ON HOPKINS' PICARD GROUP

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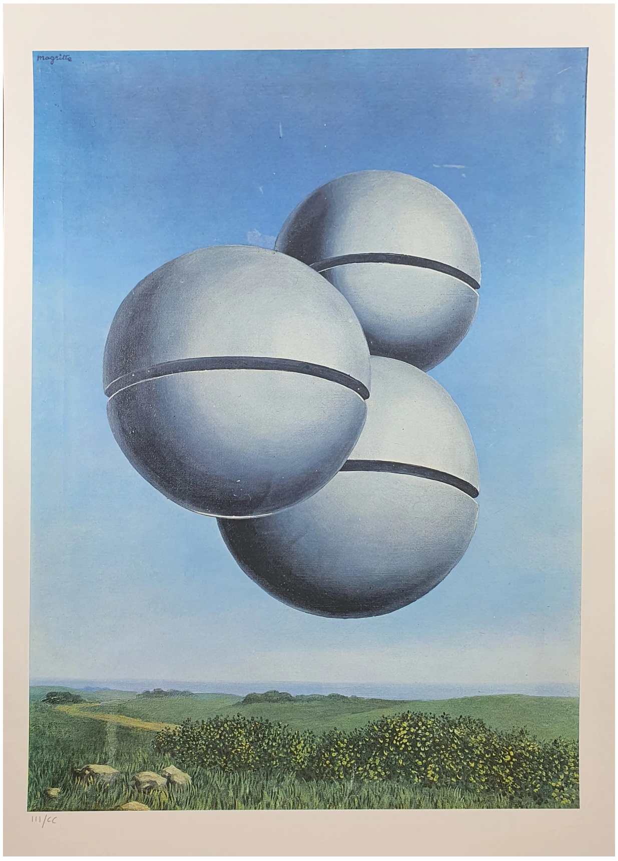
Abstract. We compute the algebraic Picard group of the category of $K(n)$-local spectra, for all heights n and all primes p. In particular, we show that it is always finitely generated over $\mathbb{Z}_p$ and, whenever $n \geq 2$, is of rank 2, thereby confirming a prediction made by Hopkins in the early 1990s. In fact, with the exception of the anomalous case $n = p = 2$, we provide a full set of topological generators for these groups. Our arguments rely on recent advances in p-adic geometry to translate the problem to a computation on Drinfeld's symmetric space, which can then be solved using results of Colmez–Dospinescu–Niziol.



René Magritte, *The Voice of Space*, 1931

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1. Introduction

1.1. **Results.** This article settles a question about the $K(n)$-local Picard group $\mathrm{Pic}_{n,p}$, relative to a prime p and an integer $n \geq 1$, formulated by Hopkins in the early 1990s ([Str92]). Here, $K(n)$ denotes Morava K-theory at height n, the prime p being implicit. Also known as Hopkins' Picard group, $\mathrm{Pic}_{n,p}$ is the group of isomorphism classes of invertible $K(n)$-local spectra, under the operation of $K(n)$-local smash product. The study of $\mathrm{Pic}_{n,p}$ was initiated by Hopkins–Mahowald–Sadovsky [HMS94], who defined an approximation to $\mathrm{Pic}_{n,p}$ in the form of a comparison map

$$\varepsilon\colon \mathrm{Pic}_{n,p} \to \mathrm{Pic}_{n,p}^{\mathrm{alg}}.$$

The target $\mathrm{Pic}_{n,p}^{\mathrm{alg}}$ of this map is defined in a purely algebraic manner in terms of equivariant line bundles on Lubin–Tate space (reviewed below). The map ε is an isomorphism for p large enough with respect to n, but generally it has a kernel $\kappa_{n,p}$ which is a (possibly infinite) product of cyclic p-groups with bounded exponent. Our main theorem gives a complete description of $\mathrm{Pic}_{n,p}^{\mathrm{alg}}$ together with the image of ε (up to a $\mathbb{Z}/2$ ambiguity in the case $(n,p) = (2,2)$).

Theorem A (Theorem 2.4.3)**.** *Let p be a prime, assume $n \geq 2$, and write $\mathcal{Z}_{n,p}$ for the pro-cyclic group*[§] $\lim_k \mathbb{Z}/p^k(2p^n-2)$*. There is an isomorphism*

$$\mathrm{Pic}_{n,p}^{\mathrm{alg}} \cong \mathcal{Z}_{n,p} \oplus \mathbb{Z}_p \oplus (\mathbb{Z}/2)^{\oplus e(n,p)},$$

where

$$e(n,p) = \begin{cases} 0, & p \neq 2; \\ 2, & p = 2 \text{ and } n \geq 3; \\ 3, & p = 2 \text{ and } n = 2. \end{cases}$$

In all cases except possibly $(n,p) = (2,2)$, the map ε is surjective. In the case $(n,p) = (2,2)$, the map ε is split with cokernel of order at most 2.

The case $n = 1$ has long been understood, so we have excluded it from the statement of the theorem. The computation for $n = 2$ has also been known by work of Hopkins and Lader ($p \geq 5$), Karamanov as well as Goerss–Henn–Mahowald–Rezk ([Kar10, GHMR15] $p = 3$), and Henn ($p = 2$, unpublished). These computations rely on explicit resolutions such as those constructed in [GHMR05] in the case $n = 2$ and $p = 3$, and thus seem very difficult to extend to larger heights. Indeed, prior to the present work, there was not a single pair (n,p) with $n \geq 3$ for which $\mathrm{Pic}_{n,p}^{\mathrm{alg}}$ had been computed.

A precise version of Theorem A appears as Theorem 2.4.3, which in fact provides explicit topological generators for $\mathrm{Pic}_{n,p}^{\mathrm{alg}}$. The case $p = 2$ being rather delicate, for the moment we focus on what happens when p is odd. As a profinite group, $\mathrm{Pic}_{n,p}^{\mathrm{alg}} \cong \mathcal{Z}_{n,p} \oplus \mathbb{Z}_p$ is generated by the images under ε of the following two classes in $\mathrm{Pic}_{n,p}$: the suspended $K(n)$-local sphere $\Sigma\mathbb{S}_{K(n)}$, and the $(p-1)$-fold power of the determinant sphere $\mathbb{S}_{K(n)}[\det]$, defined in Section 2.3. (In the case $n = 1$, $\mathbb{S}_{K(1)}[\det]$ and $\Sigma^2\mathbb{S}_{K(1)}$ have the same image in $\mathrm{Pic}_{1,p}^{\mathrm{alg}}$, and $\mathrm{Pic}_{1,p}^{\mathrm{alg}} \cong \mathcal{Z}_{1,p}$ is generated by the image of $\Sigma\mathbb{S}_{K(1)}$.) Combined with the known isomorphism range of the comparison map ε, for all heights and generic primes p, we obtain a full computation of the topological $K(n)$-local Picard group, stated here only for the previously unknown cases:

Theorem B (Corollary 2.4.4)**.** *Let $n > 2$ and $p > (n^2+1)/2$. There is an isomorphism*

$$\mathrm{Pic}_{n,p} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}/(2p^n-2).$$

As a profinite group, $\mathrm{Pic}_{n,p}$ is freely generated by $\Sigma\mathbb{S}_{K(n)}$ and $\mathbb{S}_{K(n)}[\det]^{\otimes(p-1)}$.

[§]This notation allows us to deal with all primes uniformly. When p is odd, $\mathcal{Z}_{n,p} \cong \mathbb{Z}_p \oplus \mathbb{Z}/(2p^n-2)$, while $\mathcal{Z}_{n,2} \cong \mathbb{Z}_2 \oplus \mathbb{Z}/(2^n-1)$.

Theorem A can be reduced to a purely algebraic result concerning deformations of formal groups, as we now explain. Let H_n be a 1-dimensional formal group of height n over $\overline{\mathbb{F}}_p$, and define the Lubin–Tate ring A_n as the deformation ring of H_n. Then A_n is non-canonically isomorphic to a power series ring in $n-1$ variables over the ring of Witt vectors of $\overline{\mathbb{F}}_p$. Finally, define the Morava stabilizer group $\mathbb{G}_n$ as the group of automorphisms of the pair $(H_n, \overline{\mathbb{F}}_p)$. Then $\mathbb{G}_n$ is a profinite group acting continuously on the topological ring A_n.

The ring A_n admits a spectral incarnation $E = E(H_n, \overline{\mathbb{F}}_p)$, known as Morava E-theory or the Lubin–Tate spectrum. This is a $K(n)$-local commutative ring spectrum with homotopy ring $E_* = A_n[u^{\pm 1}]$ for a class u of degree -2. The spectrum E is constructed functorially from the pair $(H_n, \overline{\mathbb{F}}_p)$, so that it admits an action of $\mathbb{G}_n$. The completed E-homology of a spectrum is a graded E_*-module which is derived complete with respect to the maximal ideal of $\pi_0 E = A_n$, and which admits a continuous action of $\mathbb{G}_n$ lying over the action on E_*. We call such a structure a Morava module.

The algebraic Picard group $\mathrm{Pic}^{\mathrm{alg}}_{n,p}$ is defined as the group of isomorphism classes of invertible Morava modules. The comparison map ε appearing above is given by applying completed E-homology to invertible $K(n)$-local spectra. The algebraic Picard group sits in an exact sequence

$$0 \to H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^*) \to \mathrm{Pic}^{\mathrm{alg}}_{n,p} \to \mathbb{Z}/2 \to 0. \tag{1.1.1}$$

This is due to the fact that E_* is 2-periodic, so an invertible Morava module is supported in either odd or even degrees, giving the map to $\mathbb{Z}/2$; since the shift $E_*[1]$ is supported in odd degrees, the map to $\mathbb{Z}/2$ is surjective. The kernel of this map is isomorphic to the group of isomorphism classes of invertible A_n-modules carrying an equivariant continuous action of $\mathbb{G}_n$. Since A_n is a local ring, its Picard group is trivial. Equivariant continuous actions of $\mathbb{G}_n$ on a free A_n-module of rank 1 correspond exactly to classes in the continuous cohomology $H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^*)$, where A_n^* is the group of units in A_n.

The exact sequence (1.1.1) allows us to reduce Theorem A to a computation of $H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^*)$. Once again we assume $n \geq 2$, the case $n = 1$ being already well-understood.

Theorem C. *Assume $n \geq 2$. There is an isomorphism*

$$H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^*) \cong \mathbb{Z}_p^{\oplus 2} \oplus \mathbb{Z}/(p^n - 1) \oplus (\mathbb{Z}/2)^{\oplus e(n,p)},$$

where $e(n,p)$ is as in Theorem A.

Theorem C implies the description of $\mathrm{Pic}^{\mathrm{alg}}_{n,p}$, at least up to the question of which extension of $\mathbb{Z}/2$ by $H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^*)$ is represented by (1.1.1). What is more, we show in the course of proving Theorem C that every element of $H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^*)$ (or rather an index 2 subgroup of it, in the case $n = p = 2$) lifts to an element of $\mathrm{Pic}_{n,p}$ of even degree; this implies the properties of ε claimed by Theorem A.

Let $A_n^{**} \subset A_n^*$ be the subgroup of principal units (i.e., topologically unipotent elements). It is convenient to divide Theorem C into pro-p and prime-to-p parts using the $\mathbb{G}_n$-equivariant decomposition $A_n^* \cong A_n^{**} \oplus \overline{\mathbb{F}}_p^*$. The prime-to-$p$ part poses no difficulty (Proposition 2.5.2), so Theorem C is reduced to the claim $H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**}) \cong \mathbb{Z}_p^{\oplus 2} \oplus (\mathbb{Z}/2)^{\oplus e(n,p)}$. We prove a precise statement (Theorem 2.7.1) about $H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**})$, and show that it implies Theorem 2.4.3.

