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Refined face count in uniform triangulations of the Legendre polytope

Richard Ehrenborg*[1], Gábor Hetyei†[2], and Margaret Readdy‡[1]

[1]*Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027*
[2]*Department of Mathematics and Statistics, UNC-Charlotte, Charlotte NC 28223-0001*

Abstract. The Legendre polytope is the convex hull of all pairwise differences of the basis vectors, also known as the full root polytope of type A. We describe all flag triangulations of this polytope that are uniform in the sense that the edges may be described as a function of the relative order of the indices of the four basis vectors involved. We also determine the refined face counts of these triangulations that keeps track of the number of forward and backward arrows in each face.

Résumé. Le polytope de Legendre, aussi connu sous le nom polytope de racine pleine de type A, est l'enveloppe convexe des différences par paire des vecteurs de base. Nous décrivons toutes les triangulations drapeau de ce polytope qui sont uniformes dans le sens que les arêtes ne dependent que de l'ordre relative des quatres vecteurs de base impliqués. Nous déterminons aussi le compte raffiné de faces qui fait suivi le nombre des arêtes vers l'avant et ceux en arrière dans chaque face.

1 Introduction

Triangulations of root polytopes and of products of simplices have been a subject of intense study in recent years [1, 2, 3, 4, 8]. Motivated by an observation made in [6], the authors recently [7] established that the Simion type B associahedron [13] may be realized as a pulling triangulation of the Legendre polytope, defined as the convex hull of all differences of pairs of basis vectors in Euclidean space. These vertices can be thought of as *arrows* between numbered nodes. They also showed that all pulling triangulations are flag. The Legendre polytope is the centrally symmetric variant of the type A root polytope whose triangulations were studied by Gelfand, Graev and Postnikov [8].

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†ghetyei@uncc.edu Gábor Hetyei was partially supported by Grant #514648 of the Simons Foundation.

‡margaret.readdy@uky.edu. Margaret Readdy was partially supported by Grant #422467 of the Simons Foundation.

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A question naturally arises: Are there other reasonably uniform triangulations of the Legendre polytope?

In this work we fully answer this question. We consider flag triangulations that are uniform in the sense that the flag condition depends only on the relative order on the numbering of the basis vectors involved, and classify all such triangulations. The key tool we use is a characterization of triangulations of a product of simplices by Oh and Yoo [11] in terms of *matching ensembles*. We determine that there are three classes of triangulations: variants of the lexicographic pulling triangulation, variants of the revlex pulling triangulation, and variants of the triangulation representing the Simion type B associahedron. All triangulations of the boundary of the Legendre polytope have the same face numbers.

To distinguish between the three major classes, we introduce a refined face count which keeps track of the number of forward and backward arrows in each face. Remarkably we find the refined face count in a triangulation belonging to the lexicographic class yields the same face numbers for a fixed dimension, regardless how we fix the number of forward and backward arrows. The variants of the Simion type B associahedron all have the same refined face numbers, up to exchanging the forward and backward arrows. Finally, the refined face count for the revlex triangulation and its variants leads to considering an exponential generating function whose second partial derivative may be expressed in terms of modified Bessel functions. Weighted generalizations of the Delannoy numbers play a crucial role in the refined face count in the revlex and Simion class, whereas the Catalan numbers play a key role in the refined face count in the lex and the Simion class.

2 Preliminaries

The *Legendre polytope* P_n [9] is the convex hull of the $n(n+1)$ vertices $e_j - e_i$ where $i \neq j$ and $\{e_1, e_2, \ldots, e_{n+1}\}$ is the orthonormal basis of the Euclidean space $\mathbb{R}^{n+1}$. It was first studied by Cho [5], and it is called the "full" type A root polytope in the work of Ardila, Beck, Hoşten, Pfeifle and Seashore [1]. It contains the root polytope P_n^+, defined as the convex hull of the origin and the set of points $e_j - e_i$, where $i < j$, first studied by Gelfand, Graev and Postnikov [8] and later by Postnikov [12].

