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q-Stirling numbers: A new view ☆

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We show the classical q-Stirling numbers of the second kind can be expressed compactly as a pair of statistics on a subset of restricted growth words. The resulting expressions are polynomials in q and $1+q$. We extend this enumerative result via a decomposition of a new poset $\Pi(n, k)$ which we call the Stirling poset of the second kind. Its rank generating function is the q-Stirling number $S_q[n, k]$. The Stirling poset of the second kind supports an algebraic complex and a basis for integer homology is determined. A parallel enumerative, poset theoretic and homological study for the q-Stirling numbers of the first kind is done. Letting $t = 1 + q$ we give a bijective argument showing the (q, t)-Stirling numbers of the first and second kinds are orthogonal.

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1. Introduction

The idea of q-analogues can be traced back to Euler in the 1700's who was studying q-series, especially specializations of theta functions. The Gaussian polynomial or q-binomial is the familiar q-analogue of the binomial coefficient given by $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$, where $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]_q! = [1]_q \cdot [2]_q \cdots [n]_q$. A combinatorial interpretation due to MacMahon in 1916 [19, page 315] is

$$\sum_{\pi \in \mathfrak{S}(0^{n-k}, 1^k)} q^{\mathrm{inv}(\pi)} = \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

Here $\mathfrak{S}(0^{n-k}, 1^k)$ denotes the set of 0-1 bit strings consisting of $n-k$ zeroes and k ones, and for $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}(0^{n-k}, 1^k)$ the number of inversions is $\mathrm{inv}(\pi) = |\{(i,j) : i < j \text{ and } \pi_i > \pi_j\}|$. The inversion statistic goes back to work of Cramer (1750), Bézout (1764) and Laplace (1772). See the discussion in [22, page 92]. Netto enumerated the elements of the symmetric group by the inversion statistic in 1901 [22, Chapter 4, Sections 54 and 57], and in 1916 MacMahon [19, page 318] gave the q-factorial expansion $\sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{inv}(\pi)} = [n]_q!$.

Recent work of Fu–Reiner–Stanton–Thiem [9, Theorem 1] has expressed the classical q-binomial in terms of a pair of statistics over a *subset* of $\mathfrak{S}(0^{n-k}, 1^k)$ using powers of q and $1+q$:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\pi \in \Omega(n,k)'} q^{a(\pi)} \cdot (1+q)^{p(\pi)}. \tag{1.1}$$

They show this q-$(1+q)$-binomial is related to Ennola duality for finite unitary groups and that it counts unitary subspaces [9, Sections 4 and 6.2]. A two-variable version exhibits a cyclic sieving phenomenon involving unitary spaces [9, Sections 4 and 5].

It is from the q-binomial result (1.1) that we springboard our work. Our first goal is enumerative, that is, to discover compact encodings of classical q-analogues:

Goal 1. *Given a q-analogue*

$$f(q) = \sum_{w \in S} q^{\sigma(w)},$$

for some statistic $\sigma(\cdot)$, find a subset $T \subseteq S$ and statistics $A(\cdot)$ and $B(\cdot)$ so that the q-analogue may be expressed as

$$f(q) = \sum_{w \in T} q^{A(w)} \cdot (1+q)^{B(w)}. \tag{1.2}$$

For the q-Stirling numbers of the first and second kinds, we develop their q-$(1+q)$-analogues. Furthermore, we are able to understand these q-$(1+q)$-analogues via

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enumerative, poset theoretic and topological viewpoints. These lead to the following expanded goal:

Goal 2. *Given a q-analogue which can be written compactly as a q-$(1+q)$-analogue as in (1.2), find poset theoretic and homological reasons to explain this phenomenon.*

This paper proceeds as follows. In Section 2 we recall the notion of restricted growth words or *RG*-words to encode set partitions. A weighted version yields the usual q-Stirling numbers of the second kind; see Lemma 2.3. In Section 3 we describe a subset of *RG*-words, which we call *allowable*, whose weighting gives the q-Stirling numbers of the second kind and hence a more compact presentation of the q-Stirling numbers of the second kind; see Theorem 3.2.

We then take a poset theoretic viewpoint in Section 4 where we introduce the Stirling poset of the second kind $\Pi(n,k)$. Its rank generating function is precisely the q-Stirling number $S_q[n,k]$. Using discrete Morse theory, we show in Theorem 4.3 that the Stirling poset of the second kind has an acyclic matching. In Section 5 we give a decomposition of the Stirling poset into Boolean algebras with the minimal element of each Boolean algebra corresponding to an allowable *RG*-word; see Theorem 5.1. A generating function for the q-analogue of critical cells is provided.

In Section 6 we review the notion of an algebraic complex supported on a poset. In Theorem 6.3 we show that the Stirling poset $\Pi(n,k)$ supports an algebraic complex and give a basis for the integer homology, all of which occurs in even dimensions. We give two proofs of this result. The first uses Hersh, Shareshian and Stanton's homological interpretation of Stembridge's $q=-1$ phenomenon, while the second is an elementary proof using the poset decomposition in Section 5.

In Section 7 we review the de Médicis–Leroux rook placement interpretation of the q-Stirling numbers of the first kind. In Theorem 7.4 we show a subset of these boards, with the appropriate weighting, yields a compact representation of the q-Stirling number of the first kind. In Section 8 we introduce the Stirling poset of the first kind $\Gamma(m,n)$ whose rank generating function is precisely the q-Stirling number $c_q[n,k]$. Again, a decomposition of this graded poset is given. We show the Stirling poset of the first kind supports an algebraic complex and describe a basis for the integer homology which occurs in even dimensions. See Theorems 8.4 and 8.7. In Section 9 we introduce (q,t)-analogues of the Stirling numbers of the first and second kinds and show orthogonality holds combinatorially. We end with concluding remarks.

2. *RG*-words

Recall a *set partition* of the n elements $\{1,2,\ldots,n\}$ is a decomposition of this set into mutually disjoint nonempty sets called blocks. Unless otherwise indicated, throughout all set partitions will be written in standard form, that is, a partition into k blocks will

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be denoted by $\pi = B_1/B_2/\cdots/B_k$, where the blocks are ordered so that $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$. We denote the set of all partitions of $\{1, 2, \ldots, n\}$ by Π_n.

Given a partition $\pi \in \Pi_n$, we encode it using a *restricted growth word* $w(\pi) = w_1 w_2 \cdots w_n$, where $w_i = j$ if the element i occurs in the jth block B_j of π. For example, the partition $\pi = 14/236/57$ has *RG*-word $w = w(\pi) = 1221323$. Restricted growth words are also known as restricted growth functions. Recall a *restricted growth function* $f : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, k\}$ is a surjective map which satisfies $f(1) = 1$ and $f(i) \leq \max(f(1), f(2), \ldots, f(i-1)) + 1$ for $i = 2, 3, \ldots, n$. They have been studied by Hutchinson [13] and Milne [20,21].

Two facts about *RG*-words follow immediately from using the standard form for set partitions.

Proposition 2.1. *The following properties are satisfied by RG-words:*

1. *Any RG-word begins with the element* 1.
2. *For an RG-word w let $\epsilon(j)$ be the smallest index such that $w_{\epsilon(j)} = j$. Then the $\epsilon(j)$ form an increasing sequence, that is,*

$$\epsilon(1) < \epsilon(2) < \cdots .$$

The *q-Stirling numbers of the second kind* are defined by

$$S_q[n, k] = S_q[n-1, k-1] + [k]_q \cdot S_q[n-1, k], \text{ for } 1 \leq k \leq n, \tag{2.1}$$

with boundary conditions $S_q[n, 0] = \delta_{n,0}$ and $S_q[0, k] = \delta_{0,k}$, where $\delta_{i,j}$ is the usual Kronecker delta function. Setting $q = 1$ gives the familiar Stirling number of the second kind $S(n, k)$ which enumerates the number of partitions $\pi \in \Pi_n$ with exactly k blocks. There is a long history of studying set partition statistics [10,17,25] and q-Stirling numbers [3,5,11,21,32].

We begin by presenting a statistic on *RG*-words which generates the q-Stirling numbers of the second kind. Let $\mathcal{R}(n, k)$ denote the set of all *RG*-words of length n with maximum letter k, which corresponds to set partitions of $\{1, 2, \ldots, n\}$ into k blocks. For $w \in \mathcal{R}(n, k)$, let $m_i = \max(w_1, w_2, \ldots, w_i)$ and form the weight $\mathrm{wt}(w) = \prod_{i=1}^{n} \mathrm{wt}_i(w)$, where $\mathrm{wt}_1(w) = 1$ and for $2 \leq i \leq n$, let

$$\mathrm{wt}_i(w) = \begin{cases} q^{w_i - 1} & \text{if } m_{i-1} \geq w_i, \\ 1 & \text{if } m_{i-1} < w_i. \end{cases} \tag{2.2}$$

For example, $\mathrm{wt}(1221323) = 1 \cdot 1 \cdot q^1 \cdot q^0 \cdot 1 \cdot q^1 \cdot q^2 = q^4$. In terms of set partitions, the weight of $\pi = B_1/B_2/\cdots/B_k$ is $\mathrm{wt}(\pi) = \prod_{j=1}^{k} q^{(j-1)\cdot(|B_j|-1)}$.

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Table 1
Using RG-words to compute $S_q[4,2] = q^2 + 3q + 3$.

Partition	RG-word w	wt(w)
1/234	1222	$1 \cdot 1 \cdot q \cdot q = q^2$
12/34	1122	$1 \cdot 1 \cdot 1 \cdot q = q$
13/24	1212	$1 \cdot 1 \cdot 1 \cdot q = q$
14/23	1221	$1 \cdot 1 \cdot q \cdot 1 = q$
134/2	1211	$1 \cdot 1 \cdot 1 \cdot 1 = 1$
124/3	1121	$1 \cdot 1 \cdot 1 \cdot 1 = 1$
123/4	1112	$1 \cdot 1 \cdot 1 \cdot 1 = 1$

Proposition 2.2. *For $w = w_1 \cdots w_n \in \mathcal{R}(n,k)$ the weight is given by*

$$\mathrm{wt}(w) = q^{\sum_{i=1}^n w_i - n - \binom{k}{2}}.$$

Lemma 2.3. *The q-Stirling number of the second kind is given by*

$$S_q[n,k] = \sum_{w \in \mathcal{R}(n,k)} \mathrm{wt}(w).$$

Proof. We show RG-words $w \in \mathcal{R}(n,k)$ satisfy the recurrence (2.1). Given an RG-word $w = w_1 w_2 \cdots w_n \in \mathcal{R}(n,k)$, consider the map φ defined by removing the last letter of the word, that is, $\varphi(w) = w_1 w_2 \cdots w_{n-1}$. Clearly $\varphi : \mathcal{R}(n,k) \longrightarrow \mathcal{R}(n-1,k-1) \,\dot\cup\, \mathcal{R}(n-1,k)$. If the only occurrence of the maximum letter k in the word w is the nth position, that is, $w_n = k$, then these words are in bijection with the set $\mathcal{R}(n-1,k-1)$. Otherwise, $\varphi(w)$ is of length $n-1$ and all the letters from $\{1,2,\ldots,k\}$ occur at least once in $\varphi(w)$. In the first case $\mathrm{wt}(\varphi(w)) = \mathrm{wt}(w)$. In the second case the letter k occurs more than once in w. Given $w' = w_1 w_2 \cdots w_{n-1} \in \mathcal{R}(n-1,k)$ there are k possibilities for the nth letter x in the inverse image $\varphi^{-1}(w') = w_1 w_2 \cdots w_{n-1} x$, namely, $x \in \{1,2,\ldots,k\}$. Each possibility respectively contributes $1, q^1, \ldots, q^{k-1}$ to the weight, giving a total weighting contribution of $[k]_q$. $\square$

See Table 1 for the RG-word computation of the q-Stirling number $S_q[4,2]$.