1.2. **Main ideas of the proof.** An even invertible Morava module (meaning a class in $\mathrm{Pic}^{\mathrm{alg}}_{n,p}$ which maps to the identity in $\mathbb{Z}/2$ in (1.1.1)) corresponds to a free A_n-module with a continuous equivariant $\mathbb{G}_n$-action. We may view such modules geometrically, as line bundles on the stack $[\mathrm{LT}_n / \mathbb{G}_n]$, where $\mathrm{LT}_n = \mathrm{Spf}\, A_n$ is the Lubin–Tate deformation space, i.e., the affine formal scheme parameterizing deformations of the formal group H_n. The stack $[\mathrm{LT}_n / \mathbb{G}_n]$ arises naturally; it is isomorphic to the completion of the moduli space of formal groups at the height n point corresponding to H_n (see Proposition 3.1.2).

To access the Picard group of $[\mathrm{LT}_n/\mathbb{G}_n]$, we rely heavily on Faltings' isomorphism between the Lubin–Tate and Drinfeld towers, as interpreted as a relation between perfectoid spaces [SW13]. This isomorphism is only available after passage to the generic fiber. Therefore let K be the fraction field of the ring of Witt vectors of $\overline{\mathbb{F}}_p$, and let $\mathrm{LT}_{n,K}$ be the adic generic fiber of LT_n, so that $\mathrm{LT}_{n,K}$ is isomorphic to a rigid-analytic $(n-1)$-dimensional open ball over K.

In Theorem 3.7.3 we interpret Faltings' isomorphism as a relation between stacks on Scholze's category of diamonds:

$$[\mathrm{LT}_{n,K}^{\diamond}/\mathbb{G}_n] \cong [\mathcal{H}^{n-1,\diamond}/\operatorname{GL}_n(\mathbb{Z}_p)]. \tag{1.2.1}$$

On the left side of the isomorphism is the "diamond generic fiber" of the stack $[\mathrm{LT}_n/\mathbb{G}_n]$. Appearing on the right side is (the diamond version of) Drinfeld's symmetric space $\mathcal{H}^{n-1}$, defined as the complement in rigid-analytic $\mathbb{P}^{n-1}_{\mathbb{Q}_p}$ of all $\mathbb{Q}_p$-rational hyperplanes.

The isomorphism (1.2.1) is indispensable for the proof of Theorem C, because it trades the opaque action of $\mathbb{G}_n$ on A_n for the transparent (indeed linear) action of $\operatorname{GL}_n(\mathbb{Z}_p)$ on $\mathcal{H}^{n-1}$. We offer a summary of this proof. Recall we have already reduced Theorem C to the calculation of $H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**})$. In particular, we want to show that for $n \geq 2$, the torsion-free part of $H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**})$ is $\mathbb{Z}_p^{\oplus 2}$, generated by the pro-p parts of the even Morava modules $\varepsilon(\Sigma^2\mathbb{S}_{K(n)}[\det])$ and $\varepsilon(\mathbb{S}_{K(n)}[\det])$. The proof brings together two important constructions, accounting for each of these two generators.

The first construction involves determinants of formal groups. The idea is that in certain contexts, it is possible to take the determinant of a height n formal group to obtain a height 1 formal group. See Theorem 3.8.1 for the precise statement. One manifestation of this phenomenon is that there is a determinant homomorphism $\det\colon \mathbb{G}_n \to \mathbb{G}_1$ which is compatible with the natural inclusion map $A_1 \to A_n$; as a result there is an induced map

$$\det\nolimits_{\mathrm{LT}}^*\colon H^1_{\mathrm{cts}}(\mathbb{G}_1, A_1^{**}) \to H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**}).$$

The source of this map can be calculated directly,

$$H^1_{\mathrm{cts}}(\mathbb{G}_1, A_1^{**}) \cong \begin{cases} \mathbb{Z}_p, & p \neq 2; \\ \mathbb{Z}_2 \oplus (\mathbb{Z}/2)^{\oplus 2}, & p = 2, \end{cases} \tag{1.2.2}$$

and the image of the generator of the free part of this group under $\det_{\mathrm{LT}}^*$ is the pro-p part of $\varepsilon(\mathbb{S}_{K(n)}[\det])$, the Morava module of the determinant sphere.

The second construction is the one that leverages (1.2.1). In Section 4 we define a sheaf $\mathcal{O}^{**}$ of principal units on an adic space, and we observe that $A_n^{**} \cong H^0(\mathrm{LT}_{n,K}, \mathcal{O}^{**})$. Let C be the completion of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and let $\mathcal{H}^{n-1}_C$ be the base change of $\mathcal{H}^{n-1}$ to C. Another version of (1.2.1) is

$$[\mathrm{LT}_{n,K}^{\diamond}/\mathbb{G}_n] \cong [\mathcal{H}_C^{n-1,\diamond}/(\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \times \operatorname{GL}_n(\mathbb{Z}_p))].$$

As a formal consequence of this isomorphism, we obtain a map

$$b\colon H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**}) \to H^1_{\mathrm{pro\acute{e}t}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})^{\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\times \operatorname{GL}_n(\mathbb{Z}_p)}.$$

The following theorem appears as Theorem 4.3.13 combined with Theorem 5.4.1.

Theorem D. *Assume $(n,p) \neq (2,2)$. The maps $\det_{\mathrm{LT}}^*$ and b fit into an exact sequence*

$$0 \to H^1_{\mathrm{cts}}(\mathbb{G}_1, A_1^{**}) \xrightarrow{\det_{\mathrm{LT}}^*} H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**}) \xrightarrow{b} H^1_{\mathrm{pro\acute{e}t}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})^{\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\times \operatorname{GL}_n(\mathbb{Z}_p)} \to 0.$$

If $(n,p) = (2,2)$, there is a similar exact sequence, except the first nonzero term in the sequence is $H^1_{\mathrm{cts}}(\mathbb{G}_1, A_1^{**}) \oplus \mathbb{Z}/2$.

Interestingly, the reason that $(n,p)=(2,2)$ is treated differently in Theorem A traces back to the fact that it is the only case where there are nontrivial continuous homomorphisms $\mathrm{SL}_n(\mathbb{Z}_p) \to \mathbb{Q}_p^{**}$ (see Lemma 4.3.7).

At this point we apply the recent advances of Colmez–Dospinescu–Niziol [CDN20, CDN21] on the cohomology of $\mathcal{H}_C^{n-1}$, which seem to be tailor-made for our purposes. From their work we deduce (see Corollary 5.3.3) that if $n \geq 2$, then $H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\times \mathrm{GL}_n(\mathbb{Z}_p)}$ is canonically isomorphic to $\mathbb{Z}_p$. The final challenge (Theorem 5.4.1) is to show that the map b carries the pro-p part of $\varepsilon(\Sigma^2\mathbb{S}_{K(n)})$ onto a generator of this $\mathbb{Z}_p$, so that b is surjective.

Combining Theorem D with (1.2.2) shows that $H^1_{\text{cts}}(\mathbb{G}_n, A_n^{**}) \cong \mathbb{Z}_p^{\oplus 2} \oplus (\mathbb{Z}/2)^{\oplus e(n,p)}$, which completes the proof of Theorem C.

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Contents

2. The $K(n)$-local Picard group

The purpose of this section is two-fold. First, it reviews some background material on Picard groups in chromatic homotopy theory. We then explain how to reduce the determination of Hopkins' Picard group to a cohomological computation (Theorem 2.7.1), which will then be tackled in the remainder of this paper.

2.1. **Motivation: invertible spectra and degree.** Let $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal ∞-category. The *Picard group* $\mathrm{Pic}(\mathcal{C})$ of $\mathcal{C}$ is defined as the collection of isomorphism classes of $\otimes$-invertible objects in $\mathcal{C}$, i.e., those objects $X \in \mathcal{C}$ for which there exists $y \in \mathcal{C}$ with $x \otimes y \simeq \mathbb{1}$. If $\mathcal{C}$ is presentably symmetric monoidal, then $\mathrm{Pic}(\mathcal{C})$ is a set [MS16, Remark 2.1.4]. The symmetric monoidal product on $\mathcal{C}$ then equips $\mathrm{Pic}(\mathcal{C})$ with the structure of a group, with unit $\mathbb{1}$.

If $\mathcal{C} = (\mathrm{Sp}, \otimes, \mathbb{S})$ is the symmetric monoidal ∞-category of spectra, then we can classify invertible objects via their degree. Indeed, for any $X \in \mathrm{Pic}(\mathrm{Sp})$, the Künneth isomorphism implies that $H_*(X;\mathbb{Z})$ is concentrated in a single degree, which we define as $\deg(X)$. A Postnikov tower argument [HMS94] then shows that the induced map

$$\deg\colon \mathrm{Pic}(\mathrm{Sp}) \longrightarrow \mathbb{Z}$$

is a bijection, with inverse $n \mapsto \mathbb{S}^n$. In particular, Pic(Sp) is generated by $\Sigma\mathbb{S}$.

Hopkins, Mahowald, and Sadofsky observed that the situation becomes more interesting when we turn to local stable homotopy theory. This article concerns the invertible $K(n)$-local spectra. Here $K(n)$ denotes Morava K-theory of height n at the (fixed) prime p. This is an associative ring spectrum with coefficients

$$\pi_* K(n) \cong \begin{cases} \mathbb{Q} & \text{if } n = 0; \\ \mathbb{F}_p[v_n^{\pm 1}] & \text{if } 0 < n < \infty; \\ \mathbb{F}_p & \text{if } n = \infty, \end{cases}$$

with v_n of degree $2(p^n - 1)$. By convention, $K(0) = H\mathbb{Q}$ independently of p, while $K(\infty) = H\mathbb{F}_p$. Our focus in this paper will be on the case of intermediate characteristic $0 < n < \infty$, and we will often keep the ambient prime p implicit in the notation. The Morava K-theories are fields in Sp, i.e., they obey a Künneth formula

$$K(n)_*(X \otimes Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y) \tag{2.1.1}$$

for any $X, Y \in \mathrm{Sp}$, so that every module spectrum over $K(n)$ is free. In fact, as a consequence of the nilpotence theorem due to Devinatz, Hopkins, and Smith [DHS88], this provides a full classification of all prime fields in Sp. For more background information on chromatic homotopy theory, we refer the reader to [BB20].

Bousfield localization constructs the category $\mathrm{Sp}_{K(n)}$ of $K(n)$-local spectra, i.e., the universal localization $(-)_{K(n)}\colon \mathrm{Sp} \to \mathrm{Sp}_{K(n)}$ which inverts all maps f of spectra for which $K(n) \otimes f$ is an equivalence. Localization admits a fully faithful right adjoint $\mathrm{Sp}_{K(n)} \subset \mathrm{Sp}$, so we may view a $K(n)$-local spectrum as a spectrum. The smash product on Sp descends to a symmetric monoidal structure on $\mathrm{Sp}_{K(n)}$; explicitly, the product of two $K(n)$-local spectra X, Y is given by

$$X \hat{\otimes} Y = (X \otimes Y)_{K(n)}.$$

Equipped with this structure, $\mathrm{Sp}_{K(n)} = (\mathrm{Sp}_{K(n)}, \hat{\otimes}, \mathbb{S}_{K(n)})$ forms a presentably symmetric monoidal stable ∞-category.

Definition 2.1.2. Fix a height $n \geq 1$ and an implicit prime p. We define the *$K(n)$-local Picard group* as

$$\mathrm{Pic}_{n,p} \coloneqq \mathrm{Pic}(\mathrm{Sp}_{K(n)}).$$

By (2.1.1), the Morava K-homology $K(n)_*(X)$ of any invertible $X \in \mathrm{Sp}_{K(n)}$ is concentrated in degrees congruent to d modulo $2(p^n - 1)$ for some $0 \leq d < 2(p^n - 1)$. Following the global story above, we define this number as the *local degree* of X: $\deg_{n,p}(X) \coloneqq d$. The next result is proven in [HS99, Proposition 14.3], relying on [HMS94].

Proposition 2.1.3 (Hopkins–Mahowald–Sadofsky, Hovey–Strickland)**.** *Let $n \geq 1$. The $K(n)$-local degree provides a surjective map*

$$\deg_{n,p}\colon \mathrm{Pic}_{n,p} \twoheadrightarrow \mathbb{Z}/2(p^n - 1)$$

with kernel an infinite abelian pro-p-group.