We use the shorthand notation (i, j) for the vertex $e_j - e_i$ We may think of these vertices as the set of all directed nonloop edges on the vertex set $\{1, 2, \ldots, n+1\}$. To avoid confusion between edges and vertices of the Legendre polytope, we will refer to the vertices of P_n as *arrows*, and we will use the notation $V_n = \{(i, j) \ : \ 1 \leq i, j \leq n+1, i \neq j\}$.

A subset of arrows is contained in some face of P_n exactly when there is no $i \in \{1, 2, \ldots, n+1\}$ that is both the head and the tail of an arrow; see [9, Lemmas 4.2 and 4.4].

Type	Order of nodes	Type B associahedron	Lexicographic pulling	Revlex pulling
$THTH$	$i_1 < j_1 < i_2 < j_2$	$i_1 \quad j_1 \quad i_2 \quad j_2$	$i_1 \quad j_1 \quad i_2 \quad j_2$	$i_1 \quad j_1 \quad i_2 \quad j_2$
$HTHT$	$j_1 < i_1 < j_2 < i_2$	$j_1 \quad i_1 \quad j_2 \quad i_2$	$j_1 \quad i_1 \quad j_2 \quad i_2$	$j_1 \quad i_1 \quad j_2 \quad i_2$
$THHT$	$i_1 < j_1 < j_2 < i_2$	$i_1 \quad j_1 \quad j_2 \quad i_2$	$i_1 \quad j_1 \quad j_2 \quad i_2$	$i_1 \quad j_1 \quad j_2 \quad i_2$
$HTTH$	$j_1 < i_1 < i_2 < j_2$	$j_1 \quad i_1 \quad i_2 \quad j_2$	$j_1 \quad i_1 \quad i_2 \quad j_2$	$j_1 \quad i_1 \quad i_2 \quad j_2$
$TTHH$	$i_1 < i_2 < j_1 < j_2$	$i_1 \quad i_2 \quad j_1 \quad j_2$	$i_1 \quad i_2 \quad j_1 \quad j_2$	$i_1 \quad i_2 \quad j_1 \quad j_2$
$HHTT$	$j_1 < j_2 < i_1 < i_2$	$j_1 \quad j_2 \quad i_1 \quad i_2$	$j_1 \quad j_2 \quad i_1 \quad i_2$	$j_1 \quad j_2 \quad i_1 \quad i_2$

Table 1: Pairs of arrows that are edges in three triangulations of the boundary ∂P_n of the Legendre polytope.

Equivalently, the faces are products of two simplices [7, Lemma 2.2]: we may write them as $\Delta_I \times \Delta_J$ where $I, J \neq \emptyset$, $I \cap J = \emptyset$ and Δ_K denotes the convex hull of the set $\{e_i : i \in K\}$ for $K \subseteq \{1, 2, \ldots, n+1\}$. Affine independent subsets of vertices of faces of the Legendre polytope are described as follows. A set $S = \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ is a $(k-1)$-dimensional simplex if and only if, disregarding the orientation of the directed edges, the set S contains no cycle, that is, it is a forest [9, Lemma 2.4]. The analogous observations were made for the root polytope P_n^+ in [8] and for products of simplices in [10, Lemma 6.2.8] (see also [2, Lemma 2.1]).

In a recent paper [7], the authors have shown that the Simion type B associahedron [13] is combinatorially equivalent to a pulling triangulation of the boundary of the Legendre polytope. For an exact definition of a pulling triangulation we refer the reader to [7]. Here we only recall the following key observation [7, Theorem 3.1]: every pulling triangulation of the boundary of the Legendre polytope P_n is *flag*, that is, a subset of vertices is a face exactly when each pair of vertices in the subset is an edge. Thus the triangulation giving rise to a combinatorial equivalent of the Simion type B associahedron is completely determined by the rules given in the associated column of Table 1.

The last two columns in Table 1 are the analogous rules for two other pulling triangulations of the boundary of the Legendre polytope P_n, also discussed in [7]. These are the lexicographic (lex) and revlex pulling orders, introduced in [9]. Their restriction to the root polytope P_n^+ are called the antistandard, respectively standard triangulations in [8]. The reader may take Table 1 as a definition of these flag complexes.

