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The Yuri Manin Ring and Its $\mathcal{B}_n$-Analogue

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The Manin ring is a family of quadratic algebras describing pointed stable curves of genus zero whose homology gives the solution of the Commutativity Equations. This solution was first observed by the physicist Losev. We show the Manin ring is the Stanley–Reisner ring of the standard triangulation of the n-cube modulo a system of parameters. Thus, the Hilbert series of the Manin ring is given by the Eulerian polynomial. One can also view the Manin ring as the Stanley–Reisner ring of the dual of the permutahedron modulo a system of parameters. Furthermore, we develop a $\mathcal{B}_n$-analogue of the Manin ring. In this case the signed Manin ring is the Stanley–Reisner ring of the barycentric subdivision of the n-cube (equivalently, the dual of the signed permutahedron) modulo a system of parameters and its Hilbert series is the descent polynomial of augmented signed permutations. © 2001 Academic Press

1. INTRODUCTION

There are some very exciting developments and connections being made between the areas of physics and mathematics. One such example is the solutions to the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) Equations or Associativity Equations of physics [3, 4, 24]. Witten [25] recast an enumerative problem in algebraic geometry involving a space of genus zero algebraic curves into a generating function whose coefficients have the physical meaning of "potential" or "free energy." Quantum field theorists were thus able to discover the solutions of the Associativity Equations. These solutions, also known as Frobenius manifolds, were later verified mathematically [12–14].

Losev and Manin [17] showed the solutions to the Commutativity Equations (the pencils of formal flat connections) correspond to the homology of a family of pointed stable curves of genus zero, denoted $\overline{L}_n$.

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Their work parallels that of the solutions to the Associativity Equations. The Losev–Manin solution of the Commutativity Equations required verifying the dimension of $\overline{L}_n$ is $n!$. Again, the solutions had been anticipated by string theorists [15, 16].

In this paper we determine the Hilbert series of this family of commutative quadratic algebras which we call the Manin ring and hence conclude the dimension result. We also give a combinatorial description of this algebra as well as a $\mathcal{B}_n$-analogue. The techniques we use are familiar to combinatorialists, but do not seem as well-known in the commutative algebra community. It is our hope that this paper will remedy this deficiency and at the same time give insight to develop and study new interesting quadratic algebras.

Recall for Δ a simplicial complex with vertex set $V(\Delta)$, the Stanley–Reisner ring $\mathbf{k}[\Delta]$ is the polynomial ring in the variables x_v for $v \in V(\Delta)$ modulo the ideal of non-faces of Δ. Stanley used this ring to prove the Upper Bound Theorem for simplicial spheres [20]. In the case that a simplicial complex is shellable, and hence, having Stanley–Reisner ring which is Cohen–Macaulay, Kind and Kleinschmidt [11] gave necessary and sufficient conditions for $(\theta_0, \ldots, \theta_d)$ to be a linear system of parameters. Additionally, they exhibited an explicit basis for the Stanley–Reisner ring of Δ modulo a system of parameters which comes from the shelling.

Using the Kind–Kleinschmidt characterization, we show the Manin ring $\overline{L}_n$ is the familiar Stanley–Reisner ring of the standard triangulation of the n-cube modulo a system of parameters. As this simplicial complex is shellable, it is then a straightforward matter to compute the Hilbert series of the associated ring. We do this in Section 3.

In Section 4 we give a $\mathcal{B}_n$-analogue of the Manin ring, denoted $\overline{L}_n^{\pm}$. Geometrically this signed Manin ring is the Stanley–Reisner ring of the barycentric subdivision of the n-cube modulo a system of parameters. As in the case of the Manin ring, we again compute the associated Hilbert series. It would be interesting to see if this signed analogue has any physical meaning as was known for the Commutativity Equation solutions [15, 16]. In the last section of the paper we suggest some possible directions of generalization.

2. DEFINITIONS

We now review some elementary concepts in commutative algebra which can be found in [23]. Throughout $[n]$ will denote the set $[n] = \{1, \ldots, n\}$. Let $\mathbf{k}$ be a field of characteristic zero and let $A = \bigoplus_{i \geq 0} A_i$ be a *standard graded* $\mathbf{k}$*-algebra*, that is, A is generated by a finite number of degree 1 homogeneous elements. The *Hilbert series* is defined to be

$$\mathscr{H}(A) = \sum_{i \geq 0} \dim_{\mathbf{k}}(A_i) \cdot t^i.$$

A basic result of Noether's is the existence of a system of parameters. Namely, for A a standard graded $\mathbf{k}$-algebra, there exists a finite number of homogeneous degree 1 elements $\theta_0, \ldots, \theta_d$ which are algebraically independent over $\mathbf{k}$. Furthermore, there exists a finite number of homogeneous elements $\eta_1, \ldots, \eta_s$ such that for all $x \in A$,

$$x = \sum_{i=1}^{s} \eta_i \cdot p_i(\theta_0, \ldots, \theta_d), \tag{2.1}$$

where p_i is a polynomial in the $\theta_0, \ldots, \theta_d$ depending on x. The elements $\theta_0, \ldots, \theta_d$ are called a *(linear) system of parameters*, abbreviated s.o.p. An s.o.p. is *regular* if for some choice of homogeneous elements $\eta_1, \ldots, \eta_s$ the representation in (2.1) is unique, while a standard $\mathbf{k}$-algebra is *Cohen–Macaulay* if some (equivalently, every) s.o.p. is regular.