In particular, in contrast to the case of Pic(Sp), the naive notion of degree in the $K(n)$-local setting is only a rather coarse approximation to the Picard group. In the next subsection, we will review a refined invariant which provides a much closer algebraic approximation to $\mathrm{Pic}_{n,p}$.

2.2. **Morava modules.** As the starting point for the construction of a good algebraic approximation to $\mathrm{Pic}_{n,p}$, we take the following fundamental result of Devinatz–Hopkins [DH95]. A general account of the Galois theory of commutative ring spectra was developed by Rognes [Rog08].

Let H_n be a 1-dimensional formal group of height n over $\overline{\mathbb{F}}_p$. Let E_n be Morava E-theory of height n associated with H_n. This a $K(n)$-local commutative ring spectrum with

$$\pi_* E_n = A_n[u^{\pm 1}],\ |u| = -2.$$

Here $\pi_0 E_n = A_n$ is the deformation ring of H_n, known as the Lubin–Tate deformation ring. There is a (non-canonical) isomorphism $A_n \cong W(\bar{\mathbb{F}}_p)[\![u_1, \ldots, u_{n-1}]\!]$, with $W(\bar{\mathbb{F}}_p)$ the ring of Witt vectors of $\bar{\mathbb{F}}_p$. Let H_n^{univ} be the universal deformation of H_n; then $\pi_{-2} E_n = \mathrm{Lie}\, H_n^{\mathrm{univ}} = A_n u$ is the Lie algebra of H_n^{univ}. The spectrum E_n admits a continuous action of the Morava stabilizer group $\mathbb{G}_n = \mathrm{Aut}(H_n, \overline{\mathbb{F}}_p)$.

Theorem 2.2.1 (Devinatz–Hopkins)**.** *The unit map $\mathbb{S}_{K(n)} \to E_n$ witnesses the target as a $K(n)$-local Galois extension with Galois group $\mathbb{G}_n$. In particular, there is a canonical equivalence $\mathbb{S}_{K(n)} \simeq E_n^{h\mathbb{G}_n}$.*

We can twist the equivalence of this theorem by any $X \in \mathrm{Sp}_{K(n)}$, resulting in a new equivalence $X \simeq (E_n \hat{\otimes} X)^{h\mathbb{G}_n}$, where X is equipped with trivial $\mathbb{G}_n$-action. This in turn gives rise to a (Bousfield–Kan) descent spectral sequence of signature (see [BH16])

$$E_2^{s,t}(X) := H^s(\mathbb{G}_n, E_{n,t}(X)) \implies \pi_{t-s} X \tag{2.2.2}$$

that has excellent convergence properties. Indeed, since $\mathbb{S}_{K(n)} \to E_n$ is a descendable morphism of commutative ring spectra by the smash product theorem [Rav92, Theorem 7.5.6] (see also [Lur10, Lecture 31]), the spectral sequence converges strongly and collapses on a finite page with a horizontal vanishing line. When $p-1$ does not divide n, the stabilizer group $\mathbb{G}_n$ is p-torsion free and thus has finite cohomological dimension (as opposed to merely finite virtual cohomological dimension), in which case (2.2.2) collapses already on the E_2-page.

The input to the spectral sequence (2.2.2) is the continuous $\mathbb{G}_n$-cohomology of $E_*(X) = \pi_* L_{K(n)}(E_n \otimes X)$, which by a mild abuse of notation here refers to the completed E-homology of X. This structure is captured algebraically in the category of *Morava modules*, $\mathrm{Mod}_{E_*}^{\circlearrowleft \mathbb{G}_n}$, defined as follows: Let $\mathfrak{m} = (p, u_1, \ldots, u_{n-1}) \subset \pi_0 E_n$ be the maximal ideal of $\pi_0 E_n$. A Morava module is a derived $\mathfrak{m}$-complete graded E_*-module equipped with a continuous semi-linear action of $\mathbb{G}_n$. Morphisms of Morava modules are continuous equivariant maps. The category of Morava modules can be modeled as $(\pi_* E_n, E_* E)$-comodules whose underlying E_*-module is derived $\mathfrak{m}$-complete. Completing the usual tensor product induces a symmetric monoidal structure on $\mathrm{Mod}_{E_*}^{\circlearrowleft \mathbb{G}_n}$, denoted $\hat{\otimes}$. With these definitions, we obtain a canonical lift

$$\begin{tikzcd} & \mathrm{Mod}_{E_*}^{\circlearrowleft \mathbb{G}_n} \arrow[d, "U"] \\ \mathrm{Sp}_{K(n)} \arrow[ur, "E_*(-)"] \arrow[r, "E_*(-)"'] & \mathrm{Mod}_{E_*}, \end{tikzcd} \tag{2.2.3}$$

where U denotes the evident forgetful functor, which forgets the action by $\mathbb{G}_n$ on the given Morava module. Note that $E_*(-)$ is in general not symmetric monoidal. However, when restricted to spectra X with $E_*(X)$ projective (as E_*-module), it is symmetric monoidal. In particular, this applies to invertible $K(n)$-local spectra, in light of the following characterization:

Lemma 2.2.4. *For $X \in \mathrm{Sp}_{K(n)}$, the following conditions are equivalent:*

(1) X is invertible;
(2) $E_*(X)$ is free of rank 1 as a E_*-module;
(3) $K(n)_*(X)$ is free of rank 1 as a $K(n)_*$-module.

Definition 2.2.5. The *algebraic Picard group* $\mathrm{Pic}_{n,p}^{\mathrm{alg}}$ is defined as the group of isomorphism classes of $\hat{\otimes}$-invertible Morava modules. The Morava module functor (2.2.3) then induces a *comparison map*

$$\varepsilon\colon \mathrm{Pic}_{n,p} \longrightarrow \mathrm{Pic}_{n,p}^{\mathrm{alg}} := \mathrm{Pic}(\mathrm{Mod}_{E_*}^{\circlearrowleft \mathbb{G}_n}), \quad X \mapsto E_*(X). \tag{2.2.6}$$

The comparison map ε turns out to be a close approximation to the topological Picard group, essentially due to the excellent properties of the descent spectral sequence (2.2.2). The next result summarizes the situation:

Proposition 2.2.7 (Hopkins–Mahowald–Sadofsky, Pstragowski, Goerss–Hopkins)**.** *Let p be prime such that $2p-2 > \max\{n^2-1, 2n\}$, then the comparison map is bijective:*

$$\varepsilon\colon \mathrm{Pic}_{n,p} \xrightarrow{\cong} \mathrm{Pic}_{n,p}^{\mathrm{alg}}.$$

The injectivity part of this result was established in [HMS94], while surjectivity was proven under slightly stronger hypothesis in [Pst22]; the version given here is due to Goerss and Hopkins, [GH22, Remark 2.4].

In general, however, the map ε is not injective; non-trivial elements in the kernel are called *exotic*. This can only occur in the non-algebraic realm, i.e., when the descent spectral sequence (2.2.2) for $X \in \mathrm{Sp}_{K(n)}$ does not collapse for cohomological reasons.

Definition 2.2.8. The kernel of ε is the *exotic Picard group* $\kappa_{n,p}$, fitting into an exact sequence

$$0 \longrightarrow \kappa_{n,p} \longrightarrow \mathrm{Pic}_{n,p} \xrightarrow{\varepsilon} \mathrm{Pic}_{n,p}^{\mathrm{alg}}. \tag{2.2.9}$$

Remark 2.2.10. For any n and p, the group $\kappa_{p,n}$ is an abelian pro-p group of finite exponent. It is pro-p as a consequence of Proposition 2.1.3. The finiteness of its exponent can be seen as follows: Consider the descent spectral sequence (2.2.2). Since it collapses at a finite page with a horizontal vanishing line, uniformly for any $X \in \mathrm{Pic}_{n,p}$, there is an induced descent filtration on $\kappa_{n,p}$ with subquotients $E_r^{r,r-1}(\mathbb{S})$, see for example [BBG+22, Section 3.3]. By [GH22, Remark 1.8], the latter groups are all p-torsion of finite exponent, hence so is $\kappa_{n,p}$. It follows from the Prüfer–Baer theorem that $\kappa_{p,n}$ is then a (possibly infinite) product of finite cyclic p-groups. For odd p, this has already been observed by Heard [Hea14, Theorem 4.4.1].

2.3. **Two distinguished spheres.** We discuss here the two distinguished classes in the local Picard group $\mathrm{Pic}_{n,p}$ which make an appearance in Theorem 2.4.3.

The first of these classes is simply the suspension $\Sigma\mathbb{S}_{K(n)}$. The Morava module $E_{n,*}(\Sigma\mathbb{S}_{K(n)})$ is odd, but its tensor square $E_{n,*}(\Sigma^2\mathbb{S}_{K(n)})$ is

$$\varepsilon(\Sigma^2\mathbb{S}_{K(n)}) = \pi_{-2}E_n \cong \mathrm{Lie}\, H_n^{\mathrm{univ}},$$

where $\mathrm{Lie}\, H_n^{\mathrm{univ}}$ is the Lie algebra of the the universal deformation H_n^{univ} of the height n formal group H_n. Since $K(n)_*(\Sigma^{2p^n-2}\mathbb{S}_{K(n)}) \cong K(n)_*$, the local degree of $\Sigma^{2p^n-2}\mathbb{S}_{K(n)}$ is zero. In light of Proposition 2.1.3, this implies that the assignment $k \mapsto \Sigma^{(2p^n-2)k}\mathbb{S}_{K(n)}$ extends to a homomorphism

$$\iota_{\mathrm{can}}\colon \mathbb{Z}_p \longrightarrow \mathrm{Pic}_{n,p} \longrightarrow \mathrm{Pic}_{n,p}^{\mathrm{alg}}.$$

In fact, we can do better, extending to a larger procyclic group generated by $\Sigma^1\mathbb{S}_{K(n)}$. To do so, we will introduce an auxiliary piece of notation (inspired by [GH22]) that allows us to treat the cases of odd and even primes uniformly:

Notation 2.3.1. For each prime p and height n, we set $\mathcal{Z}_{n,p} := \lim_k \mathbb{Z}/(p^k(2p^n-2))$.

Remark 2.3.2. For any $n \geq 1$ and prime p, the group $\mathcal{Z}_{n,p}$ is procyclic, isomorphic to

$$\mathcal{Z}_{n,p} \cong \begin{cases} \mathbb{Z}_p \oplus \mathbb{Z}/2(p^n-1) & \text{if } p > 2; \\ \mathbb{Z}_2 \oplus \mathbb{Z}/(2^n-1) & \text{if } p = 2. \end{cases}$$

For example, if p is odd, a generator for the torsion summand in this presentation is given by the element $\lim_k p^{nk}$.

Based on the periodicity theorem [HS98], Hopkins, Mahowald, and Sadofsky [HMS94], one can use "v_n-adic interpolation" to construct an extension of ι_{can}, which we denote by the same symbol:

$$\iota_{\mathrm{can}} \colon \mathcal{Z}_{n,p} \xrightarrow{1 \mapsto \Sigma^1 \mathbb{S}_{K(n)}} \mathrm{Pic}^{\mathrm{alg}}_{n,p}. \tag{2.3.3}$$

Here, the map $\mathbb{Z}_p \to \mathcal{Z}_{n,p}$ is induced by maps $\mathbb{Z}/p^k \to \mathbb{Z}/(p^k(2p^n-2))$ determined by $1 \mapsto 2p^n-2$. In particular, for p odd, this splits the presentation of Remark 2.3.2. However, for $p=2$, this embedding includes $\mathbb{Z}_2$ as an index 2 subgroup of the $\mathbb{Z}_2$ appearing in Remark 2.3.2.