3 Classifying uniform flag triangulations of the Legendre polytope

A common property of all three flag complexes described in Table 1 is that the edges are defined in a *uniform fashion*:

Definition 3.1. *A flag simplicial complex $\triangle_n$ on the vertex set V_n is a* uniform flag complex *if determining whether or not a pair of vertices $\{(i_1, j_1), (i_2, j_2)\}$ forms an edge depends only on the equalities and inequalities between the values of i_1, i_2, j_1 and j_2.*

To facilitate making statements, we introduce some new terminology and notation. We use the letter T to mark the tail of each arrow and the letter H to mark the head. For each pair of arrows on four nodes, we will indicate the relative order of the two heads and two tails by writing down the appropriate letters left to right in the order as they occur. We will refer to the resulting word as the *type* of the pair of arrows. After that we will simply state in words the condition that a pair of arrows of a given type must satisfy to be an edge of the triangulation. Three examples of this convention are given in Table 1. Our main classification result is the following.

Theorem 3.2. *Let $\triangle_n$ be a uniform flag complex on the vertex set V_n for some $n \geq 5$ that satisfies the necessary conditions stated in Proposition 3.3 below. Then the complex $\triangle_n$ represents a triangulation of the boundary ∂P_n of the Legendre polytope if and only if it satisfies exactly one of the following conditions:*

1. *Both $THTH$ and $HTHT$ types of pairs of arrows do not nest, and both $HTTH$ and $THHT$ types of arrows do not cross. Uniform flag triangulations induced by the lexicographic pulling order belong to this class, hence we call it* the lex class.

2. *Both $THTH$ and $HTHT$ types of pairs of arrows nest, and both $HTTH$ and $THHT$ types of arrows cross. Uniform flag triangulations induced by the revlex pulling order belong to this class, hence we call it* the revlex class*.*

3. *Exactly one of the $THTH$ and $HTHT$ types of pairs of arrows nest. Furthermore, if both $THHT$ and $HTTH$ types of pairs cross then both $TTHH$ and $HHTT$ types of pairs nest. The Simion type B associahedron belongs to this class, and hence we call it* the Simion class*.*

The necessity part of Theorem 3.2 depends on a few technical propositions and the following geometric observation.

Proposition 3.3. *Let $\triangle_n$ be a uniform flag complex on the vertex set V_n. Identify each vertex $(i,j) \in V_n$ with the vertex $e_j - e_i$ of the Legendre polytope P_n. If the complex $\triangle_n$ represents a triangulation of the boundary ∂P_n of the Legendre polytope then it satisfies the following criteria:*

1. *There is no edge of the form $\{(i,j),(j,k)\}$ in the complex $\triangle_n$.*
2. *For each three-element subset $\{i,j,k\}$ of $\{1,2,\ldots,n+1\}$, the two sets $\{(i,j),(i,k)\}$ and $\{(j,i),(k,i)\}$ are edges in the complex $\triangle_n$.*
3. *For each four-element subset $\{i_1,i_2,j_1,j_2\}$ of $\{1,2,\ldots,n+1\}$, exactly one of the two sets $\{(i_1,j_1),(i_2,j_2)\}$ and $\{(i_1,j_2),(i_2,j_1)\}$ is an edge in the complex $\triangle_n$.*

The key tool in verifying sufficiency is the following result.

Theorem 3.4. *Let $\triangle_n$ be a uniform flag complex on the vertex set V_n satisfying the conditions of Proposition 3.3. Let $\mathcal{M}$ be the family of all faces that are matchings, that is, set*

$$\mathcal{M} = \{\{(i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k)\} \in \triangle_n : |\{i_1,j_1,i_2,j_2,\ldots,i_k,j_k\}| = 2k\}.$$

Identify each vertex (i,j) with the vertex $e_j - e_i$ of the Legendre polytope. Then the complex $\triangle_n$ represents a triangulation of the boundary ∂P_n of the Legendre polytope if and only if the family of matchings $\mathcal{M}$ satisfies the following two properties:

(SA) For each $I, J \subset \{1,2,\ldots,n+1\}$ satisfying $I \cap J = \emptyset$ and $|I| = |J|$ there is a unique $\sigma \in \mathcal{M}$ such that $\sigma \subseteq I \times J$ and $|\sigma| = |I|$;

(LA) Assume $I, J \subset \{1,2,\ldots,n+1\}$ satisfies $I \cap J = \emptyset$ and let σ be a non-empty matching in $\mathcal{M}$ such that $\sigma \subseteq I \times J$. Then for each $k \notin I \cup J$ there is an edge $(i,j) \in \sigma$ such that $(\sigma - \{(i,j)\}) \cup \{(k,j)\} \in \mathcal{M}$. Also, for each $k \notin I \cup J$ there is an edge $(i,j) \in \sigma$ such that $(\sigma - \{(i,j)\}) \cup \{(i,k)\} \in \mathcal{M}$.

The proof of Theorem 3.4 relies on a nontrivial revamping of the characterization of triangulations of a product of two simplices given by S. Oh and H. Yoo [11]. See also [2] and [3, Lemma 2.5]. In the proof at some point we use a slight generalization of [12, Lemma 12.6]. The labels (SA) and (LA) refer to the terms *support axiom* and *linkage axiom* used by S. Oh and H. Yoo. They also introduce a *closure axiom* which is trivially verified in our setting.

4 Tools for refined face enumeration

In this section we list a few tools we repeatedly used to prove our enumerative results. Uniform triangulations of ∂P_n are defined by a set of six rules, independent of the dimension. For a fixed set of rules, we will simultaneously consider each triangulation

$\triangle_n$ determined on the vertex set V_n defined by these rules, for each $n \geq 0$. Note that $V_0 = \emptyset$. Let $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots)$ be a sequence of collections of arrows, where $\mathcal{F}_n$ is a family of subsets of V_n. We define the associated generating function

$$F(\mathcal{F}, x, y, t) = \sum_{n,i,j \geq 0} f(\mathcal{F}_n, i, j) \cdot x^i y^j t^n$$

where $f(\mathcal{F}_n, i, j)$ is the number of sets in $\mathcal{F}_n$ consisting of i forward arrows and j backward arrows. Our interest is to compute this generating function when $\mathcal{F}$ is the collection $(\triangle_0, \triangle_1, \ldots)$. The number of cases to be considered can be reduced by extending the notion of the *dual* and *reflected dual* triangulations to all families:

Lemma 4.1. *Let $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots)$ be a sequence of collections of arrows, where $\mathcal{F}_n \subseteq 2^{V_n}$. Let $\mathcal{F}^* = (\mathcal{F}_0^*, \mathcal{F}_1^*, \ldots)$ be the* dual family*, obtained by reversing all arrows in all sets, and let $\overline{\mathcal{F}} = (\overline{\mathcal{F}_0}, \overline{\mathcal{F}_1}, \ldots)$ be the* reflected dual family*, obtained by reversing each arrow, and replacing each node i in V_n with $n+2-i$. Then we have*

$$F(\mathcal{F}^*, x, y, t) = F(\mathcal{F}, y, x, t) \quad \textit{and} \quad F(\overline{\mathcal{F}}, x, y, t) = F(\mathcal{F}, x, y, t).$$

The families of uniform flag triangulations defined by a set of rules are *coherent* in the sense that they are closed under the insertion and removal of isolated nodes. To make this informal observation precise, consider the map $\pi_k : (\mathbb{N} - \{k\}) \to \mathbb{N}$ given by $\pi_k(m) = m$ for $m < k$ and $\pi_k(m) = m - 1$ for $m > k$.

