QUANTUM A-CURVES OF TORUS KNOTS

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Abstract.

1. Introduction

2. The Chern-Simons Line Bundle

The goal of this section is to defined a bundle over the character variety of the torus boundary of a 3-manifold.

2.1. The Representation and Character Varieties. Let $G = \langle a_i \,|\, r_j \rangle$ be a finitely generated group with n generators and m relations. A representation $\rho: G \to SL_2(\mathbb{C})$ is a homomorphism determined by a choice of matrices $A_i \in SL_2(\mathbb{C})$ such that the image of each relation evaluates to the identity in $SL_2(\mathbb{C})$. Denote by $\mathrm{Rep}(G) \subset \prod_{i=1}^{n} SL_2(\mathbb{C})$ the space of all representations of G under the relations r_j and embed $\mathrm{Rep}(G)$ into $\mathbb{C}^{4n}$ by $\rho \mapsto (\rho(a_1), \rho(a_2), ..., \rho(a_n))$. Under this embedding, $\mathrm{Rep}(G)$ is an algebraic variety called the **representation variety**. Specifically, $\mathrm{Rep}(G)$ is cut out by $4m + n$ equations where $4m$ equations come from the m relations and n equations from the fact that $\det(\rho(a_i)) = 1$ for each $1 \leq i \leq n$.

There is an action of $SL_2(\mathbb{C})$ on $\mathrm{Rep}(G)$ by conjugation. The quotient of $\mathrm{Rep}(G)$ by this action does not yield a Hausdorff space. To resolve this problem, identify representations of G that have the same character. A **character** of a representation $\rho \in \mathrm{Rep}(G)$ is a homomorphism $\chi_\rho : G \to \mathbb{C}$ defined by $\chi_\rho(g) = \mathrm{tr}(\rho(g))$, for each $g \in G$, where $\mathrm{tr}(\rho(g))$ is the trace of the matrix $\rho(g)$. The **character variety** of G, denoted $\chi(G)$, is the space of all characters of elements of $\mathrm{Rep}(G)$. The variety $\chi(G)$ can be thought of as the categorical quotient $\mathrm{Rep}(G)/SL_2(\mathbb{C})$ where elements of $\mathrm{Rep}(G)$ with the same character have been identified. The fact that $\chi(G)$ is an algebraic variety does not follow as easily as with $\mathrm{Rep}(G)$, but with some effort it can be shown [10].

2.2. The Bundle. [9] Let M be a 3-manifold with torus boundary T. Let $\{\mu, \lambda\}$ be the standard basis for $\pi_1(T)$ and denote by $\chi(T)$ (respectively $\chi(M)$) the character variety $\chi(\pi_1(T))$ (respectively $\chi(\pi_1(M))$). Define a map $v : \mathrm{Hom}(\pi_1(T), \mathbb{C}) \to \chi(T)$ by $v(f) = (\alpha \mapsto e^{2\pi i f(\alpha)})$. This is a 2:1 branch covering map with covering group $G \cong \mathbb{Z} \oplus \mathbb{Z} \rtimes \mathbb{Z}_2$ which has presentation,

$$G = \langle x, y, b \,|\, xy - yx = bxbx = byby = b^2 = 1 \rangle.$$

Send each $f \in \mathrm{Hom}(\pi_1(T), \mathbb{C})$ to the pair $(f(\mu), f(\lambda)) \in \mathbb{C} \times \mathbb{C}$. With this identification, the action of G on $\mathrm{Hom}(\pi_1(T), \mathbb{C})$ is

$$x(z,w) = (z+1, w), \quad y(z,w) = (z, w+1), \quad b(z,w) = (-z,-w).$$

Extend the this action to the trivial bundle $\mathrm{Hom}(\pi_1(T), \mathbb{C}) \times \mathbb{C}^*$, where $\mathbb{C}^*$ are the nonzero complex numbers, by

$$x(z,w,\zeta) = (z+1, w, \zeta e^{2\pi i w}),\ y(z,w,\zeta) = (z, w+1, \zeta e^{-2\pi i z}),\ b(z,w,\zeta) = (-z,-w,\zeta).$$