The second important class is the *determinant sphere*. Let $\mathbb{S}(1)$ be the p-complete sphere with its natural continuous $\mathbb{Z}_p^*$-action. The stabilizer group then acts on $\mathbb{S}(1)$ through the determinant map $\det \colon \mathbb{G}_n \to \mathbb{Z}_p^*$. The determinant sphere is defined as

$$\mathbb{S}_{K(n)}[\det] := (E_n \otimes \mathbb{S}(1))^{h\mathbb{G}_n}, \tag{2.3.4}$$

where $\mathbb{G}_n$ acts diagonally on the p-completed smash product $E_n \otimes \mathbb{S}(1)$. As shown in [BBGS22], the determinant sphere topologically realizes the Morava module $A_n[\det]$, meaning the free A_n-module of rank 1 where the $\mathbb{G}_n$-action has been twisted by det. Thus:

$$\varepsilon(\mathbb{S}_{K(n)}[\det]) = A_n[\det].$$

Remark 2.3.5. The significance of having a concrete topological model for the determinant sphere comes from Gross–Hopkins duality, which describes the Morava module of the dualizing module for the $K(n)$-local category. The latter is by definition the Brown–Comenetz dual of the n-th monochromatic layer of the sphere $M_n\mathbb{S} = \mathrm{fib}(L_n\mathbb{S} \to L_{n-1}\mathbb{S})$, denoted $I_n := IM_nS^0$. Here, Brown–Comenetz duality I is a lift of Pontryagin duality to the category of spectra. Gross and Hopkins [HG94b, HG94a] prove that $I_n \in \mathrm{Pic}_{n,p}$ with Morava module $E_*(I_n) \simeq \pi_{n-n^2}E_n[\det]$, see also [Str00]. We may then restate this algebraic identification as a $K(n)$-local equivalence

$$I_n \simeq \Sigma^{n^2-n}\mathbb{S}_{K(n)}[\det]\hat{\otimes}P_{n,p},$$

where $P_{n,p} \in \kappa_{n,p}$. This formula is a source for the construction of exotic elements in the Picard group, see for example [BBS19, HLS21].

2.4. **Conjectures and results.** The following conjecture, dating from the early 1990s and due to Hopkins, summarizes the expected behaviour of the sequence (2.2.9):

Conjecture 2.4.1. *For all heights $n \geq 1$ and primes p:*

(1) $\kappa_{n,p}$ is a finite p-group;
(2) $\mathrm{Pic}^{\mathrm{alg}}_{n,p}$ is finitely generated over $\mathbb{Z}_p$.

Taken together, the two parts of the conjecture imply that $\mathrm{Pic}_{n,p}$ is finitely generated over $\mathbb{Z}_p$. In fact, this is almost an 'if and only if' statement, up to the following

Question 2.4.2. *Is the map ε in (2.2.9) surjective?*

Conjecture 2.4.1 has been confirmed at heights 1 and 2 and all primes p through explicit computations; the following table summarizes the state of the art before the present work:

n	p	$\mathrm{Pic}_{n,p}$	$\mathrm{Pic}^{\mathrm{alg}}_{n,p}$	$\kappa_{n,p}$	Reference
1	≥ 3	$\mathbb{Z}_p \oplus \mathbb{Z}/2(p-1)$	$\mathbb{Z}_p \oplus \mathbb{Z}/2(p-1)$	0	[HMS94]
1	2	$\mathbb{Z}_2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$	$\mathbb{Z}_2 \oplus (\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	[HMS94]
2	≥ 5	$\mathbb{Z}_p^2 \oplus \mathbb{Z}/2(p^2-1)$	$\mathbb{Z}_p^2 \oplus \mathbb{Z}/2(p^2-1)$	0	Hopkins, [Lad13]
2	3	$\mathbb{Z}_3^2 \oplus \mathbb{Z}/16 \oplus (\mathbb{Z}/3)^2$	$\mathbb{Z}_3^2 \oplus \mathbb{Z}/16$	$(\mathbb{Z}/3)^2$	[Kar10, GHMR15, KS04]
2	2	(?)	$\mathbb{Z}_2^2 \oplus \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^3$	$(\mathbb{Z}/8)^2 \oplus (\mathbb{Z}/2)^3$	[BBG+22], Henn
≥ 3	p	??	??	??	

TABLE 1. Comments: The case $n = 2$ and $p \geq 5$ is due to Hopkins and was later part of Lader's thesis [Lad13]. The computation of $\mathrm{Pic}^{\mathrm{alg}}_{2,2}$ has been completed by Hans-Werner Henn, but at the time of writing, remains unpublished.

The main result of the present paper is a complete calculation of the algebraic Picard group of $\mathrm{Sp}_{K(n)}$, for all heights n and all primes p; in particular, we recover the the computations of $\mathrm{Pic}^{\mathrm{alg}}_{n,p}$ for $n \leq 2$ summarized in the table above. In addition, we provide explicit topological elements which under the comparison map ε generate the algebraic Picard group with the possible exception of the case $n = p = 2$. The following is Theorem A from the introduction:

Theorem 2.4.3. *For any prime p and any height $n \geq 2$, the algebraic Picard group of the category of $K(n)$-local spectra is given by:*

$$\mathrm{Pic}^{\mathrm{alg}}_{n,p} \cong \begin{cases} \mathcal{Z}_{n,p} \oplus \mathbb{Z}_p & \text{if } p > 2 \text{ and } n \geq 2; \\ \mathcal{Z}_{n,2} \oplus \mathbb{Z}_2 \oplus (\mathbb{Z}/2)^{\oplus 2} & \text{if } p = 2 \text{ and } n > 2; \\ \mathcal{Z}_{n,2} \oplus \mathbb{Z}_2 \oplus (\mathbb{Z}/2)^{\oplus 2} \oplus \mathbb{Z}/2 & \text{if } p = 2 \text{ and } n = 2, \end{cases}$$

with $\mathcal{Z}_{n,p}$ as in Notation 2.3.1. For all primes p, the summands $\mathcal{Z}_{n,p}$ and $\mathbb{Z}_p$ are generated by $\varepsilon(\Sigma^1 \mathbb{S}_{K(n)})$ and $\varepsilon(\mathbb{S}_{K(n)}[\det]^{\otimes(p-1)})$, respectively. For $p = 2$, generators for the additional $(\mathbb{Z}/2)^{\oplus 2}$ summand are described in Section 2.8.

In the range where the comparison map ε is an isomorphism (Proposition 2.2.7), we thus obtain a complete computation of the Picard group of $\mathrm{Sp}_{K(n)}$; this is Theorem B.

Corollary 2.4.4. *When $2p - 2 > \max(n^2 - 1, 2n)$, there is an isomorphism*

$$\mathrm{Pic}_{n,p} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}/(2p^n - 2),$$

with generators given by $\Sigma \mathbb{S}_{K(n)}$ and $\mathbb{S}_{K(n)}[\det]^{\otimes(p-1)}$.

Another direct consequence of Theorem 2.4.3 is the surjectivity of the comparison map in all cases except for possibly the anomalous one when $n = p = 2$; see also Remark 2.8.2.

Corollary 2.4.5. *For all heights $n \geq 1$ and all primes p with the possible exception of $n = p = 2$, the comparison map $\varepsilon\colon \mathrm{Pic}_{n,p} \to \mathrm{Pic}^{\mathrm{alg}}_{n,p}$ is surjective.*

The goal of the remainder of this section is to reduce Theorem 2.4.3 to a computation in p-adic geometry, stated as Theorem 2.7.1 below, which we then carry out in the rest of this paper.

2.5. **The even algebraic local Picard group.** We now begin our reduction of Theorem 2.4.3 to a purely cohomological statement. Observe that the $K(n)$-local Picard group and its algebraic counterpart are naturally $\mathbb{Z}/2$-graded, since E_* is concentrated in even degrees and is 2-periodic. Let $\mathrm{Pic}^{\mathrm{alg},0}_{n,p}$ be the *even algebraic Picard group*, consisting of those invertible Morava modules whose underlying E_*-module is *even*, i.e., concentrated in even degrees. Thus there is a short exact sequence

$$0 \longrightarrow \mathrm{Pic}^{\mathrm{alg},0}_{n,p} \longrightarrow \mathrm{Pic}^{\mathrm{alg}}_{n,p} \longrightarrow \mathbb{Z}/2 \longrightarrow 0. \tag{2.5.1}$$

The even algebraic Picard group has a simple description in terms of continuous group cohomology. Let $M \in \mathrm{Pic}^{\mathrm{alg},0}_{n,p}$ be an even invertible Morava module. We can think of M as an invertible A_n-module with continuous $\mathbb{G}_n$-action. Since A_n is a local ring, the underlying A_n-module of M is free of rank 1, say with generator m. Define a continuous 1-cocycle $\phi_M\colon \mathbb{G}_n \to A_n^*$ by

$$g \cdot m = \phi_M(g)m$$

for any $g \in \mathbb{G}_n$. The class of ϕ_M in $H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^*)$ does not depend on the chosen generator. Conversely, a continuous cocycle in $H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^*)$ determines an even Morava module by the same formula, and we obtain:

Proposition 2.5.2. *The assignment $M \mapsto \phi_M$ induces an isomorphism*

$$\mathrm{Pic}^{\mathrm{alg},0}_{n,p} \cong H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^*).$$

Remark 2.5.3. Let $\mathrm{Pic}^0_{n,p}$ be the even part of the local Picard group, i.e., those invertible $K(n)$-local spectra X with even Morava module $E_*(X)$. We can translate the Morava modules of the two distinguished spheres of Section 2.3 into a 1-cocycle description. Indeed, for each prime p and height n, under the composite

$$\mathrm{Pic}^0_{n,p} \xrightarrow{\varepsilon} \mathrm{Pic}^{\mathrm{alg},0}_{n,p} \xrightarrow[\cong]{\phi} H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^*)$$

we have:

(1) since $\varepsilon(\Sigma^2\mathbb{S}_{K(n)}) \cong \mathrm{Lie}\, H$, it maps under ϕ to the class of the cocycle $t_0\colon \mathbb{G}_n \to A_n^*$, where t_0 is determined by the formula $g \cdot u = t_0(g)u$ for $g \in \mathbb{G}_n$ and u a generator of $\pi_{-2}E_n = \mathrm{Lie}\, H$.
(2) by construction, $\mathbb{S}_{K(n)}[\det]$ maps to the composition of the determinant map $\det\colon \mathbb{G}_n \to \mathbb{Z}_p^*$ with the inclusion map $\mathbb{Z}_p^* \to A_n^*$.

In order to better understand these elements and in particular their relation, it will be useful to separate the *prime-to-p* part of $\mathrm{Pic}^{\mathrm{alg},0}_{n,p}$ from its *principal* part. Recall that we consider Morava E-theory based on $\bar{\mathbb{F}}_p$. As a topological ring,

$$A_n \cong W[\![u_1, \dots, u_{n-1}]\!],$$

where $W = W(\overline{\mathbb{F}}_p)$ is the ring of Witt vectors. The ring A_n is endowed with the $\mathfrak{m}$-adic topology, where $\mathfrak{m} = (p, u_1, \dots, u_{n-1})$ is the maximal ideal. The quotient map $A_n \to A_n/\mathfrak{m} \cong \overline{\mathbb{F}}_p$ admits a multiplicative splitting via the Teichmüller lift $\overline{\mathbb{F}}_p \to W$. Thus there is a $\mathbb{G}_n$-equivariant isomorphism of topological groups:

$$A_n^* \cong A_n^{**} \oplus \overline{\mathbb{F}}_p^*,$$

where $A_n^{**} = 1 + \mathfrak{m}$. We call A_n^{**} the group of *principal units*; principal units will appear in various contexts throughout the article; we will always use the double-star notation to refer to them.

Accordingly, the even algebraic Picard group decomposes into prime-to-p and principal parts:

$$\mathrm{Pic}^{\mathrm{alg},0}_{n,p} \cong H^1_{\mathrm{cts}}(\mathbb{G}_n, \overline{\mathbb{F}}_p^*) \oplus H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**}). \tag{2.5.4}$$

With respect to this decomposition, let us write

$$\varepsilon = \varepsilon^p \oplus \varepsilon_p,$$

where ε^p, resp. ε_p, refers to the composition of ε with the projection onto the prime-to-p part, resp. the principal part.

2.6. **The prime-to-p part and a relation.** We begin by determining the prime-to-p part of $\mathrm{Pic}_{n,p}^{\mathrm{alg},0}$.

