Proof Strategies
August 5th

Proofs

Usually, when you are working on a proof, you will be given certain information and have a specific goal in mind. Based on the information given and the goal, you can choose the proof strategy that is most appropriate for the situation. Here is a summary of the various forms that givens and goals might take. Use capitol letters to represent a statement.

- Not A ($\neg A$)
  - Use proof by contradiction. Assume A is true and try to reach a contradiction in the given setting.
- A implies B ($A \implies B$)
  - Assume A is true and prove B true
  - Assume B is false and prove A is false
- A and B ($A \wedge B$)
  - Prove A and B true separately to make conjunction true.
- A or B ($A \vee B$)
  - Prove either A or B or both true to make disjunction true.
- A if and only if B ($A \iff B$)
  - Prove implication in both directions $A \implies B$ and $B \implies A$.

Useful Notation: In mathematics, it is useful to create notation to represent statements. Here is a list of some commonly used notation.

- ϵ - element of
- $\forall$ - for all
- $\exists$ - there exists

Examples:

1. **Direct Proof**: If a and b are two natural numbers, we say that a *divides* b if there is another natural number k such that $b = ak$. Using this definition, prove the following theorem.

Theorem: If a divides b and b divides c, then a divides c.

Proof: By our assumptions, and the given definition, there are natural numbers k_1 and k_2 such that

$$b = ak_1 \quad \text{and} \quad c = bk_2$$

Replace b with ak_1 in the second equation to get

$$c = bk_2 = ak_1k_2$$

Let $k = k_1k_2$. Now k is a natural number such that $c = ak$, so by the definition of divisibility, a divides c.

2. **Contradiction**: A real number, n, is rational if there exists relatively prime natural numbers p and q such that $n = \frac{p}{q}$. The real number is irrational otherwise. Using this definition, prove the following theorem.

Theorem: $\sqrt{2}$ is irrational.

Proof: Assume $\sqrt{2}$ is not irrational (i.e rational). If $\sqrt{2}$ is rational, then there exist relatively prime natural numbers p, q such that

$$\sqrt{2} = \frac{p}{q}$$

$$\implies \sqrt{2}q = p$$
$$\implies 2q^2 = p^2$$

The last equation implies that p^2 is even which means p is even. This means there exists a natural number r such that $p = 2r$. Substituting this into the last equation gives

$$\implies 2q^2 = (2r)^2$$
$$\implies 2q^2 = 4r^2$$
$$\implies q^2 = 2r^2$$

This means q^2 is even and therefore q is even. This is a contradiction to p and q being relatively prime since they both share a factor of 2.

Functions, Introduction to Limits and Continuity

Functions

Let D and C be two sets. The notation $f : D \to C$ represents a function between C and D. A function is a rule that assigns to each element of D exactly one element of C.

Vocabulary:

- Domain: Valid inputs

- Codomain: Possible outputs
- Range: Actual outputs (for $x \in D$, $f(x) \in C$ is called the *image* of x under f)

Examples:

1. Let $f(x) = x^2$. All real numbers can be inputted into the function, but only non-negative numbers are outputs. The domain is $\mathbb{R}$, the codomain is $\mathbb{R}$ and the range is $[0, \infty)$.

2. Let $f(x) = \lceil x \rceil$. This is called the ceiling function where every real number is rounded up to the nearest integer. The domain is $\mathbb{R}$ the codomain is $\mathbb{R}$ and the range is $\mathbb{Z}$ the integers.

Limits:

Limits are the tool most often used to define continuity and differentiability of functions.

Definition: The limit of $f(x)$ as x approaches a value c equals a value L is written as follows.

$$\lim_{x \to c} f(x) = L$$

To show this is a true statement, you need to verify that $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - L| < \epsilon$.