Define the Chern-Simons line bundle over the character variety $\chi(T)$ as the quotient bundle

$$CS(T) = \mathrm{Hom}(\pi_1(T), \mathbb{C}) \times \mathbb{C}^*/G.$$

As explained in [9], although the action has been defined by a fixed basis of $\pi_1(T)$, the action only depends on the orientation of T. Therefore, for the remainder of the paper, elements of $CS(T)$ are written $[z, w, \zeta]$ with the assumption of a fixed standard basis $\{\mu, \lambda\}$.

The Chern-Simons section is a map $CS_M : \chi(M) \to CS(T)$.

$$CS_M : \rho \mapsto \left[z, w, e^{2\pi i cs(\rho)}\right] \tag{2.1}$$

where $cs(\rho)$ is the Chern-Simons invariant associated to the representation ρ.

The following theorem shows how to calculate the change in the Chern-Simons invariant along a path of representations.

Theorem 2.1 (P. Kirk, E. Klassen [9])**.** *Let M denote an oriented 3-dimensional manifold whose boundary $\partial M = T$ consists of a 2-dimensional torus. Let $\{\mu, \lambda\}$ denote an oriented basis for $\pi_1(T)$. Let $\rho(t) : \pi_1(M) \to SL_2((C))$, $t \in [0,1]$, be a path of representations where $(z(t), w(t))$ denote a lift of $\rho(t)_{|\pi_1(T)}$ to $\mathbb{C}^2$. Suppose*

$$CS_M(\rho(t)) = [z(t), w(t), cs(z(t))]$$

for all t. Then,

$$cs(z(1)) \cdot cs(z(0))^{-1} = e^{2\pi i \int_0^1 z(t)w'(t) - z'(t)w(t)dt}$$

and if $z(0)$ corresponds to the trivial representation, $cs(z(0)) = 1$.

Assuming a path of representations is followed, the formula from Theorem 2.1 can be rewritten as

$$cs(z) = cs(z_0) \cdot e^{2\pi i \int_{z_o}^{z} z dw - w dz} \tag{2.2}$$

which gives a local expression of the Chern-Simons section as a function of z in a neighborhood of z_0.

3. The A-Polynomial

The A-polynomial is the defining polynomial of an algebraic curve in $\mathbb{C}^* \times \mathbb{C}^*$ where $\mathbb{C}^*$ are the nonzero complex numbers [1]. Let $K \subset S^3$ be a knot and M be the complement of a regular neighborhood of K. Then M is a compact manifold with boundary homeomorphic to a torus, $\partial M = T$.

The fundamental group $\pi_1(T)$ is a free abelian group with two generators. Let $\{\mu, \lambda\}$ be the standard basis for $\pi_1(T)$. Consider the subset $\mathrm{Rep}^{\Delta}(\pi_1(M))$ of $\mathrm{Rep}(\pi_1(M))$ consisting of upper triangular $SL_2(\mathbb{C})$ representations. Set

$$\rho(\mu) = \begin{pmatrix} m & \star \\ 0 & m^{-1} \end{pmatrix} \qquad \text{and} \qquad \rho(\lambda) = \begin{pmatrix} l & \star \\ 0 & l^{-1} \end{pmatrix}$$

and let $\epsilon : \mathrm{Rep}^{\Delta}(\pi_1(M)) \to \mathbb{C}^* \times \mathbb{C}^*$ be the eigenvalue map defined by $\epsilon(\rho) = (m, l)$. Let Z be the Zariski closure of $\epsilon(\mathrm{Rep}^{\Delta}(\pi_1(M)))$ in $\mathbb{C}^* \times \mathbb{C}^*$. Each of the components of Z are one dimensional [5]. The components are hyper-surfaces and can be cut out by a single polynomial unique up to multiplication by a constant. The A-**polynomial**, $A_K(m, l)$, is the product of all such defining polynomials. The A-polynomial can be taken to have relatively prime integer coefficients and is well defined up to a unit. The abelian component of Z will have defining polynomial $l - 1$ and thus the A-polynomial can be factored as $A_K(m, l) = (l - 1)A'_K(m, l)$ [1].