Definition 4.2. *Let $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots)$ be a collection of families of sets such that for each n the family $\mathcal{F}_n$ consists of subsets of V_n. We call such a collection* coherent *if for each subset σ of V_n and each $k \in \{1, 2, \ldots, n+1\}$ that is not incident to any arrow in σ, the set $\pi_k(\sigma) = \{(\pi_k(i), \pi_k(j)) : (i,j) \in \sigma\}$ belongs to $\mathcal{F}_{n-1}$ if and only if σ belongs to $\mathcal{F}_n$. In particular, for a coherent collection $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots)$, the empty set either belongs to all $\mathcal{F}_n$ or it belongs to none of them.*

By abuse of terminology we will say that $\mathcal{F}$ *contains the empty set* if all families $\mathcal{F}_n$ in it contain it. Sets of arrows in coherent collections may be enumerated by counting only the *saturated* sets in the families, which we now define.

Definition 4.3. *For $n \geq 1$, a subset σ of V_n is* saturated *if the set of endpoints of its arrows is the set $\{1, 2, \ldots, n+1\}$. We also consider the empty set to be a saturated subset of $V_0 = \emptyset$. Given a coherent collection $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots)$, we denote the family of saturated sets in $\mathcal{F}_n$ by $\widehat{\mathcal{F}}_n$.*

Note that $\widehat{\mathcal{F}_0} = \{\emptyset\}$ exactly when $\mathcal{F}$ contains the empty set. One of our key counting tools is the following observation.

Lemma 4.4. *Given a coherent collection of families $\mathcal{F}$ of arrows, the face generating function satisfies*

$$F(\mathcal{F}, x, y, t) = -\frac{t}{(1-t)^2} \cdot \delta_{\mathcal{F}_0, \{\emptyset\}} + \frac{1}{(1-t)^2} \cdot F\left(\widehat{\mathcal{F}}, x, y, \frac{t}{1-t}\right), \tag{4.1}$$

$$F(\widehat{\mathcal{F}}, x, y, z) = \frac{z}{1+z} \cdot \delta_{\widehat{\mathcal{F}}_0, \{\emptyset\}} + \frac{1}{(1+z)^2} \cdot F\left(\mathcal{F}, x, y, \frac{z}{1+z}\right), \tag{4.2}$$

where δ denotes the Kronecker delta function.

Using Lemma 4.4 it is always possible, albeit sometimes tedious, to compute the generating function of all faces. An interesting special case is counting all facets with a given number of forward and backward arrows. The following statement is straightforward.

Lemma 4.5. *Let $\triangle_n$ be a uniform flag triangulation of the boundary ∂P_n of the Legendre polytope. Then a face $\sigma \in \triangle_n$ is a facet if and only if it is saturated and contains no isolated nodes, that is, every $i \in \{1, 2, \ldots, n+1\}$ is incident to some element of σ.*

In other words, as a subset of V_n, a facet is the set of edges of a forest with no isolated nodes. Such a forest has $n+1$ nodes and n arrows. If the number of forward arrows is i then the number of backward arrows is $n-i$ in such a forest, contributing a term $x^i y^{n-i} t^n$ to the generating function of all faces and the term $x^i y^{n-i} z^n$ to generating function of all saturated faces.

Corollary 4.6. *Let $\mathcal{F} = (\triangle_0, \triangle_1, \ldots)$ be a coherent family of uniform flag triangulations. Then the facet generating function $\sum_{n\ge 0}\sum_{i=0}^{n} f(\triangle_n, i, n-i) x^i y^{n-i} z^n$ may be obtained by substituting x/w into x, y/w into y and wz into t in $F(\mathcal{F}, x, y, t)$ and then evaluating the resulting expression at $w = 0$. Alternatively it may also be obtained by substituting x/w into x, y/w into y and wz into z in $F(\widehat{\mathcal{F}}, x, y, z)$ and then evaluating the resulting expression at $w = 0$.*

For any uniform flag triangulation of the boundary ∂P_n of the Legendre polytope, the vertex sets consisting only of forward (backward) arrows form subcomplexes whose face numbers are easier to count. We first review the rephrasing of a known result.