3. Allowable *RG*-words

Mirroring the q-$(1+q)$-binomial, in this section we define a subset of RG-words and two statistics $A(\cdot)$ and $B(\cdot)$ which generate the classical q-Stirling number of the second kind as a polynomial in q and $1+q$. We will see in Sections 4 through 6 that this has poset and topological implications.

Definition 3.1. An RG-word $w \in \mathcal{R}(n,k)$ is *allowable* if every even entry appears exactly once. Denote by $\mathcal{A}(n,k)$ the set of all allowable RG-words in $\mathcal{R}(n,k)$.

Another way to state that $w \in \mathcal{R}(n,k)$ is an allowable RG-word is that it is an initial segment of an infinite word of the form

Table 2
Allowable RG-words in $\mathcal{A}(n,k)$ and their weights for $1 \le k \le n \le 5$.

	w	$\mathrm{wt}'(w)$		w	$\mathrm{wt}'(w)$
$\mathcal{A}(1,1)$	1	1	$\mathcal{A}(5,3)$	12311	$(1+q)^2$
$\mathcal{A}(2,1)$	11	1		12131	$(1+q)^2$
$\mathcal{A}(2,2)$	12	1		12113	$(1+q)^2$
$\mathcal{A}(3,1)$	111	1		12133	$(1+q)\cdot q^2$
$\mathcal{A}(3,2)$	121	$1+q$		12313	$(1+q)\cdot q^2$
	112	1		12331	$q^2\cdot(1+q)$
$\mathcal{A}(3,3)$	123	1		12333	$q^2\cdot q^2$
$\mathcal{A}(4,1)$	1111	1		11213	$(1+q)$
$\mathcal{A}(4,2)$	1211	$(1+q)^2$		11231	$(1+q)$
	1121	$(1+q)$		11233	q^2
	1112	1		11123	1
$\mathcal{A}(4,3)$	1213	$(1+q)$	$\mathcal{A}(5,4)$	12341	$(1+q)$
	1231	$(1+q)$		12343	$q^2(1+q)$
	1233	q^2		12134	$(1+q)$
	1123	1		12314	$(1+q)$
$\mathcal{A}(4,4)$	1234	1		12334	q^2
$\mathcal{A}(5,1)$	11111	1		11234	1
$\mathcal{A}(5,2)$	12111	$(1+q)^3$	$\mathcal{A}(5,5)$	12345	1
	11211	$(1+q)^2$			
	11121	$(1+q)$			
	11112	1			

$$w = u_1 \cdot 2 \cdot u_3 \cdot 4 \cdot u_5 \cdots,$$

where u_{2i-1} is a word on the alphabet of the odd integers $\{1, 3, \ldots, 2i-1\}$. In terms of set partitions, an RG-word is allowable if in the corresponding set partition every even indexed block is a singleton block. See Table 2.

For an RG-word $w = w_1 \cdots w_n$ define $\mathrm{wt}'(w) = \prod_{i=1}^{n} \mathrm{wt}'_i(w)$, where for $m_i = \max(w_1, \ldots, w_i)$

$$\mathrm{wt}'_i(w) = \begin{cases} q^{w_i - 1} \cdot (1+q) & \text{if } m_{i-1} > w_i, \\ q^{w_i - 1} & \text{if } m_{i-1} = w_i, \\ 1 & \text{if } m_{i-1} < w_i \text{ or } i = 1. \end{cases} \tag{3.1}$$

For completeness, we decompose the wt′ statistic into two statistics on RG-words. Let

$$A_i(w) = \begin{cases} w_i - 1 & \text{if } m_{i-1} \ge w_i, \\ 0 & \text{if } m_{i-1} < w_i \text{ or } i = 1, \end{cases} \quad \text{and}$$

$$B_i(w) = \begin{cases} 1 & \text{if } m_{i-1} > w_i, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Define

$$A(w) = \sum_{i=1}^{n} A_i(w) \quad \text{and} \quad B(w) = \sum_{i=1}^{n} B_i(w).$$

Theorem 3.2. *The q-Stirling numbers of the second kind can be expressed as a weighting over the set of allowable RG-words as follows:*

$$S_q[n,k] = \sum_{w \in \mathcal{A}(n,k)} \mathrm{wt}'(w) = \sum_{w \in \mathcal{A}(n,k)} q^{A(w)} \cdot (1+q)^{B(w)}. \tag{3.3}$$

Hence evaluating the q-Stirling number at $q = -1$ gives the number of weakly increasing allowable words in $\mathcal{A}(n,k)$.

Proof. We proceed by induction on n and k. Clearly the result holds for $S_q[n,1]$ and $S_q[n,n]$ as the corresponding allowable words are $11\cdots1$ and $12\cdots n$, each of weight 1.

For the general case it is enough to show that (3.3) satisfies the defining relation (2.1) for the q-Stirling numbers of the second kind. We first consider the case when k is even. We split the allowable words according to the value of the last letter, that is, we write $w = u \cdot w_n$. Observe that $\mathrm{wt}'(w) = \mathrm{wt}'(u) \cdot \mathrm{wt}'_n(w)$. We have

$$\begin{aligned}
\sum_{w \in \mathcal{A}(n,k)} \mathrm{wt}'(w) &= \sum_{\substack{u \in \mathcal{A}(n-1,k-1) \\ w_n = k \\ m_{n-1} = k-1}} \mathrm{wt}'(u) \cdot \mathrm{wt}'_n(w) + \sum_{\substack{u \in \mathcal{A}(n-1,k) \\ w_n < k \\ m_{n-1} = k}} \mathrm{wt}'(u) \cdot \mathrm{wt}'_n(w) \\
&= 1 \cdot S_q[n-1,k-1] + ((1+q) + q^2 \cdot (1+q) + \cdots \\
&\quad + q^{k-2} \cdot (1+q)) \cdot S_q[n-1,k] \\
&= S_q[n-1,k-1] + [k]_q \cdot S_q[n-1,k],
\end{aligned}$$

where in the second sum the last letter w_n is odd. For the case when k is odd there is a similar computation, except then there are three cases:

$$\begin{aligned}
\sum_{w \in \mathcal{A}(n,k)} \mathrm{wt}'(w) = &\sum_{\substack{u \in \mathcal{A}(n-1,k-1) \\ w_n = k \\ m_{n-1} = k-1}} \mathrm{wt}'(u) \cdot \mathrm{wt}'_n(w) + \sum_{\substack{u \in \mathcal{A}(n-1,k-1) \\ w_n = k \\ m_{n-1} = k}} \mathrm{wt}'(u) \cdot \mathrm{wt}'_n(w) \\
&+ \sum_{\substack{u \in \mathcal{A}(n-1,k-1) \\ w_n < k \\ m_{n-1} = k}} \mathrm{wt}'(u) \cdot \mathrm{wt}'_n(w).
\end{aligned}$$

Here in the second and third sums the last letter w_n is odd. In both parity cases for k, the result is equal to the q-Stirling number of the second kind $S_q[n,k]$, as desired. $\square$

See Table 2 for the allowable *RG*-words for $1 \le n \le 5$.

Denote by $a(n,k) = |\mathcal{A}(n,k)|$ the cardinality of allowable words, and call it the *allowable Stirling number of the second kind*. The following holds.

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Table 3
The allowable Stirling numbers of the second kind $a(n,k)$, the allowable Bell numbers $a(n)$ and the classical Bell numbers $b(n)$ for $0 \le n \le 10$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	$a(n)$	$b(n)$
0	1											1	1
1	0	1										1	1
2	0	1	1									2	2
3	0	1	2	1								4	5
4	0	1	3	4	1							9	15
5	0	1	4	11	6	1						23	52
6	0	1	5	26	23	9	1					65	203
7	0	1	6	57	72	50	12	1				199	877
8	0	1	7	120	201	222	86	16	1			654	4140
9	0	1	8	247	522	867	480	150	20	1		2296	21147
10	0	1	9	502	1291	3123	2307	1080	230	25	1	8569	115975

Proposition 3.3. *The allowable Stirling numbers of the second kind satisfy the recurrence*

$$a(n,k) = a(n-1,k-1) + \lceil k/2 \rceil \cdot a(n-1,k) \quad \text{for } n \ge 1 \text{ and } 1 \le k \le n,$$

with the boundary conditions $a(n,0) = \delta_{n,0}$.

Proof. By definition each allowable word $w \in \mathcal{A}(n,k)$ corresponds to a set partition of $\{1,2,\dots,n\}$ into k nonempty subsets where each block with an even label has exactly one element in it. Let $p(w)$ be the corresponding set partition.

There are two cases. If n occurs as a singleton block in $p(w)$, then after deleting the element n we obtain a set partition of the elements $\{1,2,\dots,n-1\}$ into $k-1$ blocks. This corresponds to a word in $\mathcal{A}(n-1,k-1)$. Otherwise assume the element n occurs in a block with more than one element. We can first build an allowable set partition of $\{1,2,\dots,n-1\}$ into k blocks and then put the element n into one of the k blocks. Notice that n can only be placed into an odd numbered block, so we have $\lceil k/2 \rceil$ possible blocks to assign the element n. This gives $\lceil k/2 \rceil \cdot a(n-1,k)$ possibilities. □

We call the sum $a(n) = \sum_{k=0}^{n} a(n,k)$ the *nth allowable Bell number*. See Table 3. The following properties are straightforward to verify.

Proposition 3.4. *The allowable Stirling numbers of the second kind satisfy*

$$a(n,2) = n-1, \tag{3.4}$$

$$a(n,n-1) = \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil. \tag{3.5}$$

Proof. By definition any $w \in \mathcal{A}(n,2)$ is a word of length n consisting of exactly $n-1$ 1's and one 2. Since the initial letter must be 1, there are $n-1$ choices to assign the location of 2. Thus (3.4) follows.