3.1. Torus Knots.

Denote by $T(a, b)$ the (a, b)-torus knot. The A-polynomials of torus knots are [1]

$$A_{T(p,q)}(m, l) = \begin{cases} (l-1)(lm^{2b} + 1) & : p = 2, b > 2 \\ \\ (l-1)(l^2 m^{2ab} - 1) & : a, b > 2 \end{cases}$$

The A-polynomial gives a parameterization of the representation space. Lift each component of the zero locus of the A-polynomial to a curve in $\mathbb{C} \times \mathbb{C}$ by using logarithmic coordinates. Specifically, let $m = e^{2\pi i z}$ and $l = -e^{2\pi i w}$. Using the principal branch of log, the A-curves of the (a, b)-torus knots are cut out by

$$A_{T(a,b)}(z, w) = \begin{cases} w(w + 2bz) & : a = 2, b > 2 \\ w(w + abz)(w + \frac{1}{2} + abz) & : a, b > 2 \end{cases} \tag{3.1}$$

4. Quantum Curves

4.1. Quantization.

In the language of physics, the A-polynomial, $A(z, w)$, cuts out a Lagrangian subvariety of $\mathbb{C} \times \mathbb{C}$ endowed with the symplectic form

$$2\pi i h \, dz \wedge dw. \tag{4.1}$$

The A-curve is the phase space of analytically continued Chern-Simons theory [8] with a classical state being a $SL_2(\mathbb{C})$ representation up to trace equivalence. The goal is to promote the A-curve to an operator $\hat{A}(q, M, L)$ that will annihilate $cs(z)$ (more precisely, the section CS_M) (2.1) for some operators M, L quantizing m, l. This is reminiscent of the AJ-conjecture [6] where the recurrence relation, $\widetilde{\alpha}_K(q, M, L)$ [7], of the colored Jones function is expected to semi-classically limit to the A-polynomial. In that setting, the operators M and L are elements of a ring called the quantum torus and satisfy the relation $LM = q^2 ML$. In the case at hand, it will be shown that the following operators acting on holomorphic functions lead to the same non-commutativity relation.

$$M = e^{2\pi i z} \quad \text{and} \quad L = e^{h\frac{d}{dz} + 2\pi i w} \quad (q = e^{\frac{\pi i}{h}}) \tag{4.2}$$

Notice that as $h \to 0$, $M \to e^{2\pi i z} = m$ and $L \to e^{2\pi i w} = -l$. In this sense, the operators are a coherent quantization of the classical coordinates. It will be seen that in the case of torus knots, the annihilator of certain power of the Chern-Simons section naturally limits to the A-polynomial.

4.2. **The Operator L.** The action of $M = e^{2\pi i z}$ on holomorphic functions of z is by multiplication. The action of $L = e^{h\frac{d}{dz} + 2\pi i w(z)}$ needs more clarification.

Lemma 4.1. *The operator $L = e^{h\frac{d}{dz} + 2\pi i w(z)}$ acts on holomorphic functions of z as*

$$L(f(z)) = f(z+h) \exp\left(\frac{2\pi i}{h} \int_z^{z+h} w(u) du\right).$$

Proof: If $g(z,t) = e^{t\left(h\frac{d}{dz} + 2\pi i w(z)\right)} f(z)$, then $g(z,t)$ satisfies the partial differential equation

$$\frac{\partial g}{\partial t} - h\frac{\partial g}{\partial z} = 2\pi i w \cdot g$$

with boundary condition $g(z, 0) = f(z)$.