A useful way to express our results is in terms of the generating function for the Catalan numbers:

$$C(u) = \sum_{n\ge 0} C_n \cdot u^n = \frac{1-\sqrt{1-4u}}{2u}.$$

Proposition 4.7. *Let $\mathcal{F}$ be a family of uniform flag triangulations defined by a set of rules that contains the rule that $HTHT$ types of pairs of arrows do not nest. Then the following two identities hold:*

$$F(\mathcal{F}, 0, y, t) = \frac{1-t-\sqrt{1-(4y+2)t+t^2}}{2yt}, \tag{4.3}$$

$$F(\widehat{\mathcal{F}}, 0, y, z) = \frac{1}{1+z} \cdot \left(C(yz(z+1)) + z\right). \tag{4.4}$$

Corollary 4.8. *Let $\mathcal{F}$ be a family of uniform flag triangulations such that $HTHT$ types of pairs of arrows do not nest. Then the sum over all forests F consisting of $k \geq 1$ backward arrows, no forward arrows and no isolated nodes is*

$$G_k(z) = \sum_F z^{\#\text{nodes of } F} = C_k \cdot z^{k+1} \cdot (z+1)^{k-1}. \tag{4.5}$$

Similarly, if the uniform flag triangulations $\mathcal{F}$ satisfies the requirement that $THTH$ types of pairs of arrows do not nest, then the sum over all forests F consisting of $k \geq 1$ forward arrows, no backward arrows and no isolated nodes also yields the identity (4.5).

We will also use the following refined variant of Proposition 4.7.

Proposition 4.9. *Let $\mathcal{F}$ be a family of uniform flag triangulations defined by a set of rules that contains the rule that $HTHT$ type of pairs of arrows do not nest. For each $n \geq 0$, let $\mathcal{F}_n^{(i)}$ denote the set of all faces where the sequence of heads and tails, listed in increasing order, satisfies the condition that the i smallest nodes are heads and the next node is a tail. Then the two resulting collections $\mathcal{F}^{(i)} = (\mathcal{F}_0^{(i)}, \mathcal{F}_1^{(i)}, \ldots)$ and $\widehat{\mathcal{F}^{(i)}} = (\widehat{\mathcal{F}_0^{(i)}}, \widehat{\mathcal{F}_1^{(i)}}, \ldots)$ satisfy*

$$F(\mathcal{F}^{(i)}, 0, y, t) = \frac{y^i t^i F(\mathcal{F}, 0, y, t)^i}{(1-t)^{i+1}}, \tag{4.6}$$

$$F(\widehat{\mathcal{F}^{(i)}}, 0, y, z) = \frac{1}{1+z} \cdot \left(yz(1+z)C(yz(z+1))\right)^i. \tag{4.7}$$

When $HTHT$ pairs of arrows do not nest, we obtain a very different expression enumerating the faces containing backward arrows only. We employ its *dual form*, obtained after reversing all arrows, using a generalization of the Delannoy numbers. Recall a *Delannoy path* from $(0,0)$ to (a,b) is a lattice path consisting of North steps $(0,1)$, East steps $(1,0)$ and NE steps $(1,1)$. The number of Delannoy paths from $(0,0)$ to (a,b) is the Delannoy number $D_{a,b}$.

Definition 4.10. *Given two non-negative integers a and b, the* Delannoy polynomial $D_{a,b}(x)$ *is the total weight of all Delannoy paths from $(0,0)$ to (a,b), where each step contributes a factor of x. Thus the coefficient of x^j in $D_{a,b}(x)$ is the number of Delannoy paths from $(0,0)$ to (a,b) having j steps.*

Proposition 4.11. *Let $\mathcal{F}$ be a family of uniform flag triangulations defined by a set of rules requiring that $THTH$ type of pairs of arrows nest. Then the collection $\widehat{\mathcal{F}}$ of families of saturated faces satisfies*

$$F(\widehat{\mathcal{F}}, x, 0, z) = 1 + xz \cdot \sum_{a,b \geq 0} D_{a,b}(x) \cdot z^{a+b}. \tag{4.8}$$

In particular, the contribution to $F(\widehat{\mathcal{F}}, x, 0, z)$ of all saturated faces having $a+1$ tails and $b+1$ heads is $D_{a,b}(x) \cdot xz^{a+b+1}$.