For identity (3.5) we wish to count allowable words of length n with maximal entry $n-1$. By definition of an allowable word, there will be exactly one odd integer that appears twice and all other integers appear exactly once in such a word. In other words,

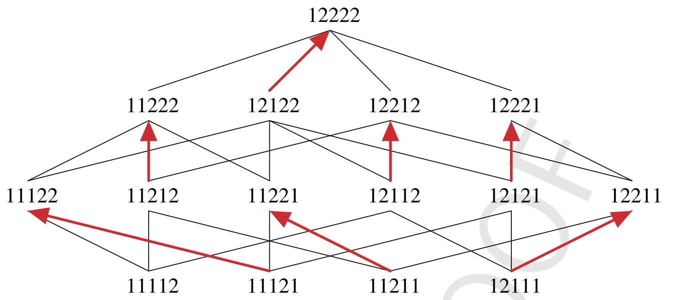

Fig. 1. The matching of the Stirling poset $\Pi(5, 2)$.

given the word $12 \cdots (n-1)$, we need to insert an odd integer less than or equal to $n-1$ so that the resulting word is still allowable. There are $\lceil (n-1)/2 \rceil = \lfloor n/2 \rfloor$ choices for such an odd integer. We can place this odd integer anywhere after its initial appearance in the word $12 \cdots (n-1)$. Thus we have in total $(n-1)+(n-3)+\cdots+(n-(2 \cdot \lceil (n-1)/2 \rceil - 1)) = \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$ ways to obtain a word in $\mathcal{A}(n, n-1)$. $\square$

Homological underpinnings of Theorem 3.2 will be discussed in Section 6.

4. The Stirling poset of the second kind

In order to understand the q-Stirling numbers more deeply, we give a poset structure on $\mathcal{R}(n, k)$, which we call *the Stirling poset of the second kind*, denoted by $\Pi(n, k)$, as follows. For $v, w \in \mathcal{R}(n, k)$ let $v = v_1 v_2 \cdots v_n \prec w$ if $w = v_1 v_2 \cdots (v_i + 1) \cdots v_n$ for some index i. It is clear that if $v \prec w$ then $\mathrm{wt}(w) = q \cdot \mathrm{wt}(v)$, where the weight is as defined in (2.2). The Stirling poset of the second kind is graded by the degree of the weight function wt. Thus the rank of the poset $\Pi(n, k)$ is $(n-k)(k-1)$ and its rank generating function is given by $S_q[n, k]$. For basic terminology regarding posets, we refer the reader to Stanley's treatise [27, Chapter 3]. See Figs. 1 and 2 for two examples of the Stirling poset of the second kind.

We next review the notion of a Morse matching [15,16]. This will enable us to find a natural decomposition of the Stirling poset of the second kind, and to later be able to draw homological conclusions. A *partial matching* on a poset P is a matching on the underlying graph of the Hasse diagram of P, that is, a subset $M \subseteq P \times P$ satisfying (i) the ordered pair $(a, b) \in M$ implies $a \prec b$, and (ii) each element $a \in P$ belongs to at most one element in M. When $(a, b) \in M$, we write $u(a) = b$ and $d(b) = a$. A partial matching on P is *acyclic* if there does not exist a cycle

$$a_1 \prec u(a_1) \succ a_2 \prec u(a_2) \succ \cdots \succ a_n \prec u(a_n) \succ a_1$$

with $n \geq 2$, and the elements $a_1, a_2, \ldots, a_n$ are distinct.

An alternate manner is to orient all the edges in the Hasse diagram of a poset downwards and then reorient all the edges occurring in the matching upwards. The acyclic

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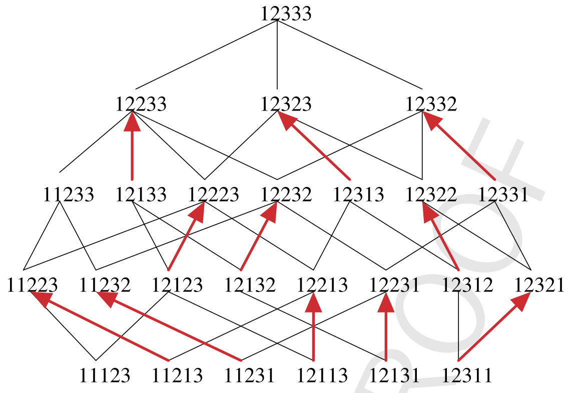

Fig. 2. The Stirling poset $\Pi(5, 3)$ and its discrete Morse matching. The rank generating function is the q-Stirling number $S_q[5, 3] = q^4 + 3q^3 + 7q^2 + 8q + 6$. The matched elements are indicated by arrows. The unmatched elements are 11123, 11233 and 12333, and the sum of their weights is $1 + q^2 + q^4$.

condition is simply that there is no cycle on the directed Hasse diagram. For the matched edge (a, b) the notation $u(a) = b$ and $d(b) = a$ denotes the fact that in the edge oriented from a to b the element b is "upwards" from a and similarly the element a is "downwards" from b. One can use the terminology of a *gradient path* or *V-path* consisting alternatively of matched and unmatched elements from the poset [7]. See Fig. 3. A *discrete Morse matching* is one where no gradient path forms a cycle.

We define a matching M on the Stirling poset $\Pi(n, k)$ in the following manner. Let w_i be the first entry in $w = w_1w_2 \cdots w_n \in \mathcal{R}(n, k)$ such that w is weakly decreasing, that is, $w_1 \le w_2 \le \cdots \le w_{i-1} \ge w_i$ and where we require the inequality $w_{i-1} \ge w_i$ to be strict unless both w_{i-1} and w_i are even. We have two subcases. If w_i is even then let $d(w) = w_1w_2 \cdots w_{i-1}(w_i - 1)w_{i+1} \cdots w_n$. In this case we have $\mathrm{wt}(d(w)) = q^{-1} \cdot \mathrm{wt}(w)$. Otherwise, if w_i is odd then let $u(w) = w_1w_2 \cdots w_{i-1}(w_i + 1)w_{i+1} \cdots w_n$ and we have $\mathrm{wt}(u(w)) = q \cdot \mathrm{wt}(w)$. If w is an allowable word which is weakly increasing, then w is unmatched in the poset. Again, we refer to Figs. 1 and 2.

Lemma 4.1. *For the partial matching M described on the poset $\Pi(n, k)$ the unmatched words $U(n, k)$ are of the form*

$$w = \begin{cases} u_1 \cdot 2 \cdot u_3 \cdot 4 \cdot u_5 \cdot 6 \cdots u_{k-1} \cdot k & \text{for } k \text{ even}, \\ u_1 \cdot 2 \cdot u_3 \cdot 4 \cdot u_5 \cdot 6 \cdots (k-1) \cdot u_k & \text{for } k \text{ odd}, \end{cases}$$

where $u_{2i-1} = (2i-1)^{j_i}$, that is, u_{2i-1} is a word consisting of $j_i \ge 1$ copies of the odd integer $2i - 1$.

Proof. The result follows by observing the unmatched elements of the Stirling poset $w(n, k)$ consist of RG-words in $\mathcal{R}(n, k)$ which are always increasing and have no repeated even-valued entries. $\square$

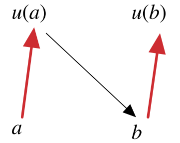

Fig. 3. First three steps of a gradient path.

Lemma 4.2. *Let a and b be two distinct elements in the Stirling poset of the second kind $\Pi(n,k)$ such that $a \prec u(a) \succ b \prec u(b)$. Then the element a is lexicographically larger than the element b.*

Proof. Suppose on the contrary that $a <_{\text{lex}} b$ with $a = a_1 \cdots a_n$. Assume that $u(a) = a_1 a_2 \cdots (a_i + 1) \cdots a_n$. Then a_i is odd and the strict inequality $a_{i-1} > a_i$ holds. Since a is lexicographically smaller than b and the element b is obtained by decreasing an entry in $u(a)$ by one, the element b must be of the form $b = a_1 \cdots (a_i + 1) \cdots (a_j - 1) \cdots a_n$ for some index $j > i$. The first i entries in b satisfy $a_1 \leq a_2 \leq \cdots \leq a_{i-1} \geq (a_i + 1)$ and $a_i + 1$ is even, so by definition the element b is matched to an element of lower rank, contradicting the fact that $(b, u(b))$ is a matched pair in M. $\square$

Theorem 4.3. *The matching M described for $\Pi(n,k)$ is an acyclic matching, that is, it is a discrete Morse matching.*

Proof. By Lemma 4.2 one cannot find a gradient cycle of the form

$$x_1 \prec u(x_1) \succ x_2 \prec u(x_2) \succ \cdots \succ x_k \prec u(x_k) \succ x_1$$

since the elements $x_1, \ldots, x_k$ must satisfy $x_1 >_{\text{lex}} x_2 >_{\text{lex}} \cdots >_{\text{lex}} x_k >_{\text{lex}} x_1$, which is impossible. $\square$

We end this section with enumeration of the words which are left unmatched in the discrete Morse matching. We will see in Section 6 that the unmatched words will provide a basis for the integer homology of the algebraic complex supported by the Stirling poset of the second kind.

Lemma 4.4. *The weighted generating function of the unmatched words $U(n,k)$ in $\Pi(n,k)$ is given by the q^2-binomial coefficient*

$$\sum_{u \in U(n,k)} \text{wt}(u) = \begin{bmatrix} n - 1 - \lfloor \frac{k}{2} \rfloor \\ \lfloor \frac{k-1}{2} \rfloor \end{bmatrix}_{q^2}.$$

Proof. Let $u = u_1 \cdots u_n \in U(n,k)$ be an unmatched word. Recall the weight is given by reading the word from left to right and gaining a multiplicative factor $q^{u_i - 1}$ for all values of i with $u_{i-1} = u_i$. Since $u_{i-1} = u_i$ can only appear when u_i is odd, the weight of an unmatched word is always q^{2m} for some non-negative integer m.

We claim that each $u \in U(n,k)$ of weight q^{2m} corresponds to an integer partition of $2m$ with at most $n-k$ parts where each part is even and where each part is at most $\rho = \lfloor (k-1)/2 \rfloor \cdot 2$. The correspondence is as follows. For each word u satisfying the condition with the odd integer j appearing m_j times, map these odd integers to $m_j - 1$ copies of $j-1$. The resulting partition of $2m$ is of the form

$$2m = \underbrace{2 + \cdots + 2}_{m_3 - 1} + \underbrace{4 + \cdots + 4}_{m_5 - 1} + \cdots + \underbrace{\rho + \cdots + \rho}_{m_\sigma - 1},$$

where σ is the largest occurring odd integer in the original RG-word u and $\rho = \sigma - 1$. For example, the word 112333455 corresponds to the partition $8 = 2 + 2 + 4$. Note that the unmatched word 1 corresponds to the empty partition $\emptyset$.

