Homework 7 - Diagonalization and Orthogonal Projection Practice

Make sure to justify your solution for each problem.

1. Let $A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$. Diagonalize A, then find a formula for A^k.

2. Let $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$. Diagonalize A, then find a formula for A^k.

3. Construct a nondiagonal 2×2 matrix that is diagonalizable but not invertible.

4. For an $n \times n$ matrix M, we define

$$e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!}.$$

   Find a formula for e^A where A is the 2×2 matrix from problem 1.

5. Let $\mathbf{y} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of two orthogonal vectors, one in Span$(\mathbf{u})$ and the other orthogonal to $\mathbf{u}$.

6. Let $\mathbf{y} = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u_1} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$, and $\mathbf{u_2} = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$.

   (a) Verify that $\{\mathbf{u_1}, \mathbf{u_2}\}$ is an orthogonal set.

   (b) Find the orthogonal projection of $\mathbf{y}$ onto Span$\{\mathbf{u_1}, \mathbf{u_2}\}$.

   (c) Find the distance from $\mathbf{y}$ to the plane in $\mathbb{R}^3$ spanned by $\mathbf{u_1}$ and $\mathbf{u_2}$.

7. (Gram-Schmidt process) Let W be a subspace of $\mathbb{R}^4$ with basis $\{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}\}$, where

$$\mathbf{x_1} = \begin{bmatrix} 3 \\ 6 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x_2} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{x_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

   Suppose you wish to construct an orthogonal basis for W. Show that $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is an orthogonal basis for W with

$$\mathbf{u_1} = \mathbf{x_1}, \qquad \mathbf{u_2} = \mathbf{x_2} - \frac{\mathbf{x_2} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}}\mathbf{u_1}, \qquad \mathbf{u_3} = \mathbf{x_3} - \frac{\mathbf{x_3} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}}\mathbf{u_1} - \frac{\mathbf{x_3} \cdot \mathbf{u_2}}{\mathbf{u_2} \cdot \mathbf{u_2}}\mathbf{u_2}.$$