Let $z(t) = z - ht$ and $G(t) = g(z(t), t)$. With this substitution, the above PDE can be written as the ODE $G'(t) = 2\pi i w(z(t)) \cdot G(t)$ where $G(0) = f(z)$. The solution is

$$G(t) = f(z) \exp\left(2\pi i \int_0^t w(z(s)) ds\right).$$

Replacing z with $z + th$ and setting $t = 1$ yields

$$g(z, 1) = f(z+h) \exp\left(2\pi i \int_0^1 w(z + (1-s)h) ds\right).$$

Now substitute $u = z + (1-s)h$ to conclude

$$L(f(z)) = g(z,1) = f(z+h) \exp\left(\frac{2\pi i}{h} \int_z^{z+h} w(u) du\right). \qquad \square$$

Corollary 4.1. *The operators M and L acting on holomorphic functions of z satisfy the relation $LM = q^2 ML$ where $q = e^{\frac{\pi i}{h}}$.*

Proof: Let $f(z)$ be a holomorphic function over $\mathbb{C}$. By Lemma 4.1 and the definition of M,

$$LM(f(z)) = L(e^{2\pi iz}f(z)) = e^{2\pi i(z+h)}f(z+h)\exp\left(\frac{2\pi i}{h}\int_z^{z+h} w(u)du\right)$$
$$= q^2 ML(f(z))$$

□

5. The $\hat{A}$ curve of torus knots

Let $A \subset \mathbb{C}\times\mathbb{C}$ be the zero locus of the A-polynomial in logarithmic coordinates $(z,w) \in \mathbb{C}\times\mathbb{C}$. There is a projection map $\pi : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ onto the first factor given by $\pi(z,w) = z$. There are two kinds of "singular points" on A, those where A is genuinely singular as an algebraic curve and those where the projection map restricted to A is not a local submersion. Away from the finite set of singular points there is a unique tangent vector $\widetilde{\frac{d}{dz}}$ such that $\pi_*\left(\widetilde{\frac{d}{dz}}\right) = \frac{d}{dz}$. Define an operator L acting on local holomorphic sections of the Chern-Simons line bundle over A by

$$L = e^{h\widetilde{\frac{d}{dz}}+2\pi iw(z)}. \tag{5.1}$$

where $w(z)$ is a local parameterization of the A-curve. This operator acts locally on holomorphic sections of the Chern-Simons bundle in the same way that L from Lemma 4.1 acts on holomorphic functions of z.

5.1. **T(2,b) knots.** The A-polynomial of the $(2,b)$-torus knot (3.1) has two factors. The factor w corresponds to the abelian component of the character variety while $w + 2bz$ defines the *geometric* component denoted for now by A_g. On the geometric component, there is a local expression $w(z) = -2bz$ which defines a plane curve with no singular points. Fix a point $z_0 \in A_g$. From Theorem 2.1 there is a local expression for the Chern-Simons section given by

$$cs(z) = cs(z_0)\cdot e^{2\pi i\int_{z_o}^{z} zdw-wdz}$$

where the integral is assumed to be over a path from z_0 to z contained in A_g. Let $h = \frac{1}{N}$ with $N \in \mathbb{N}$ and consider $cs^{\frac{1}{h}}(z)$ as a section of the N-fold tensor power of the bundle $CS(T)$ defined in 2.2. The natural number N represents the level of quantization.

Lemma 5.1. *The operator $L = e^{h\frac{d}{dz}+2\pi iw(z)}$ acts on $cs^{\frac{1}{h}}(z)$ as*

$$L\left(cs^{\frac{1}{h}}(z)\right) = cs^{\frac{1}{h}}(z)\exp\left(\frac{2\pi i}{h}\int_z^{z+h} zdw\right)$$

Proof: The lemma follows from a direct application of Lemma 4.1.