An alternate way to describe these partitions is to form a partition of m into at most $n-k$ parts with each part at most $\lfloor (k-1)/2 \rfloor$. By doubling each part, we obtain the above-mentioned partition. However, by [27, Proposition 1.7.3] the sum of the weight of partitions that fit into a rectangle of size $n-k$ by $\lfloor (k-1)/2 \rfloor$ is given by the Gaussian polynomial $\begin{bmatrix} n-k+\lfloor \frac{k-1}{2} \rfloor \\ \lfloor \frac{k-1}{2} \rfloor \end{bmatrix}_q$. By the substitution $q \mapsto q^2$, the result follows. $\square$

Corollary 4.5. *The number of unmatched words of length n, that is, $U(n) = \sum_{k=1}^{n} |U(n,k)|$ is given by the Fibonacci number F_n, where $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ and $F_0 = F_1 = 1$.*

Proof. Substituting $q^2 = 1$, that is, $q = -1$ in Lemma 4.4 gives the number of unmatched words $|U(n,k)|$ in the Stirling poset of the second $\Pi(n,k)$. Hence,

$$U(n) = \sum_{k=1}^{n} |U(n,k)| = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} = F_n,$$

where the last equality is a well-known binomial coefficient expansion for the Fibonacci number F_n arising from compositions of n using 1's and 2's. $\square$

5. Decomposition of the Stirling poset of the second kind

We next decompose the Stirling poset $\Pi(n,k)$ into Boolean algebras indexed by the allowable words. This gives a poset explanation for the factorization of the q-Stirling number $S_q[n,k]$ in terms of powers of q and $1+q$.

To state this decomposition, we need two definitions. For $w \in \mathcal{A}(n,k)$ an allowable word let $\mathrm{Inv_r}(w) = \{i : w_j > w_i \text{ for some } j < i\}$ be the set of all indices in w that contribute to the right-hand element of an inversion pair. For $i \in \mathrm{Inv_r}(w)$ such an entry w_i must be odd since in a given allowable word any entry occurring to the left of an even entry must be strictly less than it. Finally, for $w \in \mathcal{A}(n,k)$ let $\alpha(w)$ be the word formed by incrementing each of the entries indexed by the set $\mathrm{Inv_r}(w)$ by one. Additionally, for

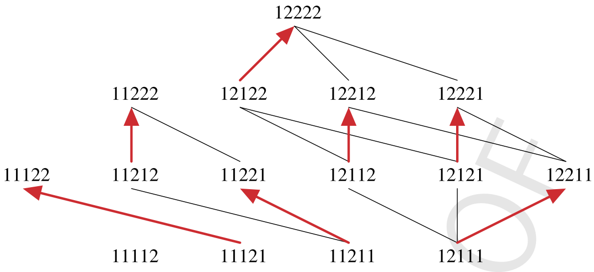

Fig. 4. The decomposition of the Stirling poset $\Pi(5,2)$ into Boolean algebras B_i for $i = 0, 1, 2, 3$. Arrows indicate the elements matched from the discrete Morse matching. Based on the ranks of the minimal elements in each Boolean algebra, one obtains the weight of the poset is $S_q[5,2] = 1 + (1+q) + (1+q)^2 + (1+q)^3$.

$w \in \mathcal{A}(n,k)$ and any $I \subseteq \mathrm{Inv_r}(w)$, the word formed by incrementing each of the entries indexed by the set I by one are elements of $\mathcal{R}(n,k)$ since if $i \in \mathrm{Inv_r}(w)$ then there is an index $h < i$ with $w_h = w_i$. This follows from Proposition 2.1 part (ii).

Theorem 5.1. *The Stirling poset of the second kind $\Pi(n,k)$ can be decomposed as the disjoint union of Boolean intervals*

$$\Pi(n,k) = \dot{\bigcup_{w \in \mathcal{A}(n,k)}} [w, \alpha(w)].$$

Furthermore, if an allowable word $w \in \mathcal{A}(n,k)$ has weight $\mathrm{wt}'(w) = q^i \cdot (1+q)^j$, then the rank of the element w is i and the interval $[w, \alpha(w)]$ is isomorphic to the Boolean algebra on j elements.

Proof. Let $w \in \mathcal{A}(n,k)$ with $\mathrm{wt}'(w) = q^i \cdot (1+q)^j$ and $|\mathrm{Inv_r}(w)| = m$. It directly follows from the definitions that the interval $[w, \alpha(w)]$ is isomorphic to the Boolean algebra B_m. With the exception of the element w, all the other elements in the interval $[w, \alpha(w)]$ are not allowable words in $\Pi(n,k)$ since all of the newly incremented entries will have at least two equal even entries. We also claim $m = j$, since $\mathrm{wt}'(w)$ picks up a factor of $1+q$ for each index i satisfying $w_i < m_{i-1} = \max(w_1, \ldots, w_{i-1})$. These indices are exactly the set $\mathrm{Inv_r}(w)$.

We claim every element of $\Pi(n,k)$ occurs in some Boolean algebra in the decomposition. This is vacuously true if $w \in \mathcal{A}(n,k)$. Otherwise since w is not an allowable word, it has even entries which are repeated. Decrease all occurrences of these repeated entries by one except for the first occurrence of each even integer. This is the allowable *RG*-word associated to w. $\square$

See Figs. 4 and 5 for examples of this decomposition for the posets in Figs. 1 and 2, respectively.

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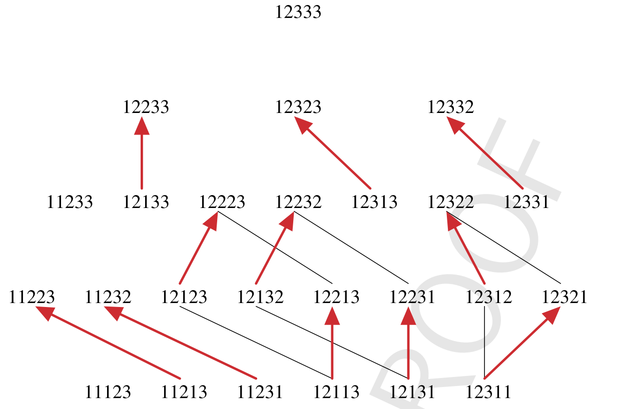

Fig. 5. The decomposition of the Stirling poset $\Pi(5,3)$ into Boolean algebras. Again, the matched elements are indicated with arrows. The weight of the poset is $S_q[5,3] = 1+2(1+q)+3(1+q)^2+q^2+3q^2(1+q)+q^4$.

6. Homological $q = -1$ phenomenon

Stembridge's $q = -1$ phenomenon [28,29] and the more general cyclic sieving phenomenon of Reiner, Stanton and White [24] count symmetry classes in combinatorial objects by evaluating their q-generating series at a primitive root of unity. Recently Hersh, Shareshian and Stanton [12] have given a homological interpretation of the $q = -1$ phenomenon by viewing it as an Euler characteristic computation on a chain complex supported by a poset. In the best scenario, the homology is concentrated in dimensions of the same parity and one can identify a homology basis. For further information about algebraic discrete Morse theory, see [14,15,26].

We will see the graded poset $\Pi(n,k)$ supports an algebraic complex $(\mathcal{C}, \partial)$. The aforementioned matching for $\Pi(n,k)$ (Theorem 4.3) is a discrete Morse matching for this complex and the unmatched elements occur in even ranks of the poset. Hence using standard discrete Morse theory [8], we can give a basis for the homology.

We now review the relevant background. We follow [12] here. See also [14,26]. Let P be a graded poset and W_i denote the rank i elements. We say the *poset P supports a chain complex* $(\mathcal{C}, \partial)$ of $\mathbb{F}$-vector spaces C_i if each C_i has basis indexed by the rank i elements W_i and $\partial_i : W_i \to W_{i-1}$ is a boundary map. Furthermore, for $x \in W_i$ and $y \in W_{i-1}$ the coefficient $\partial_{x,y}$ of y in $\partial_i(x)$ is zero unless $y <_P x$.

For $w \in \Pi(n,k)$, let

$$E(w) = \{i : w_i \text{ is even and } w_j = w_i \text{ for some } j < i\}$$

be the set of all indices of repeated even entries in the word w. Define the boundary map ∂ on the elements of $\Pi(n,k)$ by

$$\partial(w) = \sum_{j=1}^{r} (-1)^{j-1} \cdot w_1 \cdots w_{i_j - 1} \cdot (w_{i_j} - 1) \cdot w_{i_j+1} \cdots w_n, \tag{6.1}$$

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where $E(w) = \{i_1 < i_2 < \cdots < i_r\}$. For example, if $w = 122344$ then $E(122344) = \{3, 6\}$ and $\partial(122344) = 121344 - 122343$. With this definition of the boundary operator ∂, we have the following lemma.

Lemma 6.1. *The map ∂ is a boundary map on the algebraic complex $(\mathcal{C}, \partial)$ with the poset $\Pi(n, k)$ as support.*

Proof. By definition of ∂, we have

$$\partial^2(w) = \sum_{i_r < i_j} (-1)^{j-1} \cdot (-1)^{r-1} \cdot w_1 w_2 \cdots w_{i_r - 1} \cdots (w_{i_j} - 1) \cdots w_n + \sum_{i_r > i_j} (-1)^{j-1} \cdot (-1)^{r-2} \cdot w_1 w_2 \cdots w_{i_j - 1} \cdots (w_{i_r} - 1) \cdots w_n,$$

where the sum is over indices i_r and i_j with $w_{i_j}, w_{i_r} \in E(w)$. These two summations cancel since after switching r and j in the second summation, the resulting expression becomes the negative of the first. Hence we have that $\partial^2(w) = 0$. □

We have shown the graded poset $\Pi(n, k)$ supports an algebraic complex $(\mathcal{C}, \partial)$. We will need a lemma due to Hersh, Shareshian and Stanton [12, Lemma 3.2]. This is part (ii) of the original statement of the lemma.

Lemma 6.2 *(Hersh–Shareshian–Stanton). Let P be a graded poset supporting an algebraic complex $(\mathcal{C}, \partial)$. Assume the poset P has a Morse matching M such that for all matched pairs (y, x) with $y \prec x$ one has $\partial_{y,x} \in \mathbb{F}^*$. If all unmatched poset elements occur in ranks of the same parity, then $\dim(H_i(\mathcal{C}, \partial)) = |P_i^{\text{un } M}|$, that is, the number of unmatched elements of rank i.*

We can now state our result.