In our formulas we will often use the following bivariate generating function of the Delannoy polynomials. We set

$$D(u,v,x) = \sum_{a,b\geq 0} u^a v^b \cdot D_{a,b}(x) = \frac{1}{1-x(u+v+uv)}. \tag{4.9}$$

5 Face enumeration for uniform flag triangulations

The following theorem covers half of the uniform flag triangulations in the Simion class. The analogous result for the remaining triangulations may be easily obtained using Lemma 4.1.

Theorem 5.1. *Let $\mathcal{F}$ be a collection of uniform flag triangulations belonging to the Simion class and let $\widehat{\mathcal{F}}$ be the collection of families of saturated faces. If $THTH$ types of pairs of arrows do not nest and $HTHT$ types of pairs of arrows nest then the following identity holds:*

$$F(\widehat{\mathcal{F}}, x, y, z) = \frac{C(yz(z+1)) + z}{1+z} + \frac{xz \cdot (1 + zC(yz(z+1))) \cdot C(yz(z+1))^2}{(1+z)\cdot(1-2C(yz(z+1))xz - C(yz(z+1))^2 xz^2)}.$$

By combining Theorem 5.1 and Lemma 4.4 it is possible to give the generating function of all faces. Extracting the coefficients from the facet generating function yields the following result.

Theorem 5.2. *Let $\triangle_n$ be a uniform flag triangulation of the boundary ∂P_n of the Legendre polytope belonging to the Simion class satisfying the property that $THTH$ type of pairs of arrows nest and $HTHT$ type of arrows do not nest. Then for $i \geq 1$, the number of facets of $\triangle_n$ consisting of i forward arrows and $n-i$ backward arrows is given by*

$$f(\triangle_n, i, n-i) = 2^{i-1} \cdot \frac{(i+1)\cdot(2n-i)!}{(n-i)!\cdot(n+1)!}. \tag{5.1}$$

The number of facets of the triangulation $\triangle_n$ consisting of no forward arrows and n backward arrows is given by the Catalan number C_n.

Next we look at the revlex class.

Theorem 5.3. *Let $\mathcal{F}$ be a family of uniform flag triangulations in the revlex class. Then the collection $\widehat{\mathcal{F}}$ of families of saturated faces satisfies*

$$\begin{aligned} F(\widehat{\mathcal{F}}, x, y, z) = 1 &+ \sum_{a,b\geq 0} (x\cdot D_{a,b}(x) + y \cdot D_{a,b}(y)) \cdot z^{a+b+1} \\ &+ xy \cdot \sum_{a',b',a'',b''\geq 0} D_{a',b'}(x)\cdot D_{a'',b''}(y)\cdot z^{a'+b'+a''+b''+2}\cdot C(a',b',a'',b'',z) \end{aligned} \tag{5.2}$$

where

$$C(a',b',a'',b'',z) = \binom{a'+b''+2}{a'+1} \cdot \binom{a''+b'+2}{b'+1} \cdot z + \binom{a'+b''+1}{b''} \cdot \binom{a''+b'+1}{b'} + \binom{a'+b''+1}{a'} \cdot \binom{a''+b'+1}{a''}. \tag{5.3}$$

We obtain a more compact expression using the proof of Theorem 5.3 by introducing the following generating function.

Definition 5.4. *Let $\widehat{\mathcal{F}} = (\widehat{\mathcal{F}}_0, \widehat{\mathcal{F}}_1, \ldots)$ be a collection of families of arrows such that for each n the family $\widehat{\mathcal{F}_n}$ consists of saturated subsets of V_n. We define the* node-enriched exponential generating function *of $\widehat{\mathcal{F}}$ as follows:*

1. *The empty set (if it belongs to $\widehat{\mathcal{F}}_0$) contributes a factor of* 1.
2. *Each nonempty $\sigma \in \widehat{\mathcal{F}}_n$ contributes a term*

$$x^i y^j \cdot \frac{u^{a+1} \cdot v^{b+1}}{(a+1)! \cdot (b+1)!} \cdot t^n,$$

where i is the number of forward arrows, j is the number of backward arrows, $a+1$ is the number of nodes that are left ends of arrows and $b+1$ is the number of nodes that are right ends of arrows.