The *Stanley–Reisner ring* or *face ring* of a finite simplicial complex Δ on the vertex set $V(\Delta)$ is the quotient ring

$$\mathbf{k}[\Delta] = \mathbf{k}[x_v : v \in V(\Delta)]/I(\Delta),$$

where $I(\Delta)$ is the face ideal generated by all squarefree monomials $x_{v_1} \cdots x_{v_j}$ satisfying the condition that the vertices $v_1, \ldots, v_j$ do not lie on a common face of Δ.

Consider the n-cube C_n as $[0, 1]^n$. The *standard triangulation of the n-cube*, denoted $\Delta(C_n)$, is the triangulation induced by cutting $[0, 1]^n$ with the $\binom{n}{2}$ hyperplanes $x_i = x_j$, $1 \leq i < j \leq n$. The resulting triangulation gives $n!$ simplices of dimension n.

The *Stanley–Reisner ring of the triangulated n-cube* is given by $\mathbf{k}[\Delta(C_n)] = \mathbf{k}[y_t : t \subseteq [n]]/I(\Delta(C_n))$, where the ideal $I(\Delta(C_n))$ is generated by the degree 2 monomials $y_t \cdot y_u$ with the vertices t and u not lying on the same facet of $\Delta(C_n)$. Alternatively, one can think of the triangulated n-cube as being the Boolean algebra B_n, that is, the lattice of all subsets of $[n]$ ordered with respect to inclusion. The n-dimensional simplices of $\Delta(C_n)$ correspond to maximal chains in the Boolean algebra. More formally, the maximal chain $c : \varnothing = P_0 \subset P_1 \subset \cdots \subset P_{n-1} \subset P_n = [n]$ in the Boolean algebra B_n corresponds to the facet F spanned by the vertices $V(F) = \{P_0, \ldots, P_n\}$. Thus the Stanley–Reisner ring is simply the polynomial ring having variables the 2^n elements of the Boolean algebra and the face ideal is generated by pairs of non-comparable elements in the poset.

Let $F_1, \ldots, F_s$ be a linear ordering of the facets of a pure simplicial complex Δ of dimension d and set $\Delta_j = \overline{F}_1 \cup \cdots \cup \overline{F}_j$. Such a linear ordering of the facets is called a *shelling order*, or *shelling*, if $\Delta_{k-1} \cap F_k$ is pure and $(d-1)$-dimensional for $2 \le k \le s$. If such a facet order exists, we say Δ is *shellable*. Given a shelling order, for a facet F_k the *restriction* $R(F_k)$ is the minimal new face added in the kth shelling step, that is, $R(F_k) = \min\{F \in \Delta_k : F \notin \Delta_{k-1}\}$.

Given a facet $F \in \Delta(C_n)$ with corresponding maximal chain $c : \emptyset = P_0 \subset \cdots \subset P_n = [n]$ in the Boolean algebra B_n, let π_i denote the unique element in the set difference $P_i - P_{i-1}$ for $1 \le i \le n$. Then $\pi(F) = \pi_1 \cdots \pi_n \in S_n$ is the *permutation representation of the facet* F. For $\pi = \pi_1 \cdots \pi_n$ a permutation in the symmetric group S_n, the *number of descents* of π, denoted $\operatorname{des}(\pi)$, is the number of indices i such that $\pi_i > \pi_{i+1}$. The *descent set* of π, denoted $\operatorname{Des}(\pi)$, is the set of indices i such that $\pi_i > \pi_{i+1}$.

A shelling order for the triangulated n-cube is to take the facet order given by the lexicographic order inherited by the permutation representation of each facet. For the 3-cube this shelling order is given in Table I. Observe there is one restriction of size 0, four of size 1, and one of size 2. In general, a result due to Björner on EL-labelings [1] implies the facet restriction $R(F_i)$ depends on the descent set of the permutation $\pi(F)$. More specifically, if $\pi(F_i) = \pi_1 \cdots \pi_n$ and $\operatorname{Des}(\pi(F_i)) = \{s_1, \ldots, s_k\}$ with $s_1 < \cdots < s_k$ then $R(F_i) = \{T_1, \ldots, T_k\}$, where $T_i = \{\pi_1, \ldots, \pi_{s_i}\}$.

One well-known example of shellable objects is due to Bruggesser and Mani, namely the boundary complex of a convex polytope [2]. Implicit from Hochster's work [10] is the result that if a simplicial complex is shellable then its Stanley–Reisner ring is Cohen–Macaulay.

Let Δ be a pure shellable simplicial complex of dimension d and $\theta_0, \ldots, \theta_d$ be homogeneous degree 1 elements from $\mathbf{k}[\Delta]$ of the form $\theta_i = \sum_{v \in V(\Delta)} m_{i,v} \cdot x_v$ with $m_{i,v} \in \mathbf{k}$. Let M denote the matrix of the linear forms θ_i, that is, $M = (m_{i,v})_{0 \le i \le d,\, v \in V(\Delta)}$. For F a facet of Δ, let $M|_F$ denote the matrix formed by restricting the θ_i to the variables x_v for $v \in F$. Observe that $M|_F$ is a square matrix of order $d+1$.

