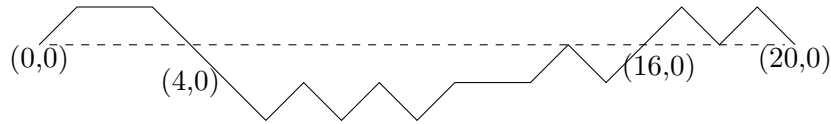


FIGURE 4. The lattice path associated to the digraph in Figure 2.

of heads. Hence, after recording the horizontal steps in step (1) we may perform all twisting operations simultaneously in step (2). We leave the remaining details to the reader.

9. CONCLUDING REMARKS

In a recent paper, Cellini and Marietti [8] used abelian ideals to produce a triangulation for various root polytopes. In the case of type A , their construction yields once again a lexicographic triangulation of each face. Restricting to the positive roots yields Gelfand, Graev and Postnikov's anti-standard tree bases for the type A positive root polytope. Is there an ideal corresponding to the reverse lexicographic triangulation?

The h -vector of Simion's type B associahedron may be computed from the f -vector using elementary operations on binomial coefficients.

Lemma 9.1 (Simion [23, Corollary 1]). *The h -vector (h_0, h_1, \dots, h_n) of the Simion type B associahedron Q_n^B satisfies*

$$(9.1) \quad h_i = \binom{n}{i}^2 \quad \text{for } 0 \leq i \leq n.$$

One would like to find an explicit shelling argument proving (9.1) by counting faces in Γ_n^B . Note the same values occur as the nonzero entries of the h -vector of the direct product $\Delta_n \times \Delta_n$ of two n -dimensional simplices in the work of Billera, Cushman and Sanders [3, Section 3]. They describe an explicit shelling using lattice paths having i corners. The fact that the number of such lattice paths is given by $\binom{n}{i}^2$ is a classical result of MacMahon from 1915 [17, Vol. I, Article 89, pp. 119–120]. Unlike Q_n^B , which is an n -dimensional polytope, the polytope $\Delta_n \times \Delta_n$ is $2n$ -dimensional and half of its h -vector entries are zero. It is a tantalizing question to explore geometric links between these two polytopes. Recent work of Ceballos, Padrol and Sarmiento have recovered our pulling triangulation of the boundary of the Legendre polytope [7, Theorem 8.5].

Finally, are there other interesting simplicial polytopes that can be better understood as pulling triangulations of less complicated polytopes?

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KY 40506-0027.
<http://www.math.uky.edu/~jrge/>, richard.ehrenborg@uky.edu.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNC-CHARLOTTE, CHARLOTTE NC 28223-0001.
<http://www.math.uncc.edu/~ghetyei/>, ghetyei@uncc.edu.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KY 40506-0027.
<http://www.math.uky.edu/~readdy/>, margaret.readdy@uky.edu.