The node-enriched exponential generating function of the saturated faces in a triangulation in the revlex class has a compact expression in terms of the following exponential generating function of the Delannoy polynomials:

$$\widetilde{D}(u,v,x) = \sum_{a,b \geq 0} \frac{D_{a,b}(x) \cdot u^{a+1} \cdot v^{b+1}}{(a+1)! \cdot (b+1)!}. \tag{5.4}$$

Theorem 5.5. *Let $\mathcal{F}$ be a family of uniform flag triangulations in the revlex class. Then the node-enriched exponential generating function of the collection $\widehat{\mathcal{F}}$ of families of saturated faces is given by*

$$1 + \frac{1}{z} \cdot \widetilde{D}(uz,vz,x) + \frac{1}{z} \cdot \widetilde{D}(vz,uz,y) + \frac{1}{z} \cdot \widetilde{D}(uz,vz,x) \cdot \widetilde{D}(vz,uz,y) + \frac{1}{z^2} \cdot \frac{\partial}{\partial u}\widetilde{D}(uz,vz,x) \cdot \frac{\partial}{\partial v}\widetilde{D}(vz,uz,y) + \frac{1}{z^2} \cdot \frac{\partial}{\partial v}\widetilde{D}(uz,vz,x) \cdot \frac{\partial}{\partial u}\widetilde{D}(vz,uz,y).$$

Theorem 5.5 motivates computing $\widetilde{D}(u,v,x)$ explicitly.

Theorem 5.6. *The exponential generating function* $\widetilde{D}(u, v, x)$ *is given by*

$$\widetilde{D}(u, v, x) = \sum_{k \geq 0} \frac{(uv \cdot (x^2 + x))^k}{k!^2} \cdot \psi_{k+1}(ux) \cdot \psi_{k+1}(vx)$$

where $\psi_{k+1}(z) = \frac{d^k}{dz^k}\left(\frac{e^z - 1}{z}\right)$.

Remark 5.7. It is a direct consequence of Theorem 5.6 that

$$\frac{\partial}{\partial u}\frac{\partial}{\partial v}\widetilde{D}(u, v, x) = \exp(x \cdot (u + v)) \cdot I_0\left(2\sqrt{(x^2 + x) \cdot uv}\right)$$

where $I_0(z)$ is the modified Bessel function of the first kind.

Using Corollary 4.6 we may count the facets.

Theorem 5.8. *Let* $\triangle_n$ *be a uniform flag triangulation of the boundary* ∂P_n *of the Legendre polytope that belongs to the revlex class. For* $1 \leq k \leq n-1$*, the number of facets consisting of* k *forward arrows and* $n-k$ *backward arrows, that is,* $f(\triangle_n, k, n-k)$*, is given by*

$$\sum_{i=1}^{k}\sum_{j=1}^{n-k}\binom{k-1}{i-1}\binom{n-k-1}{j-1}\left[\binom{n-k+i-j}{i}\binom{k-i+j}{j} + \binom{n-k+i-j}{i-1}\binom{k-i+j}{j-1}\right].$$

The number of facets with n *forward arrows and no backward arrows; and the number of facets with no forward arrows and* n *backward arrows are both equal to* 2^{n-1}.

Finally we turn our attention to the lex class.

Theorem 5.9. *Let* $\triangle_n$ *be a uniform flag triangulation of the boundary* ∂P_n *of the Legendre polytope in the lex class. Then the number of* $(k-1)$*-dimensional faces in the triangulation* $\triangle_n$ *consisting of* i *forward arrows and* $k-i$ *backward arrows is given by*

$$f(\triangle_n, i, k-i) = \frac{f_{k-1}(\triangle_n)}{k+1} = \frac{1}{k+1} \cdot \binom{n+k}{k} \cdot \binom{n}{k}.$$

Furthermore, this quantity is independent of the parameter i.

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