TABLE I
Shelling Restrictions for the 3-Cube Using the Lexicographic Shelling Order

Facet F_i	$\pi(F_i)$	$R(F_i)$
$F_1 = \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}\}$	123	$\emptyset$
$F_2 = \{\emptyset, \{1\}, \{1,3\}, \{1,2,3\}\}$	132	$\{\{1,3\}\}$
$F_3 = \{\emptyset, \{2\}, \{1,2\}, \{1,2,3\}\}$	213	$\{\{2\}\}$
$F_4 = \{\emptyset, \{2\}, \{2,3\}, \{1,2,3\}\}$	231	$\{\{2,3\}\}$
$F_5 = \{\emptyset, \{3\}, \{1,3\}, \{1,2,3\}\}$	312	$\{\{3\}\}$
$F_6 = \{\emptyset, \{3\}, \{2,3\}, \{1,2,3\}\}$	321	$\{\{3\}, \{2,3\}\}$

A beautiful result of Kind and Kleinschmidt [11] characterizes linear systems of parameters for shellable simplicial complexes.

THEOREM 2.1 (Kind–Kleinschmidt). *Let Δ be a pure shellable simplicial complex of dimension d with shelling order $F_1, \ldots, F_s$. Let $\theta_0, \ldots, \theta_d$ be homogeneous degree 1 elements in $\mathbf{k}[\Delta]$ and let M be the matrix of the θ_i's. Then*

(a) *The elements $\theta_0, \ldots, \theta_d$ are a linear system of parameters for $\mathbf{k}[\Delta]$ if and only if the matrix $M|_F$ is invertible for any facet $F \in \Delta$.*

(b) *A basis for $\mathbf{k}[\Delta]/(\theta_0, \ldots, \theta_d)$ is given by the facet restriction monomials, that is,*

$$\eta_i = \prod_{v \in R(F_i)} x_v, \qquad 1 \leq i \leq s.$$

In the case that a standard graded $\mathbf{k}$-algebra is Cohen–Macaulay, the Hilbert series has an explicit form.

THEOREM 2.2 [23, p. 35]. *Let A be a Cohen–Macaulay standard graded $\mathbf{k}$-algebra with an s.o.p. $(\theta_0, \ldots, \theta_n)$ and let $\eta_1, \ldots, \eta_s$ be homogeneous elements of A which form a $\mathbf{k}$-basis for $A/(\theta_0, \ldots, \theta_n)$. Then*

$$\mathscr{H}(A) = \frac{\sum_{i=1}^{s} t^{\deg \eta_i}}{(1-t)^{n+1}}.$$

Returning to the example, the Hilbert series of the Stanley–Reisner ring of the triangulated 3-cube is $\mathscr{H}(\mathbf{k}[\Delta(C_3)]) = (1 + 4t + t^2)/(1-t)^3$. In general, the numerator of $\mathscr{H}(\mathbf{k}[\Delta(C_n)])$ is the *Eulerian polynomial*, that is, the degree n polynomial having the coefficient of t^i equal to the number of permutations in the symmetric group having exactly i descents. This definition of the Eulerian polynomial differs slightly from the usual one which instead has each term having degree one more than we have stated. For a general reference on the Eulerian numbers and the Eulerian polynomial, we refer the reader to [21].

Putting all of these results together, we have the following theorem which is also a consequence of work of Hetyei [8].

THEOREM 2.3. *Let $\theta_0, \ldots, \theta_n$ be an s.o.p. of $\mathbf{k}[\Delta(C_n)]$, the Stanley–Reisner ring of the triangulated n-cube. The Hilbert series of the quotient ring $\mathbf{k}[\Delta(C_n)]/(\theta_0, \ldots, \theta_n)$ is given by*

$$\mathscr{H}(\mathbf{k}[\Delta(C_n)]/(\theta_0, \ldots, \theta_n)) = \sum_{\pi} t^{\mathrm{des}(\pi)},$$

where the sum is over all permutations π in the symmetric group S_n.

3. THE MANIN RING $\overline{L}_n$

Let n be an integer greater than or equal to 2 and let the index set N_n consist of all ordered pairs of sets $s = (s_1, s_2)$ with $s_1 \dot{\cup} s_2 = [n]$ and $s_1 \neq \emptyset, [n]$. Let the ideal I_n be generated by:

(1) The linear relations

$$\lambda(i, j) = \sum_{(s_1, s_2)} x_{(s_1 \cup \{i\}, s_2 \cup \{j\})} - x_{(s_1 \cup \{j\}, s_2 \cup \{i\})},$$

where $i \neq j$ are fixed integers satisfying $1 \leq i, j \leq n$, and the ordered pairs $(s_1 \cup \{i\}, s_2 \cup \{j\})$ and $(s_1 \cup \{j\}, s_2 \cup \{i\})$ belong to the index set N_n.