Lemma 2.6.1. *There are isomorphisms*

$$H^1_{\mathrm{cts}}(\mathbb{G}_n, \overline{\mathbb{F}}_p^*) \cong \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)}(\mathbb{F}_{p^n}^*, \overline{\mathbb{F}}_p^*) \cong \mathbb{Z}/(p^n-1).$$

Proof. For the proof, let us abbreviate $\Gamma = \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. We write σ for the Frobenius automorphism, so that σ generates Γ as a profinite group. We recall here some facts about the structure of the Morava stabilizer group $\mathbb{G}_n$. It is an extension

$$0 \to \mathcal{O}_D^* \to \mathbb{G}_n \to \Gamma \to 0,$$

where $\mathcal{O}_D = \mathrm{End}\, H_n$ is the ring of integers in a central simple algebra D of invariant $1/n$ over $\mathbb{Q}_p$. The ring $\mathcal{O}_D$ has a unique two-sided maximal ideal J, with quotient ring $\mathbb{F}_{p^n}$. Let $\mathcal{O}_D^{**} = 1 + J$ be the group of principal units; then $\mathcal{O}_D^{**}$ is a normal pro-p subgroup of $\mathbb{G}_n$, with quotient

$$\mathbb{G}_n/\mathcal{O}_D^{**} \cong \mathbb{F}_{p^n}^* \rtimes \Gamma.$$

The pro-p subgroup $\mathcal{O}_D^{**} \subset \mathbb{G}_n$ acts trivially on $\overline{\mathbb{F}}_p^*$, on which p acts invertibly, so we have an isomorphism for all $i \geq 0$:

$$H^i_{\mathrm{cts}}(\mathbb{G}_n, \overline{\mathbb{F}}_p^*) \cong H^i_{\mathrm{cts}}(\mathbb{F}_{p^n}^* \rtimes \Gamma, \overline{\mathbb{F}}_p^*).$$

Consider the Lyndon–Hochschild–Serre spectral sequence

$$H^i_{\mathrm{cts}}(\Gamma, H^j(\mathbb{F}_{p^n}^*, \overline{\mathbb{F}}_p^*)) \implies H^{i+j}_{\mathrm{cts}}(\mathbb{G}_n, \overline{\mathbb{F}}_p^*).$$

Since Γ is pro-cyclic, its cohomology vanishes in degrees ≥ 2, so we have a short exact sequence

$$0 \to H^1_{\mathrm{cts}}(\Gamma, \overline{\mathbb{F}}_p^*) \to H^1_{\mathrm{cts}}(\mathbb{G}_n, \overline{\mathbb{F}}_p^*) \to H^1(\mathbb{F}_{p^n}^*, \overline{\mathbb{F}}_p^*)^\Gamma \to 0.$$

The term $H^1_{\mathrm{cts}}(\Gamma, \overline{\mathbb{F}}_p^*)$ is the cokernel of $\sigma - 1$ on $\overline{\mathbb{F}}_p^*$, which is 0. The term $H^1(\mathbb{F}_{p^n}^*, \overline{\mathbb{F}}_p^*)^\Gamma$ is the group of Γ-equivariant homomorphisms $\mathbb{F}_{p^n}^* \to \overline{\mathbb{F}}_p^*$. This is a cyclic group of order $p^n - 1$, generated by the inclusion map $\mathbb{F}_{p^n}^* \hookrightarrow \overline{\mathbb{F}}_p^*$. $\square$

Next, we can determine the fate of $\Sigma^2\mathbb{S}_{K(n)}$ and $\mathbb{S}_{K(n)}[\det]$ in the prime-to-p part.

Proposition 2.6.2. *As a cyclic group of order p^n-1, $H^1_{\mathrm{cts}}(\mathbb{G}_n, \overline{\mathbb{F}}_p^*)$ is generated by $\varepsilon^p(\Sigma^2\mathbb{S}_{K(n)})$. Furthermore, the following relation holds in $H^1_{\mathrm{cts}}(\mathbb{G}_n, \overline{\mathbb{F}}_p^*)$:*

$$\varepsilon^p(\mathbb{S}_{K(n)}[\det]) = \frac{p^n-1}{p-1}\varepsilon^p(\Sigma^2\mathbb{S}_{K(n)}).$$

Proof. We use the identification of Lemma 2.6.1 and claim that this group is generated by the prime-to-p part of $\varepsilon(\Sigma^2\mathbb{S}_{K(n)})$. Recall from Section 2.3 that $\varepsilon(\Sigma^2\mathbb{S}_{K(n)})$ is the Morava module $\mathrm{Lie}\, H_n^{\mathrm{univ}}$, where H_n^{univ} is the universal deformation of the (unique up to isomorphism) formal group $H_n \cong H_n^{\mathrm{univ}} \otimes_{A_n} \overline{\mathbb{F}}_p$ of height n over $\overline{\mathbb{F}}_p$. The prime-to-p part of $\varepsilon(\Sigma^2\mathbb{S}_{K(n)})$ is the class in $H^1_{\mathrm{cts}}(\mathbb{G}_n, \overline{\mathbb{F}}_p^*)$ represented by the $\mathbb{G}_n$-equivariant $\overline{\mathbb{F}}_p$-vector space $\mathrm{Lie}\, H_n^{\mathrm{univ}} \otimes_{A_n} \overline{\mathbb{F}}_p \cong \mathrm{Lie}\, H_n$. All we need to know about the $\mathbb{G}_n$-action on $\mathrm{Lie}\, H_n$ is that the subgroup $\mathcal{O}_D^*$ acts on it through the composition $\mathcal{O}_D^* \to \mathbb{F}_{p^n}^* \hookrightarrow \overline{\mathbb{F}}_p^*$. Tracing through the calculation of $H^1_{\mathrm{cts}}(\mathbb{G}_n, \overline{\mathbb{F}}_p^*)$ in the proof of Lemma 2.6.1, we see that it is generated by the class of $\mathrm{Lie}\, H_n^{\mathrm{univ}}$.

With respect to the isomorphism $H^1_{\text{cts}}(\mathbb{G}_n, \overline{\mathbb{F}}_p^*) \cong \operatorname{Hom}_\Gamma(\mathbb{F}_{p^n}^*, \overline{\mathbb{F}}_p^*)$, the class $\varepsilon^p(\mathbb{S}_{K(n)}[\det])$ corresponds to the norm map $\mathbb{F}_{p^n}^* \to \mathbb{F}_p^* \hookrightarrow \overline{\mathbb{F}}_p^*$. The norm map is $(p^n-1)/(p-1)$ times the inclusion map, giving the relation claimed in the second part of the proposition. □

The proposition also allows us to compute the local degree of $\mathbb{S}_{K(n)}[\det]$, as we shall explain next. First, we will denote the even Picard group of the $K(n)$-local category by $\operatorname{Pic}^0_{n,p}$; by definition, this is the subgroup of $\operatorname{Pic}_{n,p}$ consisting of those elements that have even Morava module. Restricting the local degree from Proposition 2.1.3 to the even part, it factors through the inclusion $\mathbb{Z}/(p^n-1) \subseteq \mathbb{Z}/2(p^n-1)$ determined by $1 \mapsto 2$. Therefore, we obtain the *even local degree* homomorphism

$$\deg^0_{n,p} \colon \operatorname{Pic}^{\mathrm{alg},0}_{n,p} \longrightarrow \mathbb{Z}/(p^n-1).$$

We observe that the target is normalized so that $\deg^0_{n,p} \varepsilon(\Sigma^2\mathbb{S}_{K(n)}) = 1$. Moreover, the quotient map $A_n^* \to (A_n/\mathfrak{m})^* \cong \bar{\mathbb{F}}_p^*$ is $\mathbb{G}_n$-equivariant, and hence induces a reduction map on continuous cohomology, $q\colon H^1_{\text{cts}}(\mathbb{G}_n, A_n^*) \to H^1_{\text{cts}}(\mathbb{G}_n, \bar{\mathbb{F}}_p^*)$. The next result provides an alternative perspective on the even local degree:

Proposition 2.6.3. *There is a commutative diagram:*

$$\begin{tikzcd}
\operatorname{Pic}^0_{n,p} \arrow[r, "\deg^0_{n,p}"] \arrow[d, "\phi_{(-)}\circ\varepsilon"'] & \mathbb{Z}/(p^n-1) \arrow[d, "\cong"] \\
H^1_{\text{cts}}(\mathbb{G}_n, A_n^*) \arrow[r, "q"] & H^1_{\text{cts}}(\mathbb{G}_n, \bar{\mathbb{F}}_p^*).
\end{tikzcd}$$

Proof. Recall that the kernel of the even local degree map is a pro-p-group by Proposition 2.1.3, while $H^1_{\text{cts}}(\mathbb{G}_n, \bar{\mathbb{F}}_p^*)$ is a group of order prime to p, see Lemma 2.6.1. It follows that the composite $q \circ \phi_{(-)} \circ \varepsilon$ factors through $\deg^0_{n,p}$, i.e., there is a commutative square

$$\begin{tikzcd}
\operatorname{Pic}^0_{n,p} \arrow[r, "\deg^0_{n,p}"] \arrow[d, "\phi_{(-)}\circ\varepsilon"'] & \mathbb{Z}/(p^n-1) \arrow[d] \\
H^1_{\text{cts}}(\mathbb{G}_n, A_n^*) \arrow[r, "q"] & H^1_{\text{cts}}(\mathbb{G}_n, \bar{\mathbb{F}}_p^*).
\end{tikzcd}$$

We claim that the induced right vertical map is an isomorphism. Indeed, it suffices to verify that on a generator of $\mathbb{Z}/(p^n-1)$, which we may take to be $\deg^0_{n,p}(\Sigma^2\mathbb{S}_{K(n)}) = 1$. As already observed in the proof of Proposition 2.6.2, we have that $q\circ\phi_{(-)}\circ\varepsilon(\Sigma^2\mathbb{S}_{K(n)}) = \varepsilon^p(\Sigma^2\mathbb{S}_{K(n)})$ is the canonical inclusion $\mathbb{F}_{p^n}^* \hookrightarrow \overline{\mathbb{F}}_p^*$. This is our preferred generator of the cyclic group $H^1_{\text{cts}}(\mathbb{G}_n, \bar{\mathbb{F}}_p^*)$, which verifies the claim. □

Corollary 2.6.4. *The even local degree of* $\mathbb{S}_{K(n)}[\det]$ *is* $\frac{p^n-1}{p-1}$. *In particular, we have*

$$\varepsilon^p(\mathbb{S}_{K(n)}[\det]^{\otimes(p-1)}) = 0.$$

Proof. This follows from Proposition 2.6.2 and Proposition 2.6.3. □

2.7. **A reduction of Theorem 2.4.3.** The next result, whose proof will occupy the remaining sections of this paper, describes the principal part of the even algebraic Picard group. Assuming it, we can then deduce Theorem 2.4.3 up to some special features for the prime 2 that are treated separately in Section 2.8.

Theorem 2.7.1. *Let p be a prime $n \geq 2$ some height. There are isomorphisms*

$$H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**}) \cong \begin{cases} \mathbb{Z}_p^2 & \text{if } p > 2; \\ \mathbb{Z}_2^2 \oplus (\mathbb{Z}/2)^{\oplus 2} & \text{if } p = 2 \text{ and } n > 2; \\ \mathbb{Z}_2^2 \oplus (\mathbb{Z}/2)^{\oplus 3} & \text{if } p = 2 \text{ and } n = 2. \end{cases}$$

For all primes p, the free summands are generated by the 1-cocycles t_0 and det *of Remark 2.5.3, respectively. For $p = 2$, generators for the $(\mathbb{Z}/2)^{\oplus 2}$-summand are described in Section 2.8.*

Proof of Theorem 2.4.3, assuming Theorem 2.7.1. By Proposition 2.5.2, Theorem 2.7.1, Lemma 2.6.1, and (2.5.4), we have isomorphisms

$$\psi^0_{n,p} \colon \operatorname{Pic}^{\mathrm{alg},0}_{n,p} \cong H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^*) \cong \begin{cases} \mathbb{Z}_p^2 \oplus \mathbb{Z}/(p^n-1) & \text{if } p > 2; \\ \mathbb{Z}_2^2 \oplus \mathbb{Z}/(p^n-1) \oplus (\mathbb{Z}/2)^{\oplus 2} & \text{if } p = 2 \text{ and } n > 2; \\ \mathbb{Z}_2^2 \oplus \mathbb{Z}/(p^n-1) \oplus (\mathbb{Z}/2)^{\oplus 3} & \text{if } p = 2 \text{ and } n = 2. \end{cases}$$

Since the 2-torsion part for $p = 2$ is dealt with separately in Section 2.8 below, we will ignore these summands for the remainder of this proof. Keeping in mind Remark 2.5.3, it follows from the identifications of the generators in Theorem 2.7.1 together with Corollary 2.6.4 that the group $\mathbb{Z}_p \oplus \mathbb{Z}/(p^n-1)$ is generated by $\varepsilon(\Sigma^2\mathbb{S}_{K(n)})$, while the other $\mathbb{Z}_p$ summand is generated by $\varepsilon(\mathbb{S}_{K(n)}[\det])$.