Theorem 5.1. *On the component of the A-curve parameterized by $w(z) = -2bz$ (3.1), the section $cs^{\frac{1}{h}}(z)$ of the N^{th} tensor power bundle $CS^N(T)$, where T is the boundary of the $T(2,b)$ knot complement in S^3, is annihilated by the operator*

$$\hat{A} = 1 - q^{2b}M^{2b}L \tag{5.2}$$

Proof: Recall that $h = \frac{1}{N}$ and $q = e^{\pi i h}$. The parameterization $w(z) = -2bz$ gives $zdw = -2bz\,dz$. By Lemma 5.1,

$$\begin{aligned} L\left(cs^{\frac{1}{h}}(z)\right) &= cs^{\frac{1}{h}}(z)\exp\left(\frac{2\pi i}{h}\int_z^{z+h} zdw\right) \\ &= cs^{\frac{1}{h}}(z)\exp\left(-\frac{2\pi i b}{h}(2zh + h^2)\right) \\ &= cs^{\frac{1}{h}}(z)(e^{2\pi i z})^{-2b}(e^{\pi i h})^{-2b} \\ &= q^{-2b}M^{-2b}(cs^{\frac{1}{h}}(z)) \end{aligned}$$

Therefore, $(q^{-2b}M^{-2b} - L)cs^{\frac{1}{h}}(z) = 0$. Multiplying on the left by $q^{2b}M^{2b}$ gives the desired result.

□

Corollary 5.1. $\hat{A}\big|_{q=-1} = \frac{A_{T(2,b)}(m,l)}{l-1}$

Proof: When $q = -1$, $L = -l$. Therefore, $\hat{A}\big|_{q=-1} = lm^{2b} + 1$.

□

5.2. **T(a,b) knots.** The A-curve of $T(a,b)$ torus knots has two geometric components. Denote by (A_1, w_1) the component corresponding to the factor $w + \frac{1}{2} + abz$ and (A_2, w_2) the component corresponding to $w + abz$ from (3.1). The operator L (5.1) changes depending on the parameterization. Let L_i be the operator defined by the parameterization $w_i(z)$, for $i = 1, 2$. With this notation, the operators satisfy $L_1 = -L_2$. Denote by $cs_i(z)$ the Chern-Simons section over the component (A_i, w_i).

Theorem 5.2. *Over the (A_i, w_i)-component of the A-curve parameterized by $w_i(z)$ the section $cs_i^{\frac{1}{h}}(z)$ of the N^{th} tensor power bundle $CS^N(T)$, where T is the boundary of the $T(a,b)$ knot complement in S^3, is annihilated by the operator*

$$\hat{A}_i = 1 - q^{ab}M^{ab}L_i \tag{5.3}$$

Proof: The proof that $cs_i^{\frac{1}{h}}(z)$ is annihilated by $\hat{A}_i$ on either component is almost identical to that of Theorem 5.1 since $zdw_i = -abz$.

□

Corollary 5.2. *The operator $\hat{A}_1\hat{A}_2 = (1 - q^{ab}M^{ab}L_1)(1 - q^{ab}M^{ab}L_2)$ annihilates the section $cs^{\frac{1}{h}}(z)$ defined over both geometric components of the A curve.*

Proof: It must be shown that $\hat{A}_1\hat{A}_2$ annihilates both $cs_1^{\frac{1}{h}}(z)$ and $cs_2^{\frac{1}{h}}(z)$. By Theorem 5.2,

$$(1 - q^{ab}M^{ab}L_1)(1 - q^{ab}M^{ab}L_2)cs_1^{\frac{1}{h}}(z) = (1 - q^{ab}M^{ab}L_1)2cs_1^{\frac{1}{h}}(z) = 0$$

$$(1 - q^{ab}M^{ab}L_1)(1 - q^{ab}M^{ab}L_2)cs_2^{\frac{1}{h}}(z) = 0$$

□

Corollary 5.3. $(\hat{A}_1 \circ \hat{A}_2)\big|_{q=-1} = \frac{A_{T(a,b)}(m,l)}{l-1}$

Proof: $\hat{A}_1 \circ \hat{A}_2 = (1 - q^{ab}M^{ab}L_1)(1 - q^{ab}M^{ab}L_2) = (1 - q^{ab}M^{ab}L_1)(1 + q^{ab}M^{ab}L_1)$ If $q = -1$, then without loss of generality, $L_1 = -l$. After this replacement, the A-polynomial of $T(a,b)$ (without the $(l-1)$ factor) is recovered.