(2) The quadratic relations

$$x_{(s_1 \cup \{i\}, s_2 \cup \{j\})} \cdot x_{(t_1 \cup \{j\}, t_2 \cup \{i\})},$$

where $i \neq j$ are fixed integers satisfying $1 \leq i, j \leq n$, and the ordered pairs $(s_1 \cup \{i\}, s_2 \cup \{j\})$ and $(t_1 \cup \{j\}, t_2 \cup \{i\})$ belong to the index set N_n.

The *Manin ring*, denoted $\overline{L}_n$, is defined by $\overline{L}_1 = \mathbf{k}$ and for $n \geq 2$ it is the quotient

$$\overline{L}_n = \mathbf{k}[x_s : s \in N_n]/I_n.$$

Manin used the notation $\sum_s (x_{isj} - x_{jsi})$ to denote the linear relation $\lambda(i, j)$ and $x_{isj} \cdot x_{jti}$ to denote the quadratic relation for i and j fixed integers.

PROPOSITION 3.1. *The following identity holds among the differences* $\lambda(i, j)$:

$$\lambda(k, j) = \lambda(i, j) - \lambda(i, k).$$

In particular, the quantity $\lambda(i, j)$ *can be written as a linear combination of the* $n - 1$ *quantities* $\lambda(1, 2), \ldots, \lambda(1, n)$.

Proof. It is enough to observe that $\lambda(i, j) = \kappa(i) - \kappa(j)$, where κ is defined as

$$\kappa(i) = \sum_{(s_1, s_2)} x_{(s_1 \cup \{i\}, s_2)},$$

the sum taken over all ordered pairs (s_1, s_2) such that $(s_1 \cup \{i\}, s_2)$ belongs to the index set N_n. ∎

We now introduce the 2^n variables y_t where $t \subseteq [n]$. Let $y_t = x_{(t, [n]-t)}$ for $t \neq \emptyset, [n]$. Define the $n + 1$ linear forms

$$\theta_0 = y_\emptyset, \;\; \theta_i = \lambda(1, i+1) \text{ for } 1 \leq i \leq n-1, \quad \text{and} \;\; \theta_n = y_{[n]}. \tag{3.2}$$

Here the relations $\lambda(1, i+1)$ are written using the variables y_t.

We now state the main lemma of this section.

LEMMA 3.2. *Let M be the $(n+1) \times 2^n$ matrix of the θ_i given in* (3.2). *Assume the columns of the matrix M are ordered using any linear extension of the Boolean algebra B_n. Let F be a facet of the triangulated n-cube. Then*

$$\det(M|_F) = (-1)^{\pi(F)},$$

that is, the determinant of the matrix $M|_F$ is precisely the signature of the permutation $\pi(F)$.

Proof. The variables $y_\emptyset$ and $y_{[n]}$ only appear in θ_0, respectively θ_n. Thus to compute $\det(M|_F)$ it is enough to compute $\det(M')$, where M' is the $(n-1) \times (n-1)$ square submatrix formed by restricting the matrix M to the columns labeled by the vertices of F not equal to $\emptyset$ or $[n]$.

Without loss of generality, we may assume $\pi_j = 1$. Observe that the column labels of the matrix M' are written in increasing order according to the size of the subset t corresponding to the variable y_t. Let e_i denote the ith standard unit column vector in $\mathbb{R}^{n-1}$ and $\mathbf{1}$ denote the column vector of all ones in $\mathbb{R}^{n-1}$. We claim the kth column of M' has the form

$$(M')_k = \begin{cases} -\sum_{1 \le i \le k} e_{\pi_i - 1} & \text{if } k < j, \\ \mathbf{1} - \sum_{\substack{1 \le i \le k, \\ i \ne j}} e_{\pi_i - 1} & \text{if } k \ge j. \end{cases}$$

To see this, let $v = \{\pi_1, \ldots, \pi_k\}$ be the vertex of the facet F of cardinality k. First consider the case $k < j$ corresponding to the fact the element 1 does not belong to v. Then the variable x_v appears in the relations $\lambda(1, \pi_1)$ through $\lambda(1, \pi_k)$, that is, $\theta_{\pi_1 - 1}$ through $\theta_{\pi_k - 1}$, with a coefficient of -1. Similarly, when $k \ge j$ we have $\pi_j = 1$. In this case the variable x_v cannot occur in $\lambda(1, \pi_1), \ldots, \lambda(1, \pi_{j-1}), \lambda(1, \pi_{j+1}), \ldots, \lambda(1, \pi_k)$, hence it has a coefficient of zero in the corresponding rows of M'. Since $\pi_j = 1$, the variable x_v can only occur in the remaining $n - k$ rows with coefficient of $+1$. Thus the matrix M' is of the proposed form.