In order to deduce Theorem 2.4.3 from this description, it remains to solve the extension problem of (2.5.1). To this end, recall from Section 2.3 that we have constructed a map

$$\psi_{n,p} \colon \mathcal{Z}_{n,p} \oplus \mathbb{Z}_p \xrightarrow{(\Sigma^1\mathbb{S}_{K(n)}, \mathbb{S}_{K(n)}[\det]^{\otimes(p-1)})} \operatorname{Pic}^{\mathrm{alg}}_{n,p}, \, b$$

whose two components send the distinguished generators of $\mathcal{Z}_{n,p}$ and $\mathbb{Z}_p$ to $\varepsilon(\Sigma^1\mathbb{S}_{K(n)})$ and $\varepsilon(\mathbb{S}_{K(n)}[\det]^{\otimes(p-1)})$, respectively. This map fits into a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & (\mathbb{Z}_p \oplus \mathbb{Z}/(p^n-1)) \oplus \mathbb{Z}_p \arrow[r, "{(2,1)}"] \arrow[d, "\psi^0_{n,p}"'] & \mathcal{Z}_{n,p} \oplus \mathbb{Z}_p \arrow[r] \arrow[d, "\psi_{n,p}"] & \mathbb{Z}/2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \operatorname{Pic}^{\mathrm{alg},0}_{n,p} \arrow[r] & \operatorname{Pic}^{\mathrm{alg}}_{n,p} \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0, \end{tikzcd}$$

where the top left map is induced by multiplication by 2 resp. the identity map on the two cyclic summands. The left vertical map is injective with cokernel $(\mathbb{Z}/2)^{\oplus e(n,p)}$ for

$$e(n,p) = \begin{cases} 0, & p \neq 2; \\ 2, & p = 2 \text{ and } n \geq 3; \\ 3, & p = 2 \text{ and } n = 2. \end{cases}$$

Tracing $\varepsilon(\Sigma^1\mathbb{S}_{K(n)})$ in the right square, we deduce that the right vertical map is an isomorphism. Therefore, the snake lemma shows that $\psi_{n,p}$ is an isomorphism with cokernel $(\mathbb{Z}/2)^{\oplus e(n,p)}$, as desired. □

2.8. **Addenda and anomalies for $p = 2$.** The case of the even prime has some oddities, which we discuss separately in this subsection. The description of $\operatorname{Pic}^{\mathrm{alg},0}_{n,p}$ in Theorem 2.4.3 implies that, at least if $p \neq 2$, the comparison map $\varepsilon \colon \operatorname{Pic}_{n,p} \to \operatorname{Pic}^{\mathrm{alg}}_{n,p}$ is surjective, since $\operatorname{Pic}^{\mathrm{alg}}_{n,p}$ is generated as a profinite group by the images of $\Sigma\mathbb{S}_{K(n)}$ and $\mathbb{S}_{K(n)}[\det]$ under ε. For completeness' sake we discuss the surjectivity of ε when $p = 2$, referring to Remark 2.8.2 for the anomalous case $n = p = 2$.

Proposition 2.8.1. *The comparison map* $\varepsilon\colon \mathrm{Pic}_{n,2} \to \mathrm{Pic}^{\mathrm{alg}}_{n,2}$ *is surjective if* $n > 2$.

Proof. Let $n > 2$. In light of Theorem 2.4.3, it remains to find $K(n)$-local spectra which realize the 2-torsion in $\mathrm{Pic}^{\mathrm{alg}}_{n,2}$, i.e., to find topological lifts for the non-trivial elements in $\mathbb{Z}/2 \oplus \mathbb{Z}/2 = \mathrm{Pic}^{\mathrm{alg}}_{n,2}[2]$. This has already been achieved in [CSY21]; for the convenience of the reader, we sketch the construction. There is a cyclotomic $(\mathbb{Z}/8)^*$-Galois extension $\mathbb{S}_{K(n)} \to \mathbb{S}_{K(n)}[\omega_n^8]$, obtained from $\mathbb{S}_{K(n)}[B^n\mathbb{Z}/8]$ by splitting off a certain idempotent. By [CSY21, Section 5.3], the cofibers of the unit maps of the three $\mathbb{Z}/2$-Galois subextensions of $\mathbb{S}_{K(n)}[\omega_n^8]$ then give the three non-zero elements in $\mathrm{Pic}^{\mathrm{alg}}_{n,2}[2]$. $\square$

Remark 2.8.2. Combining Theorem 2.4.3 and Proposition 2.8.1 show that the comparison map ε is surjective in all cases except potentially when $(n, p) = (2, 2)$. We currently do not know if the generator of the third, "anomalous" $\mathbb{Z}/2$-summand that arises in the algebraic Picard group $\mathrm{Pic}^{\mathrm{alg}}_{2,2}$ can be realized topologically. This question has also been studied in unpublished work of Hans-Werner Henn.

2.9. **A toy case: height 1.** For completeness and because we will make use of the following lemma later in the proof of Theorem 2.7.1, we briefly include a discussion of the case $n = 1$.

Lemma 2.9.1. *There are isomorphisms*

$$H^1_{\mathrm{cts}}(\mathbb{G}_1, A_1^{**}) \cong \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^*, \mathbb{Z}_p^{**}) \cong \begin{cases} \mathbb{Z}_p & \textit{if } p > 2; \\ \mathbb{Z}_2 \oplus (\mathbb{Z}/2)^{\oplus 2} & \textit{if } p = 2. \end{cases}$$

Proof. Recall that $A_1 \cong W(\overline{\mathbb{F}}_p)$, equipped with its canonical action by the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$. We first claim that the inclusion $\mathbb{Z}_p^{**} \to A_1^{**}$ induces an equivalence $\mathbb{Z}_p^{**} \cong (A_1^{**})^{h\hat{\mathbb{Z}}}$. Indeed, taking ϕ to be a topological generator of $\hat{\mathbb{Z}}$, what we need to prove is that there is a short exact sequence

$$0 \to \mathbb{Z}_p^{**} \to W(\overline{\mathbb{F}}_p)^{**} \xrightarrow{\mathrm{Id}-\phi} W(\overline{\mathbb{F}}_p)^{**} \to 0.$$

Using the compatible p-adic filtrations on $\mathbb{Z}_p$ and A_1, it suffices to show that the corresponding sequence of associated graded pieces

$$0 \to \mathbb{F}_p \to \overline{\mathbb{F}}_p \xrightarrow{\mathrm{Id}-\phi} \overline{\mathbb{F}}_p \to 0$$

is short exact, which is indeed the case. It follows that there are isomorphisms

$$H^1_{\mathrm{cts}}(\mathbb{G}_1, A_1^{**}) \cong H^1((A_1^{**})^{h(\mathbb{Z}_p^* \times \hat{\mathbb{Z}})}) \cong H^1((\mathbb{Z}_p^{**})^{h\mathbb{Z}_p^*}) \cong \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^*, \mathbb{Z}_p^{**})$$

The calculation of $\mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^*, \mathbb{Z}_p^{**})$ is routine. $\square$

Remark 2.9.2. The calculation in the proof of Lemma 2.9.1 recovers the computation of the algebraic Picard group at height 1 due to Hopkins, Mahowald, and Sadofsky [HMS94]. Indeed, combined with Lemma 2.6.1 and Proposition 2.5.2, we get the even algebraic Picard group $\mathrm{Pic}^{0,\mathrm{alg}}_{1,p}$. The extension problem (2.5.1) is then resolved as in the proof of Theorem 2.7.1. We obtain isomorphisms

$$\mathrm{Pic}^{\mathrm{alg}}_{1,p} \cong \begin{cases} \mathbb{Z}_p \oplus \mathbb{Z}/(2p-2) & \text{if } p > 2; \\ \mathbb{Z}_2 \oplus (\mathbb{Z}/2)^{\oplus 2} & \text{if } p = 2. \end{cases}$$

For odd primes, the algebraic Picard group is generated by $\varepsilon(\Sigma^1\mathbb{S}_{K(n)})$; for the even prime, the same class generates the $\mathbb{Z}_2$-summand. Explicit generators for the 2-torsion part of $\mathrm{Pic}^{\mathrm{alg}}_{1,2}$ can again be constructed as in the proof of Proposition 2.8.1; the original reference is [HMS94, Sections 4 and 5]. For $p > 2$, the comparison map gives an isomorphism $\mathrm{Pic}_{1,p} \cong \mathrm{Pic}^{\mathrm{alg}}_{1,p}$, while the exotic part for $p = 2$ is a $\mathbb{Z}/2$, see [HMS94].

3. Moduli of formal groups

We have reduced our main topological result to an algebraic one, namely Theorem 2.7.1, which describes the continuous cohomology group $H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**})$. Its proof will require a dive into the theory of moduli of formal groups, especially those modern aspects of the theory which meet p-adic geometry and especially perfectoid spaces and diamonds. The star player in the proof is $\widehat{\mathcal{M}}^n_{\mathrm{FG}}$, the completion of the moduli stack of formal groups along the height n locus in characteristic p. It enters the picture via Proposition 3.1.2, which states that $\widehat{\mathcal{M}}^n_{\mathrm{FG}}$ is the quotient of the formal scheme $\operatorname{Spf} A_n$ by $\mathbb{G}_n$.

In this section we offer the reader an exposition of the theory of moduli of formal groups and present two results required for the proof of Theorem 2.7.1. Neither result is entirely original; our work here is more or less a direct adaptation of material appearing in [SW13] and [SW20]. Both results concern the "diamond generic fiber" of $\widehat{\mathcal{M}}^n_{\mathrm{FG}}$, which we call $\mathfrak{M}_n$. In a nutshell, $\mathfrak{M}_n$ classifies formal groups of height n over perfectoid spaces of characteristic 0.

The first result is Theorem 3.7.3, which gives an alternative description of $\mathfrak{M}_n$ in terms of linear algebra objects ("modifications of vector bundles on the Fargues–Fontaine curve"). Faltings' isomorphism between the Lubin–Tate and Drinfeld towers appears within Theorem 3.7.3.

The second result is Theorem 3.8.1, which leverages the linear-algebraic description of $\mathfrak{M}_n$ to construct a *determinant morphism* $\mathfrak{M}_n \to \mathfrak{M}_1$. This result states that if H is a 1-dimensional formal group of height n over a perfectoid space S, there exists a functorially associated 1-dimensional formal group $\bigwedge^n H$ of height 1 over S, whose associated linear algebra objects (the Tate and Dieudonné modules) are the top exterior powers of those of H. Ultimately this result will help us understand the class of the determinant sphere in $H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**})$.

3.1. The stack $\widehat{\mathcal{M}}^n_{\mathrm{FG}}$ of height n formal groups.

For an introduction to the theory of moduli of formal groups aimed at the algebraic topologist, see [Goe08]. Unless specified otherwise, all formal groups in this discussion have dimension 1.

Let $\mathcal{M}_{\mathrm{FG}}$ be the moduli stack of formal groups on the category of schemes with the fpqc topology. Then $\mathcal{M}_{\mathrm{FG}}$ admits a uniformization by an affine scheme $\operatorname{Spec} L$, where $L = \mathbb{Z}[t_1, t_2, \dots]$ is the Lazard ring. The scheme $\operatorname{Spec} L$ parameterizes formal group laws, which carry a choice of coordinate T. For each $n \in \mathbb{Z}$, let $[n](T) \in L[\![T]\!]$ be the multiplication-by-n series in the universal formal group law over L.