□

Remark: The factor $(l-1)$ of the A-polynomial corresponds to $w = 0$. In this case, $cs(z) = 1$ and $L(f(z)) = f(z+h)$. Therefore, the operator $L-1$ annihilates $cs^{\frac{1}{h}}(z)$.

6. Conclusions and Discussion

As mentioned in the introduction, the motivation for finding an annihilator of the Chern-Simons section (2.1) stems from a relationship between the Witten path integral [12] and the Jones polynomial.

6.1. Chern-Simons Theory.

Let M be a compact oriented 3-manifold with a single torus boundary and consider the principal $\mathrm{SL}_2(\mathbb{C})$-bundle, P, over M. Let A be an $sl_2(\mathbb{C})$-valued one form on M and define the *Chern-Simons action* on A by

$$cs(A) = \frac{t}{8\pi}\int_M Tr\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right) + \frac{\widetilde{t}}{8\pi}\int_M Tr\left(\overline{A} \wedge d\overline{A} + \frac{2}{3}\overline{A} \wedge \overline{A} \wedge \overline{A}\right)$$

where Tr denotes the trace and $t = N + is, \overline{t} = N - is$ are *coupling constants*. The integer N is called the level and $s \in \mathbb{R}(\text{or } i\mathbb{R})$ is introduced to ensure the action behaves consistently under a change of orientation on M [8]. Using this action, define the following partition function by means of the Feynman path integral.

$$Z(M) = \int_{\mathcal{A}} e^{ics(A)}\mathcal{D}A \tag{6.1}$$

The partition function is not rigorously defined since the measure, $\mathcal{D}A$, is postulated on the (infinite dimensional) space, $\mathcal{A}$, of connections on M. Proceeding heuristically, it was shown that in the case of compact gauge group, (6.1) satisfies the same skein relation as the colored Jones polynomial [12].

To make a more concrete connection between the partition function and the current paper, it is prudent to discuss quantum perturbation theory. The analytically continued partition function can be written as a finite sum over contributions from different critical points,

$$Z(M,h,\widetilde{h}) = \sum_{\alpha,\widetilde{\alpha}} n_{\alpha,\widetilde{\alpha}} Z^{\alpha}(M,h)\overline{Z}^{\widetilde{\alpha}}(M,\widetilde{h}) \tag{6.2}$$

where $h = \frac{1}{t}$ and $\widetilde{h} = \frac{1}{\widetilde{t}}$, $n_{\alpha,\widetilde{\alpha}} \in \mathbb{Z}$, and $\alpha, \widetilde{\alpha}$ label the local branch and conjugate branch of the A-curve. In the limit, $h \to 0$, the *holomorphic blocks*, $Z_{\alpha}(M,h)$, have asymptotic expansion given by [3],

$$Z^{\alpha}(M,h) \sim \exp\left(\frac{1}{h}S_0^{\alpha} - \frac{1}{2}\delta^{\alpha}\log h + \sum_{n=1}^{\infty} S_n^{\alpha} h^{n-1}\right) \tag{6.3}$$

The leading order term, $S_0^{\alpha}(z) = 2\int_{A(m,l)=0}^{z} w(z)dz$ is the value of the classical Chern-Simons section on the α^{th} branch of the A-curve. It is related to (2.2) by an application of integration by parts. Given a classical state, (i.e. a flat connection A) there is an associated flat bundle, E_A, over M. The term δ^{α} is the following difference in the dimension of the cohomology groups of the bundle [3] and is therefore locally constant.

$$\delta^{\alpha} = dim(H^1(M,E_A) - dim(H^0(M,E_A))$$

The term $S_1^{\alpha}(z) = \frac{1}{2}\log(T(z))$ is the twisted Reidemeister torsion with coefficients in the adjoint representation[4].