It remains to prove the main result. By definition of the signature of a permutation, we have $(-1)^\pi = \det(f_{\pi_1}, \ldots, f_{\pi_n})$, where f_i denotes the ith standard unit vector in $\mathbb{R}^n$ and we have displayed the columns in the determinant expression. Cyclically rotating the rows of this matrix once (that is, rotating the ith row to the $(i-1)$st row for $2 \le i \le n$ and the first row to the nth), distributing the $(n-1)$ minus signs to all the columns except the jth, and then switching the jth and nth columns gives

$$\begin{aligned}(-1)^\pi &= (-1)^{n-1} \det(f_{\pi_1 - 1}, \ldots, f_{\pi_{j-1} - 1}, f_n, f_{\pi_{j+1} - 1}, \ldots, f_{\pi_n - 1}) \\ &= \det(-f_{\pi_1 - 1}, \ldots, -f_{\pi_{j-1} - 1}, f_n, -f_{\pi_{j+1} - 1}, \ldots, -f_{\pi_n - 1}) \\ &= \det(-f_{\pi_1 - 1}, \ldots, -f_{\pi_{j-1} - 1}, f_{\pi_n - 1}, -f_{\pi_{j+1} - 1}, \ldots, f_n).\end{aligned}$$

Adding the absolute value of all the columns except the nth to the jth column and then expanding the determinant by the last column gives

$$\begin{aligned}(-1)^{\pi} &= \det(-e_{\pi_1 - 1}, \ldots, -e_{\pi_{j-1}-1}, \mathbf{1}, -e_{\pi_{j+1}-1}, \ldots, -e_{\pi_{n-1}-1}) \\ &= \det(M'),\end{aligned}$$

as desired. ∎

THEOREM 3.3. *Let the linear forms* $(\theta_0, \ldots, \theta_n)$ *be as in* (3.2). *Then*

(a) *The* $(\theta_0, \ldots, \theta_n)$ *form a system of parameters for* $\mathbf{k}[\Delta(C_n)]$, *the Stanley–Reisner ring of the triangulated n-cube.*

(b) $\overline{L}_n = \mathbf{k}[\Delta(C_n)]/(\theta_0, \ldots, \theta_n)$, *that is, the Manin ring is precisely the Stanley–Reisner ring of the triangulated n-cube modulo a system of parameters.*

Proof. Part (a) follows from Lemma 3.2 and the Kind–Kleinschmidt result.

By Proposition 3.1 the linear relations defining the Manin ring can be reduced to $\theta_1, \ldots, \theta_{n-1}$ as θ_0 and θ_n simply kill the extra variables $y_{\emptyset}$ and $y_{[n]}$. For (b), it is then straightforward to check that the system of parameters given and the face ideal coincide with the Manin relations. ∎

By Theorems 2.3 and 3.3 we have two immediate corollaries.

COROLLARY 3.4. *The Hilbert series of the Manin ring is given by*

$$\mathcal{H}(\overline{L}_n) = \sum_{\pi} t^{\operatorname{des}(\pi)},$$

where the sum is over all permutations π *in the symmetric group* S_n.

COROLLARY 3.5. *The dimension of the Manin ring* $\overline{L}_n$ *is given by*

$$\dim(\overline{L}_n) = n!.$$

Some values of the Hilbert series of the Manin ring are given in Table II.

TABLE II
Values of $\mathcal{H}(\overline{L}_n)$ for $1 \leq n \leq 5$

$\mathcal{H}(\overline{L}_1) = 1$
$\mathcal{H}(\overline{L}_2) = 1 + t$
$\mathcal{H}(\overline{L}_3) = 1 + 4t + t^2$
$\mathcal{H}(\overline{L}_4) = 1 + 11t + 11t^2 + t^3$
$\mathcal{H}(\overline{L}_5) = 1 + 26t + 66t^2 + 26t^3 + t^4$

As an aside, the *permutahedron* Π_n is the polytope formed by the convex hull of the vertices $(\pi_1, \ldots, \pi_n) \in \mathbb{R}^n$ with $\pi_1 \cdots \pi_n \in S_n$. Since the permutahedron is a simple polytope, its dual polytope Π_n^* is a simplicial polytope. Note that the vertices of Π_n^* correspond to ordered partitions of the elements $[n]$ into two parts, and in general, the i-dimensional faces of Π_n^* correspond to ordered partitions of the elements $[n]$ into $i+2$ parts. It is now not difficult to convince oneself of the following result.

THEOREM 3.6. *The Manin ring $\overline{L}_n$ coincides with the Stanley–Reisner ring of the dual of the permutahedron modulo the $n-1$ linear relations $\lambda(1,2)$ through $\lambda(1,n)$. That is,*