Fix a prime number p. The base change $\mathcal{M}_{\mathrm{FG}} \otimes \mathbb{Z}_{(p)}$ admits a well-known height stratification by closed subsets $\mathcal{M}^{\geq n}_{FG}$ for $0 \leq n \leq \infty$, characterized as follows. For $n \geq 0$, let v_n be the coefficient of T^{p^n} in $[p](T)$. In particular $v_0 = p$. For $1 \leq n \leq \infty$, the vanishing locus of $(v_0, v_1, \dots, v_{n-1})$ in $\operatorname{Spec} L$ is coordinate-invariant, meaning that it descends to a closed substack of $\mathcal{M}_{\mathrm{FG}} \otimes \mathbb{Z}_{(p)}$, to wit $\mathcal{M}^{\geq n}_{\mathrm{FG}}$. We refer to the locally closed substack $\mathcal{M}^n_{\mathrm{FG}} \coloneqq \mathcal{M}^{\geq n+1}_{FG} \setminus \mathcal{M}^{\geq n}_{FG}$ as the moduli stack of formal groups of height n. It is characterized this way: a formal group H of height n over a characteristic p ring R is one for which the p-torsion $H[p]$ is a locally free group scheme of rank p^n over $\operatorname{Spec} R$.

Formal groups of height n obey an important rigidity condition with respect to isomorphisms. This is proven by Goerss in [Goe08, Theorem 5.23], who attributes it to Lazard; see also [Lur10, Lecture 14].

Lemma 3.1.1. *Let R be a ring of characteristic p, and let H, H' be formal groups over R of height n, where $1 \leq n < \infty$. Let $\underline{\mathrm{Isom}}(H, H')$ denote the functor sending an R-algebra R' to the set of isomorphisms $H \otimes_R R' \to H' \otimes_R R'$. Then $\underline{\mathrm{Isom}}(H, H')$ is represented by an affine scheme which is pro-finite étale over* $\operatorname{Spec} R$.

For each $1 \le n \le \infty$, there exists a formal group H_n of height n over $\overline{\mathbb{F}}_p$. By Lemma 3.1.1 (or by choice of p-typical coordinate when $n = \infty$), H_n is unique up to isomorphism. H_∞ is the formal additive group, while for $1 \le n < \infty$, H_n is p-divisible. Assume henceforth that $1 \le n < \infty$. Recalling that $\mathbb{G}_n = \mathrm{Aut}(H_n, \overline{\mathbb{F}}_p)$, we have an isomorphism of stacks over $\operatorname{Spec} \mathbb{F}_p$:

$$\mathcal{M}^n_{\mathrm{FG}} \cong [\operatorname{Spec} \overline{\mathbb{F}}_p / \mathbb{G}_n].$$

For $1 \le n < \infty$, let $\widehat{\mathcal{M}}^n_{\mathrm{FG}}$ be the completion of $\mathcal{M}_{\mathrm{FG}}$ along the locally closed substack $\mathcal{M}^n_{\mathrm{FG}}$. This is a formal stack over $\operatorname{Spf} \mathbb{Z}_p$. We may describe it by giving its values on rings in which p is nilpotent. Equivalently, we may describe it by giving its values on admissible $\mathbb{Z}_p$-algebras R, meaning that R is complete with respect to the topology defined by an ideal containing p. For a ring R in which p is nilpotent (respectively, an admissible $\mathbb{Z}_p$-algebra R), the R-points of $\widehat{\mathcal{M}}^n_{\mathrm{FG}}$ classify formal groups H over R such that $H \otimes_R R/I$ has height n for some nilpotent ideal (respectively, an open ideal) $I \subset R$ containing p.

The stack $\widehat{\mathcal{M}}^n_{\mathrm{FG}}$ has a standard presentation as a pro-étale quotient of a finite-type formal scheme. Let

$$\mathrm{LT}_n = \mathrm{Def}(H_n)$$

be the deformation space of H_n. This means that for an Artinian local ring R with residue field $\overline{\mathbb{F}}_p$, the R-points of LT_n classify pairs (H, ρ), where H is a formal group over R, and $\rho\colon H_n \to H \otimes_R \overline{\mathbb{F}}_p$ is an isomorphism. Note that such an R is necessarily an algebra over $W = W(\overline{\mathbb{F}}_p)$. Then LT_n is pro-representable by an affine formal scheme: $\mathrm{LT}_n = \operatorname{Spf} A_n$, where A_n is an admissible $\mathbb{Z}_p$-algebra, isomorphic to $W[\![u_1, \dots, u_{n-1}]\!]$ with its $(p, u_1, \dots, u_{n-1})$-adic topology. The formal scheme LT_n admits an action of $\mathbb{G}_n$, which is compatible with the action of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ on W.

Proposition 3.1.2. *We have an isomorphism of stacks of formal schemes in the pro-étale topology over* $\operatorname{Spf} \mathbb{Z}_p$*:*

$$\widehat{\mathcal{M}}^n_{\mathrm{FG}} \cong [\mathrm{LT}_n / \mathbb{G}_n].$$

Proposition 3.1.2 is standard, but we discuss its proof in some detail as a means of building up the necessary machinery for what follows. To do so, it will be important to have a moduli interpretation for the R-points of LT_n, where R is a general W-algebra (not Artinian or even Noetherian) in which p is nilpotent. This interpretation, appearing in the moduli spaces studied by Rapoport–Zink [RZ96], shifts the focus from formal groups of finite height to the larger category of p-divisible groups, which we review in the next subsection.

3.2. p-divisible groups.

These were introduced by Tate [Tat67].

Definition 3.2.1. Let R be a ring. A *p-divisible group* (or *Barsotti–Tate group*) of height n over R is an inductive system $G = \varinjlim_{\nu \ge 1} G^\nu$ of commutative group schemes G^ν over R whose underlying scheme is locally free of rank $p^{n\nu}$ over R. It is required that $G^\nu \to G^{\nu+1}$ induces an isomorphism of G^ν with the p^ν-torsion in $G^{\nu+1}$. Let BT_R denote the category of p-divisible groups over R, with the evident notion of morphism; then BT_R is a $\mathbb{Z}_{(p)}$-linear category.

The same axioms applied to an abstract group would force $G = (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus n}$. Indeed, for whatever ring R, the *constant* group $(\mathbb{Q}_p/\mathbb{Z}_p)_R = \varinjlim_{\nu \ge 1} (p^{-\nu}\mathbb{Z}/\mathbb{Z})_R$ is an example of a p-divisible group.

There is a close link between formal groups and p-divisible groups. Let R be an admissible $\mathbb{Z}_p$-algebra, and let H/R be a formal group representing an R-point of $\widehat{\mathcal{M}}^n_{\mathrm{FG}}$: thus there is an open ideal $I \subset R$ containing p such that $H \otimes_R R/I$ has height n. Locally on $\operatorname{Spec} R$ we may choose a coordinate T for H; then in $(R/I)[\![T]\!]$, the multiplication-by-p series is a unit multiple of T^{p^n}. It follows that the torsion $H[p^\nu] = \operatorname{Spec} R[\![T]\!]/[p^\nu]_H(T)$ is a connected free group scheme over R of rank $p^{\nu h}$, and that (globally on $\operatorname{Spec} R$) $H[p^\infty] := \varinjlim_{\nu\geq 1} H[p^\nu]$ is a connected p-divisible group over R.

As an example, if H is the formal multiplicative group over R, then $H[p^\infty] = \mu_{p^\infty}$ is the group of pth power roots of unity over R.

A theorem of Tate [Tat67, §2, Proposition 1] states that if R is a Noetherian local ring of residue characteristic p, then the functor $H \mapsto H[p^\infty]$ is an equivalence between the category of formal groups over R (of whatever dimension) which are p-divisible (meaning that the p-torsion is a locally free group scheme) and the full subcategory of BT_R whose objects are connected p-divisible groups.

A p-divisible group $G = \varinjlim_\nu G^\nu$ over an admissible $\mathbb{Z}_p$-algebra R determines a functor of points valued in abelian groups: if S is an admissible R-algebra with ideal of definition I, define $G^\nu(S) = \varprojlim_m G^\nu(S/I^m)$ and $G(S) = \varinjlim_\nu G^\nu(S)$. The *Lie algebra* of G is defined via the formula

$$\operatorname{Lie} G = \ker\left(G(R[\varepsilon]/\varepsilon^2) \to G(R)\right).$$

It is a locally free R-module, whose rank is by definition the dimension of G. If $G = H[p^\infty]$ for a finite-height formal group H over R, then the functor of points and Lie algebra for G agree with the analogous constructions for H. Via the functor of points it is possible to embed both formal groups and p-divisible groups as full subcategories of the category of fppf sheaves of abelian groups. From that point of view a p-divisible formal group H is identified with its p-divisible group $H[p^\infty]$; nonetheless we shall continue to notationally distinguish the two sorts of objects.

Definition 3.2.2. Let G and G' be p-divisible groups over a ring R. An *isogeny* $f\colon G \to G'$ between p-divisible groups over R is a morphism whose kernel is a finite locally free group scheme over R. If f is an isogeny, its kernel $\ker f$ has rank p^n, where n is a locally constant function on $\operatorname{Spec} R$; we call $n = \operatorname{ht}(f)$ the *height* of f. Let $\mathrm{BT}^0_R = \mathrm{BT}_R \otimes \mathbb{Q}$ and call this the category of p-divisible groups up to isogeny. An isomorphism in BT^0_R is called a *quasi-isogeny*.

Thus the multiplication-by-p map on G is an isogeny whose height equals the height of G. Note that if $f\colon G \to G'$ is an isogeny of height n, then f factors through multiplication by p^n, so that f is also a quasi-isogeny. Conversely, a quasi-isogeny becomes an isogeny when multiplied by a sufficient power of p. There is an extension of the notion of height to quasi-isogenies, so that if f is a quasi-isogeny, its height $\operatorname{ht}(f)$ is a (possibly negative) integer, and if the composition $f' \circ f$ of isogenies is defined, then $\operatorname{ht}(f' \circ f) = \operatorname{ht}(f') + \operatorname{ht}(f)$.

Quasi-isogenies satisfy a rigidity property similar to what appears in Lemma 3.1.1. For p-divisible groups G and G' over a ring R, let $\underline{\mathrm{QIsog}}(G, G')$ be the functor which associates to an R-algebra S the set $\mathrm{QIsog}(G_S, G'_S)$, meaning the set of quasi-isogenies $G \otimes_R S \to G' \otimes_R S$. Then $\underline{\mathrm{QIsog}}(G, G')$ is formally étale over $\operatorname{Spec} R$:

Proposition 3.2.3 ([Dri76])**.** *Let R be a ring in which p is nilpotent, and let $I \subset R$ be a nilpotent ideal. Let G and G' be p-divisible groups over R. The natural map* $\mathrm{QIsog}(G, G') \to \mathrm{QIsog}(G_{R/I}, G'_{R/I})$ *is a bijection.*

Consequently, given p-divisible groups G and G' over R, an isogeny $G \otimes_R R/I \to G' \otimes_R R/I$ will lift to an isogeny $G \to G'$ once multiplied by a sufficiently high power of p.

3.3. **Lubin–Tate space as a moduli space of p-divisible groups.** Recall that H_n is the (unique up to isomorphism) 1-dimensional formal group of height n over $\overline{\mathbb{F}}_p$. Let

$$G_n = H_n[p^\infty],$$

so that G_n is a p-divisible group over $\overline{\mathbb{F}}_p$ of height n. Let $\mathcal{O}_{D_n} = \operatorname{End} H_n \cong \operatorname{End} G_n$, and let $D_n = \mathcal{O}_{D_n}[1/p]$. Then D_n is a division algebra over $\mathbb{Q}_p$ of invariant $1/n$, and $\mathcal{O}_{D_n}$ is its ring of integers. Therefore the group of automorphisms of G_n (as an object in $\mathrm{BT}_{\overline{\mathbb{F}}_p}$) is $\mathcal{O}^*_{D_n}$, and the group of quasi-isogenies of G_n (that is, automorphisms in $\mathrm{BT}^0_{\overline{\mathbb{F}}_p}$) is D_n^*. By Proposition 3.2.3, the functor $\mathrm{QIsog}(G_n, G_n)$ is formally étale over an algebraically closed field; it is therefore the constant group scheme associated to D_n^*.