Examples:

1. $\lim_{x \to 1}(x + 1) = 2$

Let $\epsilon > 0$ be given. We want to show

$$|x - 1| < \delta \implies |f(x) - 2| < \epsilon$$

Notice that $|f(x) - 2| = |x + 1 - 2| = |x - 1|$. In order to prove the limit using the definition, we choose $\delta = \epsilon$ to get

$$|f(x) - 2| = |x - 1| < \delta = \epsilon$$

The point here is that we have the freedom to choose any delta that works. In this case, we choose delta to be whatever the given epsilon is.

2. $\lim_{x \to 3}(x^2 - 9) = 0$

Let $\epsilon > 0$ be given. Notice that

$$|x - 3| < \delta \implies |x^2 - 9| = |x - 3||x + 3| < \delta|x + 3|$$

Since the limits only consider values of x that are close to 3, we can start by assuming that x is within 1 of 3. This means $|x - 3| < 1$. This means $|x + 3| < 7$. Let $\delta = \frac{\epsilon}{7}$. Then we have

$$|x-3|<\delta \implies |x^2-9| = |x-3||x+3| < \delta|x+3| < \frac{\epsilon}{7} * 7 = \epsilon$$

One Sided Limits: When we only care about numbers approaching a value from the left (respectively right), then we are considering left sided (resp. right) limits.

Left Sided: To prove $\lim_{x\to c^-} f(x) = L$ we need to show that $\forall\,\epsilon > 0\,\exists\,\delta > 0$ such that $c - x < \delta \implies |f(x) - L| < \epsilon$.

Right Sided: To prove $\lim_{x\to c^+} f(x) = L$ we need to show that $\forall\,\epsilon > 0\,\exists\,\delta > 0$ such that $x - c < \delta \implies |f(x) - L| < \epsilon$.

Example: Let $f(x) = \frac{x}{|x|}$. Then,

$$\lim_{x\to 0^-} f(x) = -1 \quad \text{and} \quad \lim_{x\to 0^+} f(x) = 1$$

Since the left and right sided limits are not equal, the two sided limit does not exist. (Note, $f(x)$ is not continuous at $x = 0$).

Definition: A function $f(x)$ is continuous at $x = c$ if $f(c)$ is defined and $\lim_{x\to c} f(x) = f(c)$.

There are other definitions of continuity, but this definition will work for this class.

Infinite Limits: When we care about how a function behaves as we plug larger and larger values into the function (end behavior), we want to find an infinite limit.

To prove $\lim_{x\to\infty} f(x) = L$, we need to show that $\forall\ \epsilon > 0\ \exists\ M > 0$ such that $x > M \implies |f(x) - L| < \epsilon$. There is a similar definition for negative end behavior.

To prove $\lim_{x\to\infty} f(x) = \infty$, we need to show that $\forall\ N > 0\ \exists\ M > 0$ such that $x > M \implies f(x) > N$.

Example: Prove $\lim_{x\to\infty} \frac{2x}{x-1} = 2$.

Let $\epsilon > 0$ be given. We need to find an M depending on ϵ so that $x > M \implies |f(x) - L| < \epsilon$. We start with some scratch work.

$$|f(x) - L| = \left|\frac{2x}{x-1} - 2\right| = \left|\frac{2}{x-1}\right|$$

We want

$$\left|\frac{2}{x-1}\right| < \epsilon$$

Equivalently,
We want

$$\frac{2}{\epsilon} + 1 < x$$

The absolute values went away because we are dealing with large positive values of x. So we let $M = \frac{2}{\epsilon} + 1$ and for $x > M$, we get

$$|f(x) - L| = \left| \frac{2x}{x-1} - 2 \right| = \left| \frac{2}{x-1} \right| < \left| \frac{2}{\frac{2}{\epsilon} + 1 - 1} \right| = \epsilon$$

The Squeeze Theorem: If $f(x) < g(x) < h(x)$ for all x such that $|x - c| < \delta$ and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} g(x) = L$.

Example: Evaluate $\lim_{x \to 0} x \cos(x)$ using the squeeze theorem.

We know that $|\cos(x)| \leq 1$ for all x. This means,

$$-x \leq x \cos(x) \leq x$$

And since $\lim_{x \to 0} \pm x = 0$, the same must hold for $x \cos(x)$.