Theorem 6.3. *For the algebraic complex $(\mathcal{C}, \partial)$ supported by the Stirling poset of the second kind $\Pi(n, k)$, a basis for the integer homology is given by the weakly increasing allowable RG-words in $\mathcal{A}(n, k)$. Furthermore, we have*

$$\sum_{i \geq 0} \dim(H_i(\mathcal{C}, \partial; \mathbb{Z})) \cdot q^i = \begin{bmatrix} n - 1 - \lfloor \frac{k}{2} \rfloor \\ \lfloor \frac{k-1}{2} \rfloor \end{bmatrix}_{q^2}.$$

Proof. By definition of the boundary map ∂, if $(x, y) \in M$ then $\partial_{y,x} = 1$ and all of the unmatched words in $\Pi(n, k)$ occur in even ranks. The conditions in Lemma 6.2 are satisfied. So $\sum_{i \geq 0} \dim(H_i(\mathcal{C}, \partial; \mathbb{Z})) \cdot q^i$ is the q^2-binomial coefficient in Lemma 4.4. □

Remark 6.4. (A second proof of Theorem 6.3.) Theorem 6.3 can be proved without resorting to Lemma 6.2 as follows. The boundary map ∂ is supported on the Boolean algebras in the poset decomposition given in Theorem 5.1. Furthermore, the restriction to one of these Boolean algebras is the natural boundary map on that Boolean algebra. Hence the algebraic complex is a direct sum of algebraic complexes of Boolean algebras. The only summands that contribute any homology are the rank 0 Boolean algebras, that is, the unmatched elements.

7. q-Stirling numbers of the first kind

The (unsigned) *q-Stirling numbers of the first kind* are defined by the recurrence formula

$$c_q[n,k] = c_q[n-1,k-1] + [n-1]_q \cdot c_q[n-1,k], \tag{7.1}$$

where $c_q[n,0] = \delta_{n,0}$. When $q = 1$, the Stirling number of the first kind $c(n,k)$ enumerates permutations in the symmetric group $\mathfrak{S}_n$ having exactly k disjoint cycles. A combinatorial way to express q-Stirling numbers of the first kind is via rook placements; see de Médicis and Leroux [4]. Throughout a staircase chessboard of length m is a board with $m-i$ squares in the ith row for $i = 1, \ldots, m-1$ and each row of squares is left-justified.

Definition 7.1. Let $\mathcal{P}(m,n)$ be the set of all ways to place n rooks onto a staircase chessboard of length m so that no two rooks are in the same column. For any rook placement $T \in \mathcal{P}(m,n)$, denote by $\mathrm{s}(T)$ the number of squares to the south of the rooks in T.

Theorem 7.2 *(de Médicis–Leroux). The q-Stirling number of the first kind $c_q[n,k]$ is given by*

$$c_q[n,k] = \sum_{T \in \mathcal{P}(n,n-k)} q^{\mathrm{s}(T)},$$

where the sum is over all rook placements of $n-k$ rooks on a staircase board of length n.

We now define a subset $\mathcal{Q}(n,n-k)$ of rook placements in $\mathcal{P}(n,n-k)$ so that the q-Stirling number of the first kind $c_q[n,k]$ can be expressed as a statistic on the subset involving q and $1+q$. The key is given any staircase chessboard, assign it a certain alternating shaded pattern.

Definition 7.3. Given any staircase chessboard, assign it a chequered pattern such that every other antidiagonal strip of squares is shaded, beginning with the lowest antidiagonal. Let

$$\mathcal{Q}(m,n) = \{T \in \mathcal{P}(m,n) : \text{all rooks are placed in shaded squares}\}.$$

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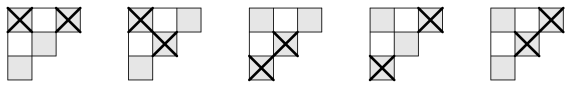

Fig. 6. Computing the q-Stirling number of the first kind $c_q[4,2]$ using $\mathcal{Q}(4,2)$.

For any rook placement $T \in \mathcal{Q}(m,n)$, let $\mathrm{r}(T)$ denote the number of rooks in T that are not in the first row. Define the weight to be $\mathrm{wt}(T) = q^{\mathrm{s}(T)} \cdot (1+q)^{\mathrm{r}(T)}$.

Theorem 7.4. *The q-Stirling number of the first kind is given by*

$$c_q[n,k] = \sum_{T \in \mathcal{Q}(n,n-k)} \mathrm{wt}(T) = \sum_{T \in \mathcal{Q}(n,n-k)} q^{\mathrm{s}(T)} \cdot (1+q)^{\mathrm{r}(T)},$$

where the sum is over all rook placements of $n-k$ rooks on an alternating shaded staircase board of length n.

Proof. We proceed by induction on n. It is straightforward to see the result holds for $n = k = 0$. Suppose the result is true for alternating shaded staircase boards of length $n-1$. Then we have

$$\begin{aligned}
\sum_{T \in \mathcal{Q}(n,n-k)} \mathrm{wt}(T) &= \sum_{\substack{T \in \mathcal{Q}(n,n-k) \\ \text{leftmost column is empty}}} \mathrm{wt}(T) + \sum_{\substack{T \in \mathcal{Q}(n,n-k) \\ \text{leftmost column is not empty}}} \mathrm{wt}(T) \\
&= \sum_{T \in \mathcal{Q}(n-1,n-k)} \mathrm{wt}(T) + \sum_{T \in \mathcal{Q}(n-1,n-k-1)} [n-1]_q \cdot \mathrm{wt}(T) \\
&= c_q[n-1,k-1] + [n-1]_q \cdot c_q[n-1,k] \\
&= c_q[n,k].
\end{aligned}$$

In the second equality, the first term follows from the fact that one can remove the leftmost column from the board, leaving a rook placement of $n-k$ rooks on a length $n-1$ shaded board. For the second term, we first consider where the rook occurs in the leftmost column. If the rook occurs in the $(2i+1)$st entry from the bottom of the leftmost column, where $0 \le i < \lfloor (n-1)/2 \rfloor$, it contributes a weight of $q^{2i} \cdot (1+q)$ since there are $2i$ squares below it and the rook does not occur in the first row. The only way a rook in the first column can also occur in the first row of a shaded staircase board is if the leftmost column has an odd number of squares, that is, n is even. In this case the rook would contribute a weight of q^{n-2}. For n even the overall weight contribution from a rook in the first column is $1 \cdot (1+q) + q^2 \cdot (1+q) + \cdots + q^{n-4} \cdot (1+q) + q^{n-2} = [n-1]_q$ and for n odd the weight contribution is $1 \cdot (1+q) + q^2 \cdot (1+q) + \cdots + q^{n-3} \cdot (1+q) = [n-1]_q$. Hence removing the first column from the staircase board along with the rook that occurs in

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Table 4
The allowable Stirling numbers of the first kind $d(n, k)$, their row sum $r(n)$ and $n!$ for $0 \le n \le 10$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	$r(n)$	$n!$
0	1											1	1
1	0	1										1	1
2	0	1	1									2	2
3	0	1	2	1								4	6
4	0	2	5	4	1							12	24
5	0	4	12	13	6	1						36	120
6	0	12	40	51	31	9	1					144	720
7	0	36	132	193	144	58	12	1				576	5040
8	0	144	564	904	769	376	106	16	1			2880	40320
9	0	576	2400	4180	3980	2273	800	170	20	1		14400	362880
10	0	2880	12576	23300	24080	15345	6273	1650	270	25	1	86400	3628800

it leaves a shaded staircase board of length $n-1$ with $n-k-1$ rooks. The total weight lost is $[n-1]_q$. Finally, the last equality is recurrence (7.1). □

See Fig. 6 for the computation of $c_q[4, 2]$ using allowable rook placements on length 4 shaded staircase boards.

When we substitute $q = -1$ into the q-Stirling number of the first kind, the weight $\mathrm{wt}(T)$ of a rook placement T will be 0 if there is a rook in T that is not in the first row. Hence the Stirling number of the first kind $c_q[n, k]$ evaluated at $q = -1$ counts the number of rook placements in $\mathcal{Q}(n, n-k)$ such that all of the rooks occur in shaded squares of the first row.

Corollary 7.5. *The q-Stirling number of the first kind $c_q[n, k]$ evaluated at $q = -1$ gives the number of rook placements in $\mathcal{Q}(n, n-k)$ where all of the rooks occur in shaded squares in the first row, that is,*

$$c_q[n, k]\big|_{q=-1} = \binom{\lfloor n/2 \rfloor}{n-k}.$$

Let $d(n, k) = |\mathcal{Q}(n, n-k)|$. We call $d(n, k)$ the *allowable Stirling number of the first kind*. See Table 4 for values.

Proposition 7.6. *The allowable Stirling numbers of the first kind $d(n, k)$ satisfy the recurrence*

$$d(n, k) = d(n-1, k-1) + \left\lceil \frac{n-1}{2} \right\rceil \cdot d(n-1, k)$$

with boundary conditions $d(n, 0) = \delta_{n,0}$, $d(n, n) = 1$ for $n \ge 0$ and $d(n, k) = 0$ when $k > n$.

Proof. For each $T \in \mathcal{Q}(n, n-k)$, there are two cases. If the leftmost column in T is empty, then after deleting this column we obtain an allowable rook placement $T' \in \mathcal{Q}(n-1, n-k)$. Otherwise assume there is a rook in the leftmost column. We can first build an allowable rook placement $T' \in \mathcal{Q}(n-1, n-k-1)$ and then add a column of $n-1$ squares with a rook in it to the left of T' to form a rook placement in $\mathcal{Q}(n, n-k)$. Notice that the rook in the

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leftmost column can only be placed into a shaded square, so there are $\lceil (n-1)/2 \rceil$ possible squares to place the rook. Overall this case gives $\lceil (n-1)/2 \rceil \cdot d(n-1,k)$ possibilities. □

Certain allowable Stirling numbers of the first kind have closed forms as follows. Here we let $r(n) = \sum_{k=0}^{n} d(n,k)$ denote the row sum of the allowable Stirling numbers of the first kind.

Proposition 7.7. *The allowable Stirling numbers of the first kind satisfy*

$$d(n,1) = \begin{cases} \left(\frac{n-1}{2}\right)!^2 & \text{for } n \text{ odd}, \\ \frac{n}{2} \cdot \left(\frac{n-1}{2}\right)!^2 & \text{for } n \text{ even}, \end{cases} \tag{7.2}$$

$$d(n,n-1) = \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil, \tag{7.3}$$

$$r(n) = d(n+2,1). \tag{7.4}$$

Proof. We first prove (7.4). Let $T \in \mathcal{Q}(n+2,1)$ be a rook placement on a shaded board. Since rooks are only allowed to be placed in shaded squares, the two rooks in the rightmost two columns must be in the bottommost antidiagonal. Delete the two longest anti-diagonals from T to obtain T'. Since the shaded squares are preserved, T' is still allowable with the longest column length n. The rightmost two rooks in T are deleted to form T', giving at most $n-1$ rooks in T'. Hence $d(n+2,1) \leq r(n)$.

On the other hand, for any rook placement T with at most $n-1$ rooks on a shaded staircase board of length n, we can add two anti-diagonals to T and place a rook in the bottom row for each empty column in the new chessboard to obtain T'. The board T' has $n+1$ rooks and $n+1$ columns, hence $r(n) \leq d(n+2,1)$. Hence we have the equality (7.4).

The expression $d(n,n-1)$ counts the number of rook placements of length n using 1 rook. This is the same as counting the number of shaded squares in a length n staircase chessboard. Counting column by column, beginning from the right, gives $1+1+2+2+\cdots+\lfloor n/2 \rfloor = \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$.