The goal is to extend the defining polynomial of the A-curve to an operator, $\hat{A}(q,M,L) = \sum_{i=0}^{\infty} a_i(q,M)L^i$, that annihilates the partition function (6.2). The equation

$$\hat{A}Z = 0 \tag{6.4}$$

leads to an infinite hierarchy of difference equations that can be solved recursively given the initial condition $S_0^{\alpha}(z)$ [3]. In the case of torus knots, the Reidemeister torsion is locally constant along the geometric component of the A-curve. Therefore, all the higher order terms in the asymptotic expansion (6.3) are left constant under the action of L. Thus the local partition function and the local operator $\hat{A}$ are completely determined by the Chern-Simons section.

6.2. **Matching Results.** One of the main goals of this paper is to set up the framework to prove the AJ-conjecture [6] which is a proposed relationship between the A-polynomial and colored Jones polynomial of a knot $K \subset S^3$. The colored Jones polynomial is a function that assigns to each $n \in \mathbb{N}$ a Laurent polynomial $J_K(n) \in \mathbb{Z}[q^{\pm 1}]$. For discrete functions $f : \mathbb{N} \to \mathbb{C}[q^{\pm 1}]$ define operators M and L acting on f by

$$M(f)(n) = q^{2n}f(n) \quad \text{and} \quad L(f)(n) = f(n+1). \tag{6.5}$$

When acting on discrete functions, these operators satisfy the same non-commutativity relation, $LM = q^2ML$, as the operators (4.2) acting on holomorphic functions. It was shown [7] that the colored Jones function of every knot $K \subset S^3$ is annihilated by an operator $\widetilde{\alpha}_K(q, M, L)$. The AJ-conjecture asserts that $\widetilde{\alpha}_K(-1, M, L) = R(M)A_K(M, L)$ where $R(M)$ is some rational function of M, and $A_K(M, L)$ is the A-polynomial of K. In the case of torus knots, the AJ-conjecture has been verified [11].

For $T(2, b)$ torus knots, the colored Jones polynomial is annihilated by the operator $\widetilde{\alpha}(q, M, L) = c_2L^2 + c_1L + c_0$ where

$$c_2 = q^2M^2 - q^{-2}M^{-2}$$
$$c_1 = q^{-2b}\left(q^{-4b}M^{-2b}(q^2M^2 - q^{-2}M^{-2}) - (q^6M^2 - q^{-6}M^{-2})\right)$$
$$c_0 = -q^{-4b}M^{-2b}(q^6M^2 - q^{-6}M^{-2}).$$

This annihilator can be factored as

$$\widetilde{\alpha}(q, M, L) = \left((q^2M^2 - q^{-2}M^{-2})L - (q^6M^2 - q^{-6}M^{-6})q^{-2b}\right)\left(L + q^{-2b}M^{-2b}\right) \tag{6.6}$$

Recall the annihilator $\hat{A}(q, M, L) = 1 + q^{2b}M^{2b}L$ from Theorem 5.1. It can be seen that these annihilators satisfy

$$\widetilde{\alpha}(q, M, L) = \left((q^2M^2 - q^{-2}M^{-2})L - (q^6M^2 - q^{-6}M^{-6})q^{-2b}\right)q^{2b}M^{2b}\hat{A}(q, M, L)$$

and

$$\widetilde{\alpha}(-1, M, L) = (M^2 - M^{-2})(L - 1)(L + M^{-2b}) = (M^2 - M^{-2})(L - 1)\hat{A}(-1, M, L).$$

In the case of $T(a, b)$ torus knots, the colored Jones polynomial is annihilated by the operator $c_3L^3 + c_2L^2 + c_1L + c_0$ where

$$c_3 = q^2(q^{2(a+b)}M^{a+b} + q^{-2(a+b)}M^{-(a+b)}) - q^{-2}(q^{2(a-b)}M^{a-b} + q^{-2(a-b)}M^{-(a-b)})$$
$$c_2 = -q^{-2ab}\left(q^2(q^{4(a+b)}M^{a+b} + q^{-4(a+b)}M^{-(a+b)}) + q^{-2}(q^{4(a-b)}M^{a-b} + q^{-4(a-b)}M^{-(a-b)}\right)$$
$$c_1 = -q^{-8ab}M^{-2ab}c_3$$
$$c_0 = -q^{-4ab}M^{-2ab}c_2.$$

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