$$\overline{L}_n = \mathbf{k}[(\Pi_n)^*]/(\lambda(1,2), \ldots, \lambda(1,n)).$$

4. A $\mathscr{B}_n$-ANALOGUE OF THE MANIN RING

We now introduce a signed analogue of the Manin ring. In order to do this, we first review the notions of signed sets and signed permutations. Let $\mathscr{S}_n^{\pm}$ denote the set of all signed subsets from the set $\{\pm 1, \pm 2, \ldots, \pm n\} = \{1, \bar{1}, 2, \bar{2}, \ldots, n, \bar{n}\}$, that is, $\{a_1, \ldots, a_k\} \in \mathscr{S}_n^{\pm}$ if $\{|a_1|, \ldots, |a_k|\}$ is a k element subset of $[n]$. We say $\sigma = \sigma_1 \cdots \sigma_n$ is a *signed permutation* if the set $\{\sigma_1, \ldots, \sigma_n\}$ has cardinality n and belongs to $\mathscr{S}_n^{\pm}$. An *augmented signed permutation* is of the form $\tau = \sigma_1 \cdots \sigma_n 0$ where the zero element has been adjoined to the end of the signed permutation σ, in other words, $\tau_{n+1} = 0$. We denote the set of all signed permutations on n elements by $S_n^{\pm}$ and the set of all augmented signed permutations by $S_{n,\mathrm{aug}}^{\pm}$. The number of descents of $\tau \in S_{n,\mathrm{aug}}^{\pm}$, denoted $\mathrm{des}(\tau)$, is the number of indices i such that $\tau_i > \tau_{i+1}$ for $1 \le i \le n$. Finally, the *signature of a signed permutation* $\sigma \in S_n^{\pm}$, denoted $(-1)^{\sigma}$, is the signature of the permutation $\pi = |\sigma_1| \cdots |\sigma_n| \in S_n$ times $(-1)^{\epsilon}$, where ϵ is the number of negative signs in σ.

Let $\mathscr{L}(C_n)$ denote the face lattice of the n-dimensional cube or cubical lattice. The dual of the rank n cubical lattice, denoted $\mathscr{L}(C_n)^*$, is simply the elements of $\mathscr{S}_n^{\pm}$ ordered by inclusion with a maximal element $\hat{1}$ adjoined. The maximal chains of the cubical lattice correspond to augmented signed permutations in the following manner. Given a maximal chain $c : \varnothing = Q_0 \subset Q_1 \subset \cdots \subset Q_n < \hat{1}$ in $\mathscr{L}(C_n)^*$, let σ_i denote the unique element in the set difference $Q_i - Q_{i-1}$ for $1 \le i \le n$. The permutation representation of the maximal chain c is the augmented signed permutation $\tau = \sigma_1 \cdots \sigma_n 0$. Note we will usually be working with its signed permutation $\sigma = \sigma_1 \cdots \sigma_n$. Geometrically, the barycentric subdivision of the n-cube gives $n! \cdot 2^n$ number of n-simplices each corresponding to a permutation in $S_n^{\pm}$.

We now define an ideal using the signed variables y_s for $s \in S_n^{\pm}$. Let $I_n^{\pm}$ be the ideal generated by:

(1) The linear relations

$$\lambda^{\pm}(i, j) = \sum_{s} \left(y_{s\cup\{i\}} + y_{s\cup\{\bar{i}\}}\right) - \left(y_{s\cup\{j\}} + y_{s\cup\{\bar{j}\}}\right),$$

where $i \neq j$ are fixed positive integers satisfying $1 \leq i, j \leq n$, and the sum is over all signed subsets s of $\mathcal{S}_n^{\pm}$ not containing the elements i, $\bar{i}$, j, or $\bar{j}$.

(2) The quadratic relations

$$y_s \cdot y_t,$$

where s and t are signed subsets of $\mathcal{S}_n^{\pm}$ that are incomparable elements in the dual of the cubical lattice.

(3) The relation

$$\lambda_n^{\pm} = \sum_{s} (-1)^{\#\text{ signs in } s} \cdot y_s,$$

where s ranges over all signed subsets of $\mathcal{S}_n^{\pm}$ with $|s| = n$.

Define the *signed Manin ring* $\overline{L}_n^{\pm}$ by $\overline{L}_0^{\pm} = \mathbf{k}$ and for $n \geq 1$ by

$$\overline{L}_n^{\pm} = \mathbf{k}[y_t : t \in \mathcal{S}_n^{\pm}]/I_n^{\pm}.$$

By a similar argument as Proposition 3.1, we have the following result.

PROPOSITION 4.1. *The following identity holds among the differences* $\lambda^{\pm}(i, j)$:

$$\lambda^{\pm}(k, j) = \lambda^{\pm}(i, j) - \lambda^{\pm}(i, k).$$

In particular, the quantity $\lambda^{\pm}(i, j)$ *can be written as a linear combination of the* $n - 1$ *quantities* $\lambda^{\pm}(1, 2), \ldots, \lambda^{\pm}(1, n)$.

Define the $n + 1$ linear forms $\theta_0^{\pm}, \ldots, \theta_n^{\pm}$ by

$$\theta_0^{\pm} = y_{\varnothing}, \quad \theta_i^{\pm} = \lambda^{\pm}(1, i + 1) \text{ for } 1 \leq i \leq n - 1, \quad \text{and} \quad \theta_n^{\pm} = \lambda_n^{\pm}. \tag{4.3}$$

LEMMA 4.2. *Let M be the matrix of the $\theta_i^{\pm}$ given in* (4.3), *where the columns are ordered using any linear extension of the dual of the cubical lattice $\mathcal{L}(C_n)^*$. Let F be an n-simplex of the barycentric subdivision of the n-cube with signed permutation representation $\sigma(F) = \sigma_1 \cdots \sigma_n$. Then*

$$\det(M|_F) = (-1)^{\sigma(F)},$$

that is, the determinant of the matrix $M|_F$ is precisely the signature of the signed permutation $\sigma(F) \in S_n^{\pm}$.