Finally, the expression $d(n,1)$ counts the number of rook placements with $n-1$ columns and $n-1$ rooks. Thus each column must have a rook. For each column with k squares, there are $\lceil k/2 \rceil$ shaded squares, hence $\lceil k/2 \rceil$ choices for the rook. This gives $((n-1)/2)!^2$ ways when n is odd and $(n/2) \cdot ((n-1)/2)!^2$ ways when n is even. □

8. Structure and topology of the Stirling poset of the first kind

We define a poset structure on rook placements on a staircase shape board. For rook placements T and T' in $\mathcal{P}(m,n)$, let $T \prec T'$ if T' can be obtained from T by either moving a rook to the left (west) or up (north) by one square. We call this poset *the Stirling poset of the first kind* and denote it by $\Gamma(m,n)$. It is straightforward to check that the poset

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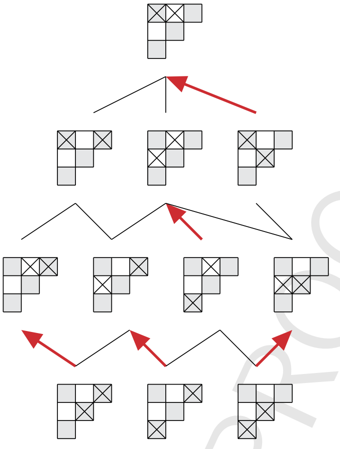

Fig. 7. Example of $\Gamma(4,2)$ with its matching. There is one unmatched rook placement in rank 2. The rank generating function of this poset is $c_q[4,2] = 3 + 4q + 3q^2 + q^3$.

$\Gamma(m,n)$ is graded of rank $(m-2)+(m-3)\cdots+(m-n-1) = (m-1)\cdot n - \binom{n+1}{2}$ and its rank generating function is $c_q[m, m-n]$. See Fig. 7 for an example.

We wish to study the topological properties of the Stirling poset of the first kind. To do so, we define a matching M on the poset as follows. Given any rook placement $T \in \Gamma(m,n)$, let r be the first rook (reading from left to right) that is not in a shaded square of the first row. Match T to T' where T' is obtained from T by moving the rook r one square down if r is not in a shaded square, or one square up if r is in a shaded square but not in the first row. It is straightforward to check that the unmatched rook placements are the ones where all of the rooks occur in the shaded squares of the first row.

As an example, the matching for $\Gamma(4,2)$ is shown in Fig. 7, where an upward arrow indicates a matching and other edges indicate the remaining cover relations. Observe the unmatched rook placements are the ones with all the rooks occurring in the shaded squares in the first row. By the way a chessboard is shaded, the unmatched rook placements only appear in even ranks in the poset.

We have a q-analogue of Corollary 7.5.

Theorem 8.1. *For the Stirling poset of the first kind $\Gamma(m,n)$ the generating function for the unmatched rook placements is*

$$\sum_{\substack{T\in\Gamma(m,n)\\ T\ unmatched}} \mathrm{wt}(T) = q^{n(n-1)} \cdot \begin{bmatrix} \lfloor \frac{m}{2} \rfloor \\ n \end{bmatrix}_{q^2}.$$

Proof. The number of unmatched rook placements in rank $2j$ in the poset $\Gamma(m,n)$ is the same as the number of integer partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ of $2j$ into n distinct non-negative even parts, with each $\lambda_i \le m-1-(2i-1)$. Alternatively, this is the

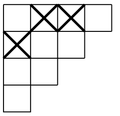

Fig. 8. A rook placement T with rook word $w_T = 3320$.

number of partitions $\delta = (\delta_1, \dots, \delta_n)$ of $2j - (0 + 2 + \cdots + (2n-2)) = 2j - n(n-1)$ into n non-negative even parts, where each part δ_i satisfies $\delta_i = \lambda_i - (2n - (2i-2)) \le m - 2n$ for $i = 1, \dots, n$. Thus we have

$$\sum_{\substack{T \in \Gamma(m,n) \\ T \text{ unmatched}}} \mathrm{wt}(T) = \sum_{j \ge 0} \sum_{\substack{(\lambda_1,\dots,\lambda_n) \vdash 2j \\ 0 \le \lambda_i \le m-1-(2i-1) \\ \lambda_i \text{ distinct even integers}}} q^{|\lambda|}$$

$$= q^{n(n-1)} \cdot \sum_{2j-n(n-1) \ge 0} \sum_{\substack{\lambda \vdash 2j-n(n-1) \\ 0 \le \lambda_i \le m-2n \\ i=1,\dots,n \\ \lambda_i \text{ even integers}}} q^{|\lambda|}$$

$$= q^{n(n-1)} \cdot \sum_{j - \frac{n(n-1)}{2} \ge 0} \sum_{\substack{\lambda \vdash j - \frac{n(n-1)}{2} \\ 0 \le \lambda_i \le \lfloor \frac{m}{2} \rfloor - n \\ i=1,\dots,n}} (q^2)^{|\lambda|} .$$

The last (double) sum is over all integer partitions into at most n parts where each part is at most $\lfloor m/2 \rfloor - n$. Hence this sum is given by the Gaussian polynomial $\begin{bmatrix} \lfloor m/2 \rfloor \\ n \end{bmatrix}_{q^2}$, proving the desired identity. □

Given a rook placement $T \in \mathcal{P}(m,n)$, we can associate to it a *rook word* $w_T = w_1 w_2 \dots w_{m-1}$ where w_i is one plus the number of squares below the column i rook. If column i is empty, let $w_i = 0$. See Fig. 8 for an example.

Lemma 8.2. *Let T and T' be two distinct elements in the Stirling poset of the first kind such that $T \prec u(T) \succ T' \prec u(T')$ is a gradient path. Then the rook words satisfy the inequality $w_T <_{\mathrm{lex}} w_{T'}$.*

Proof. Let $w_T = w_1 \cdots w_n$. Since $u(T)$ is obtained from T by shifting a rook a in column i up by one square, we have $w_{u(T)} = w_1 \cdots (w_i + 1) \cdots w_n$. By definition of the matching, in the rook placement T the rook a was in a shaded square not in the first row. In the rook placement $u(T)$ the rook a is now in an unshaded square. Furthermore, all of the rooks in the leftmost $i-1$ columns of T are in shaded squares in the first row.

The rook placement T' is obtained from $u(T)$ by shifting a rook to the right or down. We first show that T' cannot be obtained by shifting a rook in $u(T)$ down by one square.

Suppose a rook b in column $j \ne i$ of $u(T)$ is shifted down to form T'. If $j < i$ since all of the rooks in columns 1 through $i-1$ occur in shaded squares of the first row, the

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rook b is now in an unshaded square in the rook placement T'. Hence if it is matched with another rook placement, it will be of one rank lower, contradicting the fact that we assumed T' was part of a gradient path $T \prec u(T) \succ T' \prec u(T')$. If $j > i$ then the rook a in column i of T' is in an unshaded square and hence T' should be matched to a rook placement in one lower rank. Again, this contradicts our gradient path assumption. Hence this case cannot occur.

The remaining case is when a rook in $u(T)$ occurring in the jth column for some index $j < n$ is shifted to the right to form T'. Note this implies the $(j+1)$st column of T had no rooks in it. If $j < i$, then since b in column j in $u(T)$ is in a shaded square of the first row, it is shifted to an unshaded square in T' and hence T' is matched to a rook placement in one lower rank. If $j > i$ then a in T' is the first rook that does not appear in a shaded square of the first row. Hence T' is matched to some rook placement of one rank lower, contradicting the gradient path assumption.

The only remaining possibility is when $j = i$. Then the rook a in $u(T)$ is shifted to a shaded square in T', and hence $w_T = w_1 \cdots w_{i-1} \cdot w_i \cdot 0 \cdot w_{i+2} \cdots w_n >_{\text{lex}} w_1 \cdots w_{i-1} \cdot 0 \cdot (w_i - 1) \cdot w_{i+2} \cdots w_n = w_{T'}$, as desired. □

Theorem 8.3. *The matching M on the Stirling poset of the first kind $\Gamma(m,n)$ is an acyclic matching, that is, the Stirling poset has a discrete Morse matching.*

The proof is similar to that of Theorem 4.3, and thus omitted.

Next we give a decomposition of the Stirling poset of the first kind $\Gamma(m,n)$ into Boolean algebras indexed by the allowable rook placements. This will lead to a boundary map on the algebraic complex with $\Gamma(m,n)$ as the support. For any $T \in \mathcal{Q}(m,n)$, let $\alpha(T)$ be the rook placement obtained by shifting every rook that is not in the first row up by one. Then we have the following theorem.

Theorem 8.4. *The Stirling poset of the first kind $\Gamma(n,k)$ can be decomposed as the disjoint union of Boolean intervals*

$$\Gamma(m,n) = \dot{\bigcup_{T \in \mathcal{Q}(m,n)}} [T, \alpha(T)].$$

Furthermore, if $T \in \mathcal{Q}(m,n)$ has weight $\mathrm{wt}(T) = q^i \cdot (1+q)^j$, then the rank of the element T is i and the interval $[T, \alpha(T)]$ is isomorphic to the Boolean algebra on j elements.

Proof. We first show for any $T \in \mathcal{Q}(m,n)$ with $\mathrm{wt}(T) = q^i \cdot (1+q)^j$ that the interval $[T, \alpha(T)] \cong B_j$. Since $\mathrm{wt}(T) = q^i \cdot (1+q)^j$, the rank of T is i and there are j rooks in T that are not in the first row. The rank $i + l$ elements in the interval $[T, \alpha(T)]$ correspond to shifting l of those rooks up by one. It is straightforward to see that in the interval $[T, \alpha(T)]$ all of the elements except T are in $\mathcal{P}(m,n) - \mathcal{Q}(m,n)$ since the rook that is shifted up by one will not be in a shaded square.

We next need to show that every element $T \in \Gamma(m,n)$ occurs in some Boolean interval in this decomposition. This is vacuously true if $T \in \mathcal{Q}(m,n)$. Otherwise there are some rooks in T that are not in shaded squares. Shift all such rooks down by one to obtain an allowable rook placement associated to T. □

Given a rook placement $T \in \Gamma(m,n)$, let $N(T) = \{r_1, r_2, \ldots, r_s\}$ be the set of all rooks in T that are not in shaded squares, where the rooks r_i are labeled from left to right. We define the map ∂ as follows.

Definition 8.5. Let $\partial : \Gamma(m,n) \longrightarrow \mathbb{Z}[\Gamma(m,n)]$ be the map defined by

$$\partial(T) = \sum_{r_i \in N(T)} (-1)^{i-1} \cdot T_{r_i},$$

where T_{r_i} is obtained by moving the rook r_i in T down by one square.

Lemma 8.6. *The map ∂ in Definition 8.5 is a boundary map on the algebraic complex with $\Gamma(m,n)$ as the support.*

Proof. The boundary map ∂ is supported on the Boolean algebra decomposition of the Stirling poset of the first kind appearing in Theorem 8.4. The second proof of Theorem 6.3 applies again to show ∂ is a boundary map. □

Theorem 8.7. *For the algebraic complex $(\mathcal{C}, \partial)$ supported by the Stirling poset of the first kind $\Gamma(m,n)$, a basis for the integer homology is given by the rook placements in $\mathcal{P}(m,n)$ having all of the rooks occur in shaded squares in the first row. Furthermore,*

$$\sum_{i \geq 0} \dim(H_i(\mathcal{C}, \partial; \mathbb{Z})) \cdot q^i = q^{n(n-1)} \cdot \begin{bmatrix} \lfloor \frac{m}{2} \rfloor \\ n \end{bmatrix}_{q^2}.$$

Proof. The proof follows by applying Theorems 8.1 and 8.3 and Lemmas 6.2 and 8.6. □

9. (q,t)-Stirling numbers and orthogonality

In [31] Viennot has some beautiful results in which he gave combinatorial bijections for orthogonal polynomials and their moment generating functions. One well-known relation between the ordinary signed Stirling numbers of the first kind and Stirling numbers of the second kind is their orthogonality. A bijective proof of the orthogonality of their q-analogues via 0-1 tableaux was given by de Médicis and Leroux [4, Proposition 3.1].

There are a number of two-variable Stirling numbers of the second kind using bistatistics on RG-words and rook placements. See [32] and the references therein. Letting $t = 1 + q$ we define (q,t)-analogues of the Stirling numbers of the first and second kinds. We show orthogonality holds combinatorially for the (q,t)-version of the Stirling numbers via a sign-reversing involution on ordered pairs of rook placements and RG-words.

Definition 9.1. Define the (q,t)-Stirling numbers of the first and second kinds by

$$s_{q,t}[n,k] = (-1)^{n-k} \cdot \sum_{T \in \mathcal{Q}(n,n-k)} q^{\mathrm{s}(T)} \cdot t^{\mathrm{r}(T)} \tag{9.1}$$

and

$$S_{q,t}[n,k] = \sum_{w \in \mathcal{A}(n,k)} q^{A(w)} \cdot t^{B(w)}. \tag{9.2}$$

For what follows, let

$$[k]_{q,t} = \begin{cases} (q^{k-2} + q^{k-4} + \cdots + 1) \cdot t & \text{when } k \text{ is even,} \\ q^{k-1} + (q^{k-3} + q^{k-5} + \cdots + 1) \cdot t & \text{when } k \text{ is odd.} \end{cases} \tag{9.3}$$

Corollary 9.2. *The (q,t)-analogues of Stirling numbers of the first and second kinds satisfy the following recurrences:*

$$s_{q,t}[n,k] = s_{q,t}[n-1,k-1] - [n-1]_{q,t} \cdot s_{q,t}[n-1,k] \quad \text{for } n \geq 1 \text{ and } 1 \leq k \leq n, \tag{9.4}$$

and

$$S_{q,t}[n,k] = S_{q,t}[n-1,k-1] + [k]_{q,t} \cdot S_{q,t}[n-1,k] \quad \text{for } n \geq 1 \text{ and } 1 \leq k \leq n, \tag{9.5}$$

with initial conditions $s_{q,t}[n,0] = \delta_{n,0}$ and $S_{q,t}[n,0] = \delta_{n,0}$. For $k > n$, we set $s_{q,t}[n,k] = S_{q,t}[n,k] = 0$.

Proof. Immediate from Theorem 3.2 and Theorem 7.4. □

Recall the generating polynomials for the q-Stirling numbers are

$$(x)_{n,q} = \sum_{k=0}^{n} s_q[n,k] \cdot x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} S_q[n,k] \cdot (x)_{k,q}, \tag{9.6}$$

where the q-analogue of the kth falling factorial of x is given by

$$(x)_{k,q} = \prod_{m=0}^{k-1} (x - [m]_q).$$

The expressions in (9.6) are due to Carlitz [3, Section 3]. The case $q = 1$ is due to Stirling in 1730 and was his original definition for the Stirling numbers of the first and second kinds; see [30, pages 8 and 11]. We can generalize (9.6) to (q,t)-polynomials.

Theorem 9.3. *The generating polynomials for the (q,t)-Stirling numbers are*

$$(x)_{n,q,t} = \sum_{k=0}^{n} s_{q,t}[n,k] \cdot x^k \,, \tag{9.7}$$

and

$$x^n = \sum_{k=0}^{n} S_{q,t}[n,k] \cdot (x)_{k,q,t} \,, \tag{9.8}$$

where $(x)_{k,q,t} = \prod_{m=0}^{k-1} (x - [m]_{q,t})$.

Proof. Both identities follow by induction on n. It is straightforward to check the case $n = 0$, so suppose the identities are true for $n-1$. Multiply the recurrence (9.4) for the signed (q,t)-Stirling numbers of the first kind by x^k and sum over all $0 \le k \le n$ to give

$$\begin{aligned}
\sum_{k=0}^{n} s_{q,t}[n,k] \cdot x^k &= \sum_{k=0}^{n} (s_{q,t}[n-1,k-1] - [n-1]_{q,t} \cdot s_{q,t}[n-1,k]) \cdot x^k \\
&= x \cdot \sum_{k=0}^{n-1} s_{q,t}[n-1,k] \cdot x^k - [n-1]_{q,t} \cdot \sum_{k=0}^{n-1} s_{q,t}[n-1,k] \cdot x^k \\
&= (x)_{n-1,q,t} \cdot (x - [n-1]_{q,t}) \\
&= (x)_{n,q,t} \,,
\end{aligned}$$

which is the first identity. For the second identity, multiply the recurrence (9.5) for the (q,t)-Stirling number of the second kind by $(x)_{k,q,t}$ and sum over all $0 \le k \le n$ to give

$$\begin{aligned}
\sum_{k=0}^{n} S_{q,t}[n,k] \cdot (x)_{k,q,t} &= \sum_{k=0}^{n} (S_{q,t}[n-1,k-1] + [k]_{q,t} \cdot S_{q,t}[n-1,k]) \cdot (x)_{k,q,t} \\
&= \sum_{k=0}^{n} S_{q,t}[n-1,k-1] \cdot (x)_{k-1,q,t} \cdot (x - [k-1]_{q,t}) \\
&\quad + \sum_{k=0}^{n} [k]_{q,t} \cdot S_{q,t}[n-1,k] \cdot (x)_{k,q,t} \\
&= x \cdot \sum_{k=0}^{n-1} S_{q,t}[n-1,k] \cdot (x)_{k,q,t} \\
&\quad - \sum_{k=0}^{n} [k-1]_{q,t} \cdot S_{q,t}[n-1,k-1] + \sum_{k=0}^{n} [k]_{q,t} \cdot S_{q,t}[n-1,k] \,.
\end{aligned}$$

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The last two summations cancel each other by shifting indices. Apply the induction hypothesis on the remaining summation yields the desired result. □

Theorem 9.4. *The (q,t)-Stirling numbers are orthogonal, that is, for $m \le n$*

$$\sum_{k=m}^{n} s_{q,t}[n,k] \cdot S_{q,t}[k,m] = \delta_{m,n} \tag{9.9}$$

and

$$\sum_{k=m}^{n} S_{q,t}[n,k] \cdot s_{q,t}[k,m] = \delta_{m,n}. \tag{9.10}$$

Furthermore, this orthogonality holds bijectively.

Notice that orthogonality of the (q,t)-Stirling numbers follows immediately from Theorem 9.3 which gives the change of basis matrices between the ordered bases $(1, x, x^2, x^3, \ldots)$ and $((x)_{0,q,t}, (x)_{(1,q,t)}, x_{(2,q,t)}, x_{(3,q,t)}, \ldots)$ for the polynomial ring $\mathbb{Q}(q,t)[x]$. We now instead provide a bijective proof.

Proof. When $m = n$ since $s_{q,t}[n,n] = S_{q,t}[n,n] = 1$, both identities are trivial. Suppose now that $n > m$. The left-hand side of (9.9) is the total weight of the set

$$C = \bigcup_{k=m}^{n} \mathcal{Q}(n, n-k) \times \mathcal{A}(k,m),$$

where the weight of $(T, w) \in C$ is defined by

$$\mathrm{wt}(T, w) = (-1)^{n-k} \cdot \mathrm{wt}(T) \cdot \mathrm{wt}(w).$$

Here $\mathrm{wt}(w) = q^{A(w)} \cdot t^{B(w)}$ and $\mathrm{wt}(T) = q^{\mathrm{s}(T)} \cdot t^{\mathrm{r}(T)}$ where the statistics $A(\cdot)$, $B(\cdot)$, $\mathrm{s}(\cdot)$ and $\mathrm{r}(\cdot)$ are defined in Sections 3 and 7. We wish to show that $\mathrm{wt}(C) = \sum_{(T,w)\in C} \mathrm{wt}(T, w) = 0$ by constructing a weight-preserving sign-reversing involution φ on C with no fixed points.

For any pair $(T, w) \in \mathcal{Q}(n, n-k) \times \mathcal{A}(k,m)$, define the map φ as follows. Label the columns of $T \in \mathcal{Q}(n, n-k)$ from right to left with 1 through $n-1$. Let l_1 be the label of the rightmost column in T that has a rook. If T has no rooks, let $l_1 = \infty$. Denote by $\mathrm{rb}(T)$ the number of squares below the rightmost rook in T. If $l_1 = \infty$, let $\mathrm{rb}(T) = 0$. For $w \in \mathcal{A}(k,m)$, let r be the first repeating (odd) integer reading the entries of w from left to right, and let l_2 denote the number appearing to the left of the entry r in the *RG*-word w. If there is no repeating integer, let $l_2 = \infty$. Note that $\mathrm{rb}(T)$ must be even.

If $l_1 \le l_2$, remove the rightmost rook in T to form the rook placement T'. Insert the entry $\mathrm{rb}(T) + 1$ to the right of the entry l_1 to obtain the word w'. Since $l_1 \le l_2$,

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$\mathrm{rb}(T)+1 \leq l_1 \leq l_2$ and $\mathrm{rb}(T)+1$ is odd, so we have w' is an allowable word of length $k+1$. Hence $(t', w') \in \mathcal{Q}(n, n-k-1) \times \mathcal{A}(k+1, m)$. Also since we removed the rightmost rook in T to obtain T', we know $\mathrm{wt}(T) = q^{l_1} \cdot \mathrm{wt}(T')$ if $\mathrm{rb}(T)+1 = l_1$, that is, the rightmost rook is in the first row, or that $\mathrm{wt}(T) = q^{\mathrm{rb}(T)} \cdot t \cdot \mathrm{wt}(T')$ if $\mathrm{rb}(T)+1 < l_1$, that is, the rightmost rook is not in the first row. We also know that $\mathrm{wt}(w') = q^{l_1-1} \cdot \mathrm{wt}(w)$ if $l_1 = \mathrm{rb}(T)+1$, or $\mathrm{wt}(w') = q^{\mathrm{rb}(T)} \cdot t \cdot \mathrm{wt}(w)$ if $\mathrm{rb}(T)+1 < l_1$. Thus $\mathrm{wt}(T', w') = (-1)^{n-k-1} \cdot \mathrm{wt}(T') \cdot \mathrm{wt}(w') = -\mathrm{wt}(T, w)$.