Proof. Let G be the facet in the triangulated n-cube corresponding to the unsigned permutation $\pi = |\sigma_1| \cdots |\sigma_n|$ and let N be the $(n+1) \times 2^n$ matrix of the system of parameters θ_i given in (3.2). Notice that the matrix $M|_F$ is identical to the matrix $N|_G$ except in the $(n+1, n+1)$ entry. This entry has value $(-1)^\epsilon$, where ϵ is the number of negative signs appearing in the signed permutation σ. By Lemma 3.2, $\det(N|_G) = (-1)^{\pi(G)}$ and thus $\det(M|_F) = (-1)^\epsilon \cdot \det(N|_G) = (-1)^\sigma$. ∎

By virtually the same proof as Theorem 3.3, we have its signed version.

THEOREM 4.3. *Let* $(\theta_0^\pm, \ldots, \theta_n^\pm)$ *be given as in* (4.3). *Then*

(a) *The* $(\theta_0^\pm, \ldots, \theta_n^\pm)$ *form a system of parameters for* $\mathbf{k}[\mathrm{Bar}(C_n)]$, *the Stanley–Reisner ring of the barycentric subdivision of the n-cube.*

(b) $\overline{L}_n^\pm = \mathbf{k}[\mathrm{Bar}(C_n)]/(\theta_0^\pm, \ldots, \theta_n^\pm)$, *that is, the signed Manin ring is precisely the Stanley–Reisner ring of the barycentric subdivision of the n-cube modulo a system of parameters.*

We now state a signed analogue of Corollary 3.4. The fact that the sum is over all augmented signed permutations again follows from the standard EL-labeling of the cubical lattice and the aforementioned result of Björner on EL-labelings.

COROLLARY 4.4. *The Hilbert series of the signed Manin ring is given by*

$$\mathscr{H}(\overline{L}_n^\pm) = \sum_{\tau} t^{\mathrm{des}(\tau)},$$

where the sum is over all augmented signed permutations $\tau \in S_{n,\,\mathrm{aug}}^\pm$.

COROLLARY 4.5. *The dimension of the signed Manin ring* $\overline{L}_n^\pm$ *is given by*

$$\dim(\overline{L}_n^\pm) = 2^n \cdot n!.$$

The first few values of the Hilbert series of the signed Manin ring are displayed in Table III.

The *signed permutahedron* $\Pi_n^\pm$ is the polytope formed by taking the convex hull of the vertices $(\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n$ with $\sigma_1 \cdots \sigma_n \in S_n^\pm$. Its face lattice is the lattice of regions of the braid arrangement $\mathscr{B}_n$ given by $x_i = \pm x_j$, for $1 \le i < j \le n$, and $x_i = 0$, for $1 \le i \le n$. This lattice may also be described as the lattice of ordered signed partitions, denoted $\mathscr{L}(\Pi_n^\pm)$. We refer the reader to [6, Sect. 6] for a more detailed description of the ordered signed partition lattice than the one we will give here.

The elements of $\mathscr{L}(\Pi_n^\pm)$ are ordered signed partitions $Z/\widetilde{B}_1/\cdots/\widetilde{B}_k$, where Z is a (possibly empty) unsigned set called the zero set, and $\widetilde{B}_1, \ldots, \widetilde{B}_k$ are signed non-empty sets. The order relation in the lattice is merging two adjacent blocks. Hence the coatoms are of the form $Z/\widetilde{B}_1$

TABLE III
Values of $\mathscr{H}(\overline{L}_n^{\pm})$ for $0 \leq n \leq 4$

$\mathscr{H}(\overline{L}_0^{\pm}) = 1$
$\mathscr{H}(\overline{L}_1^{\pm}) = 1 + t$
$\mathscr{H}(\overline{L}_2^{\pm}) = 1 + 6t + t^2$
$\mathscr{H}(\overline{L}_3^{\pm}) = 1 + 23t + 23t^2 + t^3$
$\mathscr{H}(\overline{L}_4^{\pm}) = 1 + 76t + 230t^2 + 76t^3 + t^4$

with $Z = \{i_1, \ldots, i_m\}$ and $\widetilde{B_1} = \{i_{m+1}, \ldots, i_n\}$. Using the convention $x_{\bar{i}}$ to mean $-x_i$, this determines the ray $0 = x_{i_1} = \cdots = x_{i_m} < x_{i_{m+1}} = \cdots = x_{i_n}$ in the arrangement.

Dualizing all of this, we see the dual of the signed permutahedron has vertices corresponding to elements of the form $Z/\widetilde{B_1}$, which in turn corresponds to the variable $y_{\widetilde{B_1}}$ in the signed Manin ring. In analogy to the Manin ring, it is straightforward to verify the following result.

THEOREM 4.6. *The signed Manin ring* $\overline{L}_n^{\pm}$ *is the Stanley–Reisner ring of the dual of the signed permutahedron modulo the* n *linear relations* $\lambda^{\pm}(1,2), \ldots, \lambda^{\pm}(1,n), \lambda_n^{\pm}$, *that is,*

$$\overline{L}_n^{\pm} = \mathbf{k}[(\Pi_n^{\pm})^*]/(\lambda^{\pm}(1,2), \ldots, \lambda^{\pm}(1,n), \lambda_n^{\pm}).$$

5. CONCLUDING REMARKS

One obvious question to ask is to find other naturally occurring analogues of the Manin ring and the Commutativity Equations. The permutahedron and signed-permutahedron correspond respectively to the Weyl groups $\mathscr{A}_{n-1}$ and $\mathscr{B}_n$. In a forthcoming paper the author develops Manin-type rings arising from other Weyl groups and root systems.