On the other hand, if $l_1 > l_2$, delete the entry r in w to obtain w'. In column l_2 of T add a rook so that there are $r-1$ empty squares below it. Similarly, one can check that $(T', w') \in \mathcal{Q}(n, n-k+1) \times \mathcal{A}(k-1, m)$ and $\mathrm{wt}(T', w') = -\mathrm{wt}(T, w)$.

Since all pairs $(T, w) \in \mathcal{Q}(n, n-k) \times \mathcal{A}(k, m)$ are mapped under φ, there are no fixed points in C, hence (9.9) is true.

The proof of the second identity (9.10) follows in a similar fashion. The left-hand side of (9.10) is the total weight of the set

$$D = \bigcup_{k=m}^{n} \mathcal{A}(n, k) \times \mathcal{Q}(k, k-m),$$

where $\mathrm{wt}(w, T) = (-1)^{k-m} \cdot \mathrm{wt}(w) \cdot \mathrm{wt}(T)$. We show that $\mathrm{wt}(D) = \sum_{(w,T)\in D} \mathrm{wt}(w, T) = 0$ by constructing a weight-preserving sign-reversing involution ψ on D with no fixed points.

For $(w, T) \in \mathcal{A}(n, k) \times \mathcal{Q}(k, k-m)$, define the following. Let $w_i = r_1$ be the last repeated odd integer in w reading from left to right, and let l_1 be the maximum entry in w occurring before w_i. If there is no repeated entry in w, let $l_1 = 0$. Let l_2 be the label of the leftmost column in T with a rook in it and let r_2 be the number of squares above that rook. If there are no rooks in T let $l_2 = 0$. As before, we are labeling the columns right to left with 1 through $n-1$.

The bijection is built as follows. If $l_1 > l_2$, raise $w_i = r_1$ to l_1+1 and increase all of the entries to the right of w_i by 1. Denote the new word by w'. Since w_i is the last repeated odd entry, the *RG*-word w is of the form $w = \cdots l_1 \cdots r_1(l_1+1)(l_1+2)\cdots k$. Then by definition, the new word w' is of the form $w' = \cdots l_1 \cdots (l_1+1)(l_1+2)(l_1+3)\cdots(k+1)$. This still is an allowable word since the first $i-1$ entries in w' are the same as those in w and the remaining entries form an increasing sequence. So $w' \in \mathcal{A}(n, k+1)$. Also, in w the entries after w_i do not contribute to $\mathrm{wt}(w)$ since there are no repeated entries. When w_i is raised to l_1+1, the weight loss is q^{r_1-1} if $r_1 = l_1$ or $q^{r_1-1} \cdot t$ if $r_1 < l_1$. In the staircase board T, form a new rook placement T' by first adding a column of length k to the left, and then placing a rook in column l_1 counting from right to left such that there are $r_1 - 1$ squares below the rook. Clearly T' has k columns and $k+1-m$ rooks. Since the new rook was placed so that there are now an even number of squares below it, this rook is in a shaded square. Also since $l_1 > l_2$, there is no other rook in column l_1. Hence $T' \in \mathcal{Q}(k+1, k+1-m)$. Observe when we add a rook to obtain T', if the new rook is added in the first row, that is, $r_1 = l_1$ then the weight is increased by q^{r_1-1}. If the

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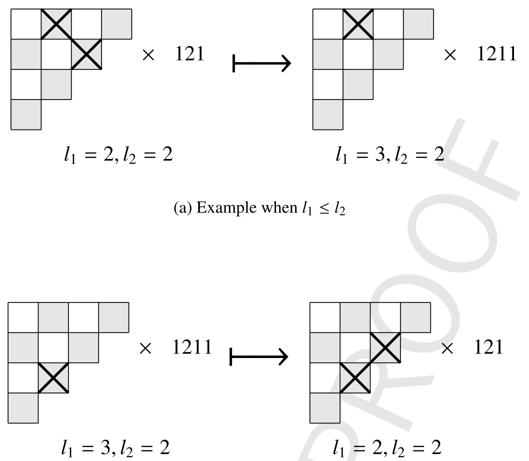

(a) Example when $l_1 \le l_2$

(b) Example when $l_1 > l_2$.

Fig. 9. Examples of the bijection proving the identity (9.9).

new rook is not in the first row, that is, $r_1 < l_1$ then the weight is increased by $q^{r_1-1} \cdot t$. Hence $\mathrm{wt}(w', T') = -\mathrm{wt}(w, T)$.

If $l_1 \le l_2$, replace the entry $w_j = l_2 + 1$ in w by $l_2 - r_2$ and subtract 1 from all of the entries to the right of w_j to obtain w'. Since $w = \cdots l_1 \cdots r_1(l_1+1) \cdots k$ and $l_1 \le l_2 \le k-1$, we have that $w_j = l_2 + 1$ appears to the right of w_i and hence such an entry is unique. Also $r_2 + 1 \le l_2$ gives $l_2 - r_2 \ge 1$. This difference is always odd by the fact that the rook is in a shaded square. So $w' = \cdots l_1 \cdots l_2(l_2 - r_2)(l_2+1) \cdots (k-1)$ is an RG-word with even integers appearing just once, hence $w' \in \mathcal{A}(n, k-1)$. The entry $w'_{j-1} = l_2$, and $w'_j = l_2 - r_2$ contributes a weight of $q^{l_2-r_2-1}$ if $l_2 = l_2 - r_2$, that is, $r_2 = 0$ or $q^{l_2-r_2-1} \cdot t$ if $r_2 > 0$. Delete the column l_2 in T and delete one square from the bottom in all columns to the left of column l_2 to make the new staircase chessboard T'. It is straightforward to check that $T' \in \mathcal{Q}(k-1, k-1-m)$. Deleting the rook in T will decrease its weight by $q^{l_2-(r_2+1)}$ if the rook is in the first row, that is, $r_2 = 0$ or by $q^{l_2-r_2-1} \cdot t$ if the rook is not in the first row, that is, $r_2 > 0$. Hence $\mathrm{wt}(w', T') = -\mathrm{wt}(w, T)$. The map we described is a weight-preserving sign-reversing involution with no fixed points, so the orthogonality in (9.10) follows. □

See Figs. 9 and 10 for examples of the bijections occurring in the proof of Theorem 9.4.

10. Concluding remarks

The Stirling numbers of the first kind and second kind are specializations of the homogeneous and elementary symmetric functions:

$$S(n,k) = h_{n-k}(x_1, \ldots, x_k), \qquad c(n,k) = e_{n-k}(x_1, \ldots, x_{n-1}), \tag{10.1}$$

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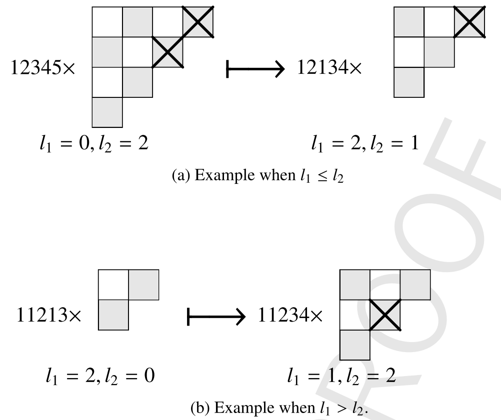

Fig. 10. Examples of the bijection proving the identity (9.10).

where $x_m = m$. The q-Stirling numbers are also specializations of these symmetric functions with $x_m = [m]_q$. See [18, Chapter I, Section 2, Example 11]. For the (q,t)-versions take $x_m = [m]_{q,t}$ as defined in (9.3). A more general statement of orthogonality is

$$\sum_{k=j}^{n} (-1)^{n-k} \cdot e_{n-k}(x_1, \ldots, x_{n-1}) \cdot h_{k-j}(x_1, \ldots, x_j) = \delta_{n,j}. \tag{10.2}$$

The specializations imply orthogonality of the (q,t)-Stirling numbers, though not combinatorially as in Theorem 9.4. It remains to find a combinatorial proof of Theorem 9.3.

Stembridge's $q = -1$ phenomenon [28,29], and the more general cyclic sieving phenomenon of Reiner, Stanton and White [24] count symmetry classes in combinatorial objects by evaluating their q-generating series at a primitive root of unity. Is there a cyclic sieving phenomenon for the q-Stirling numbers of the first and second kinds?

Are there other classical q-analogues which can be viewed naturally as q-$(1+q)$-analogues as in Goals 1 and 2? Ehrenborg and Readdy [6] have recently discovered a *symmetric* q-$(1+q)$-analogue of the q-binomial which is more compact than the Fu et al. construction.

Garsia and Remmel [10] have a more general notion of the q-Stirling number of the second kind as enumerating non-attacking rooks on a general Ferrers' board. This will be the subject of another paper.

It would be interesting to look deeper into the poset structure of the Stirling posets of the first and second kinds, such as the interval structure and the f- and h-vectors of each poset. Park has a notion of the Stirling poset which arises from the theory of P-partitions [23]. It has no connection with the Stirling posets in this paper.

The q-binomial has the combinatorial interpretation of counting certain subspaces over a finite field with q elements as well as the corresponding subspace lattice. Milne [20] has an interpretation of the q-Stirling number of the second kind as sequences of lines in a vector space over the finite field with q elements. Is there an analogous interpretation

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for the (q,t)-Stirling numbers of the second kind? Bennett, Dempsey and Sagan [1] construct families of posets which include Milne's construction. One would like a similar construction for the q-Stirling numbers of the first kind.

In [32] Wachs and White have discovered many other statistics on RG-words which generate the q-Stirling numbers. In particular, their ls and lb statistics are defined by $ls(w) = \prod_{i=1}^{n} q^{w_i - 1}$ and $lb(w) = \prod_{i=1}^{n} lb_i(w)$ where $lb_i(w) = q^{m_{i-1} - w_i}$ if $m_{i-1} \geq w_i$ and $lb_i(w) = 1$ if $m_{i-1} < w_i$. The ls statistic and the wt statistic in (2.2) are related by $ls(w) = q^{\binom{k}{2}} \cdot \mathrm{wt}(w)$. The authors are currently looking at these statistics, as well as White's interpolations [33] between these statistics, in view of the first Goal 1, as well as poset theoretic and homological consequences of Goal 2. The first author has considered the q-binomial via the major index in terms of this research program [2].

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