Two other examples which may lead to further results are worth mentioning. First, the notion of augmented signed permutations has a generalization to augmented **r**-signed permutations which corresponds to the **r**-cubical lattice; see [5]. Second, Reiner and Ziegler [19] studied the Coxeter-associahedra, a class of convex polytopes interpolating between the permutahedron and associahedron. Can anything be said about associahedron types of examples?

The Hetyei ring [8] is a cubical analogue of the Stanley–Reisner ring defined for cubical polytopes, and more generally, cubical complexes. It is an example of a ring whose defining ideal is binomial. See [18] for further investigations by the author regarding systems of parameters of the

Hetyei ring, as well as [7, 9] for the relation of the Hetyei ring with the Ron Adin h-vector. In [22] Stanley generalizes the Hetyei ring to polytopal complexes.

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REFERENCES

1. A. Björner, Shellable and Cohen–Macaulay partially ordered sets, *Trans. Amer. Math. Soc.* **260** (1980), 159–183.
2. H. Bruggesser and P. Mani, Shellable decompositions of cells and spheres, *Math. Scand.* **29** (1971), 197–205.
3. R. Dijkgraaf, E. Verlinde, and H. Verlinde, Topological strings in $d < 1$, *Nuclear Phys. B* **352** (1991), 59–86.
4. R. Dijkgraaf and E. Witten, Mean field theory, topological field theory, and multi-matrix models, *Nuclear Phys. B* **342** (1990), 486–522.
5. R. Ehrenborg and M. Readdy, The r-cubical lattice and a generalization of the **cd**-index, *European J. Combin*. **17** (1996), 709–725.
6. R. Ehrenborg and M. Readdy, On flag vectors, the Dowling lattice and braid arrangements, *Discrete Comput. Geom.* **21** (1999), 389–403.
7. J. Haglund, On the nonnegativity of the coefficients of some polynomials occurring in the theory of cubical spheres, *Ann. Sci. Math. Québec* **1** (1998), 43–45.
8. G. Hetyei, On the Stanley ring of a cubical complex, *Discrete Comput. Geom.* **14** (1995), 305–330.
9. G. Hetyei, Invariants des complexes cubiques, *Ann. Sci. Math. Québec* **20** (1996), 35–52.
10. M. Hochster, Rings of invariants of tori, Cohen–Macaulay rings generated by monomials, and polytopes, *Ann. of Math.* **96** (1972), 318–337.
11. B. Kind and P. Kleinschmidt, Schälbare Cohen–Macauley–Komplexe und ihre Parametrisierung, *Math Z.* **167** (1979), 173–179.
12. M. Kontsevich and Y. Manin, Gromov–Witten classes, quantum cohomology and enumerative geometry, *Comm. Math. Phys.* **164** (1994), 525–562.
13. M. Kontsevich and Y. Manin, Quantum cohomology of a product (with Appendix by R. Kaufmann), *Invent Math.* **124** (1996), 313–339.
14. M. Kontsevich and Y. Manin, Relations between the correlators of the topological sigma-model coupled to gravity, *Comm. Math. Phys.* **196** (1998), 385–398.
15. A. Losev, Commutativity Equations, operator-valued cohomology of the "sausage" compactification of $(\mathbb{C}^*)^N/\mathbb{C}^*$ and SQM, preprint, ITEP-TH-84/98, LPTHE-61/98.
16. A. Losev, Passing from solutions to Commutativity Equations to solutions to Associativity Equations and background independence for gravitational descendents, preprint, ITEP-TH-85/98, LPTHE-62/98.
17. A. Losev and Y. Manin, New modular spaces of pointed curves and pencils of flat connections, preprint, arXiv: math.AG/0001003 January 1, 2000.

18. M. Readdy, Systems of parameters for cubical complexes, preprint, 1999.
19. V. Reiner and G. Ziegler, Coxeter-associahedra, *Mathematika* **41** (1994), 364–393.
20. R. P. Stanley, The upper bound conjecture and Cohen–Macaulay rings, *Stud. Appl. Math.* **54** (1975), 135–142.
21. R. P. Stanley, “Enumerative Combinatorics,” Vol. I, Wadsworth and Brooks/Cole, Pacific Grove, CA, 1986.
22. R. P. Stanley, A monotonicity property of h-vectors and h^*-vectors, *European J. Combin.* **14** (1993), 251–258.
23. R. P. Stanley, “Combinatorics and Commutative Algebra,” 2nd ed., Birkhäuser, Boston, 1996.
24. E. Witten, On the structure of the topological phase of two-dimensional gravity, *Nuclear Phys. B* **340** (1990), 281–332.
25. E. Witten, Two-dimensional gravity and intersection theory on moduli space, *in* “Surveys in Differential Geometry,” Cambridge, MA, 1990, pp. 243–310, Lehigh Univ., Bethlehem, PA, 1991.