We can view the Morava stabilizer group $\mathbb{G}_n$ as the group of automorphisms of the pair $(G_n, \overline{\mathbb{F}}_p)$. Let $\mathbb{G}_n^0$ denote the group of automorphisms of the pair $(G_n, \overline{\mathbb{F}}_p)$ in the isogeny category. Explicitly, an element of $\mathbb{G}_n^0$ is a pair (τ, u), where $\tau \in \Gamma_{\mathbb{F}_p}$, and $u\colon G_n \to \tau^* G_n$ is a quasi-isogeny.

The groups discussed here fit into the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_{D_n}^* \arrow[r] \arrow[d] & D_n^* \arrow[r, "\mathrm{ht}"] \arrow[d] & \mathbb{Z} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{G}_n \arrow[r] \arrow[d] & \mathbb{G}_n^0 \arrow[r, "\mathrm{ht}"] \arrow[d] & \mathbb{Z} \arrow[r] & 0 \\
 & \Gamma_{\mathbb{F}_p} \arrow[r, "="'] \arrow[d] & \Gamma_{\mathbb{F}_p} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

in which rows and columns are exact, and $\Gamma_{\mathbb{F}_p} = \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.

The following definition is a special case of the constructions appearing in [RZ96].

Definition 3.3.1. Let RZ_n be the functor which inputs a ring R in which p is nilpotent and outputs the set of triples (G, ι, ρ), where G/R is a p-divisible group, ι is a ring homomorphism $\overline{\mathbb{F}}_p \to R/p$, and

$$\rho\colon G_n \otimes_{\overline{\mathbb{F}}_p, \iota} R/p \to G \otimes_R R/p$$

is a quasi-isogeny.

The homomorphism ι determines a homomorphism $W \to R$, so that there is a natural transformation $\mathrm{RZ}_n \to \mathrm{Spf}\, W$. There is a decomposition of functors over $\mathrm{Spf}\, W$:

$$\mathrm{RZ}_n = \coprod_{h \in \mathbb{Z}} \mathrm{RZ}_n^{(h)},$$

where $\mathrm{RZ}_n^{(h)}$ classifies those (H, ι, ρ) where ρ has constant height h.

There is an action of the group $\mathbb{G}_n$ on RZ_n lying over the action of $\Gamma_{\mathbb{F}_p}$ on $\mathrm{Spf}\, W$: An element $g = (\tau, u) \in D_n^*$ sends (G, ι, ρ) to $(G, \iota \circ \tau^{-1}, \rho \circ u^{-1})$.

Under this action, an element $g \in D_n^*$ induces an isomorphism $\mathrm{RZ}_n^{(h)} \xrightarrow{\sim} \mathrm{RZ}_n^{(h-\mathrm{ht}(g))}$. Therefore all components $\mathrm{RZ}_n^{(h)}$ are isomorphic, and in order to describe RZ_n with its $\mathbb{G}_n^0$-action, it suffices to describe $\mathrm{RZ}_n^{(0)}$ with its $\mathbb{G}_n$-action.

Recall that A_n is the universal deformation of H_n. Let $\mathfrak{m}$ be the maximal ideal of A_n, and let H_n^{univ} be the universal deformation of H_n, so that there is an isomorphism $H_n \cong H_n^{\mathrm{univ}} \otimes_{A_n} A_n/\mathfrak{m}$. Let $G_n^{\mathrm{univ}} = H_n^{\mathrm{univ}}[p^\infty]$, so that we have an isomorphism

$$G_n \cong G_n^{\mathrm{univ}} \otimes_{A_n} A_n/\mathfrak{m}.$$

For each $k \geq 1$, we apply Proposition 3.2.3 over the ring $A_n/(p, \mathfrak{m}^k)$; the above isomorphism lifts to a quasi-isogeny

$$\rho_k\colon G_n \otimes_{A_n} A_n/(p, \mathfrak{m}^k) \to G_n^{\mathrm{univ}} \otimes_{A_n} A_n/(p, \mathfrak{m}^k)$$

of height 0. Thus one obtains a morphism $\operatorname{Spec} A_n/\mathfrak{m}^k \to \mathrm{RZ}_n^{(0)}$ for all $k \geq 1$, or all the same, a morphism $\mathrm{LT}_n = \operatorname{Spf} A_n \to \mathrm{RZ}_n^{(0)}$. This morphism is $\mathbb{G}_n$-equivariant. In fact:

Proposition 3.3.2 ([RZ96, Proposition 3.79])**.** *The morphism* $\mathrm{LT}_n \to \mathrm{RZ}_n^{(0)}$ *is an isomorphism.*

We can now complete the proof of Proposition 3.1.2, which claims that $\widehat{\mathcal{M}}^n_{\mathrm{FG}} \cong [\mathrm{LT}_n/\mathbb{G}_n]$. The universal deformation of H_n determines a $\mathbb{G}_n$-equivariant morphism $\mathrm{LT}_n \to \widehat{\mathcal{M}}^n_{\mathrm{FG}}$. Composing with the isomorphism in Proposition 3.3.2, we obtain a $\mathbb{G}_n$-equivariant morphism $\mathrm{RZ}_n^{(0)} \to \widehat{\mathcal{M}}^n_{\mathrm{FG}}$, which extends to a $\mathbb{G}_n^0$-equivariant morphism $\mathrm{RZ}_n \to \widehat{\mathcal{M}}^n_{\mathrm{FG}}$. To complete the proof, we need to show that this morphism is a pro-étale $\mathbb{G}_n^0$-torsor.

Suppose R is a ring in which p is nilpotent, and that we are given a morphism $\operatorname{Spec} R \to \widehat{\mathcal{M}}^n_{\mathrm{FG}}$, corresponding to a formal group H over R such that $H \otimes_R R/I$ has height n for a nilpotent ideal I containing p. The fiber $F := \operatorname{Spec} R \times_{\widehat{\mathcal{M}}^n_{\mathrm{FG}}} \mathrm{RZ}_n$ is the functor which sends an R-algebra S to the set of pairs (ι, ρ), where $\iota\colon \overline{\mathbb{F}}_p \to S/p$ is a ring homomorphism, and

$$\rho\colon G_n \otimes_{\overline{\mathbb{F}}_p,\iota} S/p \to H[p^\infty] \otimes_R S/p$$

is a quasi-isogeny. Our objective is to show that F is a pro-étale $\mathbb{G}_n^0$-torsor over $\operatorname{Spec} R$.

First we show that $F \to \operatorname{Spec} R$ has pro-étale local sections. Let $R' = R \otimes_{\mathbb{Z}_p} W$, so that $\operatorname{Spec} R' \to \operatorname{Spec} R$ is pro-étale. By Lemma 3.1.1,

$$\underline{\mathrm{Isom}}(H_n \otimes_{\overline{\mathbb{F}}_p} R'/I, H \otimes_R R'/I)$$

is represented by an affine scheme which is pro-finite étale and surjective over $\operatorname{Spec} R'/I$; let $\operatorname{Spec} S$ be the lift of this affine scheme to $\operatorname{Spec} R'$. Let $H_n \otimes_{\overline{\mathbb{F}}_p} S/I \to H \otimes_R S/I$ be the universal isomorphism. By Proposition 3.2.3 applied to the ring S/p, the associated isomorphism of p-divisible groups lifts to a quasi-isogeny $G_n \otimes_{\overline{\mathbb{F}}_p} S/p \to H[p^\infty] \otimes_R S/p$ of height 0, determining a morphism $\operatorname{Spec} S \to F$. Therefore $F \to \operatorname{Spec} R$ admits pro-étale local sections.

Any two sections of F over an R-algebra S differ by a pair (τ, u), where $\tau \in \Gamma_{\mathbb{F}_p}$ and $u\colon G_n \otimes_{\overline{\mathbb{F}}_p} S \to \tau^* G_n \otimes_{\overline{\mathbb{F}}_p} S$ is a quasi-isogeny. The quasi-isogeny u represents an S-point of the $\overline{\mathbb{F}}_p$-scheme $Q(G_n, \tau^* G_n)$. By Proposition 3.2.3, $Q(G_n, \tau^* G_n)$ is formally étale over $\overline{\mathbb{F}}_p$, which (since the latter is algebraically closed) must be the constant scheme associated to the set of quasi-isogenies $G_n \to \tau^* G_n$. Thus any two sections of F differ by an S-point of the constant scheme associated with $\mathbb{G}_n^0$. We have shown that $F \to \operatorname{Spec} R$ is a $\mathbb{G}_n^0$-torsor, which completes the proof.

3.4. **Dieudonné modules.** The theory of Dieudonné modules [Mes72] provides a functor from p-divisible groups to linear algebra objects. It is roughly analogous to the functor that sends an abelian variety to its first de Rham cohomology group. We give here a brief summary.

Let R be a p-adically complete ring, and let G be a p-divisible group of height n over R. We can think of G as defining an fppf sheaf on the category of R-algebras. Let $R' \to R$ be a p-adically complete PD thickening, meaning that R' is p-adically complete and $R' \to R$ is a surjection of rings whose kernel is equipped with a divided power structure. Let G' be any lift of G to R', and let $EG' \to G'$ be the universal vector extension of G'. (A vector extension of G' is a surjective morphism of sheaves of abelian groups whose kernel is isomorphic to a direct sum of copies of the additive group.) The (covariant) *Dieudonné crystal* is defined as

$$M(G)(R' \to R) = \operatorname{Lie} EG'.$$

It is a projective R'-module of rank n. The Dieudonné crystal does not depend on the choice of G', in the strong sense that the formation of $M(G)(R' \to R)$ is functorial in both G and R'.

We will only deal with the case that R/p is a semiperfect ring, which means that the Frobenius endomorphism $\phi\colon R/p \to R/p$ is surjective. In this case there is a universal p-adically complete

PD thickening $A_{\mathrm{cris}}(R) \to R$. The construction goes this way: let $R^\flat = \varprojlim_\phi R/p$ be the "tilt" of R; there is a natural surjective map $W(R^\flat) \to R$, and then $A_{\mathrm{cris}}(R)$ is defined as the PD-hull of this surjection. In this situation we simply write

$$M(G) = M(G)(A_{\mathrm{cris}}(R) \to R),$$

and call this the *Dieudonné module* of G. Considering the quotient $EG' \to G'$, we find an R-linear surjective map

$$\beta_G \colon M(G) \otimes_{A_{\mathrm{cris}}(R)} R \to \operatorname{Lie} G. \tag{3.4.1}$$

Evidently $A_{\mathrm{cris}}(R)$ only depends on R/p, and $M(G)$ only depends on $G \otimes_R R/p$. We also write $B^+_{\mathrm{cris}}(R) = A_{\mathrm{cris}}(R)[1/p]$, so that $M(G)[1/p]$ is a projective $B^+_{\mathrm{cris}}(R)$-module.

The Dieudonné module $M(G)$ comes with two important structures: Frobenius and Verschiebung. To define these, we first assume that R is a semiperfect ring of characteristic p, with Frobenius map ϕ. Let $G^{(p)} = G \otimes_{R,\phi} R$, and let $F \colon G \to G^{(p)}$ be the relative Frobenius morphism. Let $V \colon G^{(p)} \to G$ be Cartier dual to the Frobenius morphism $G^\vee \to G^{\vee,(p)}$; we have the fundamental relations $VF = p$ in $\operatorname{End} G$ and $FV = p$ in $\operatorname{End} G^{(p)}$.

The Frobenius endomorphism induces by functoriality an endomorphism of $A_{\mathrm{cris}}(R)$ which we continue to call ϕ. The morphisms F and V induce maps on Dieudonné modules as in the following definition.

Definition 3.4.2. Let R be p-adically complete ring such that R/p is semiperfect. The category of Dieudonné modules DMod_R is the category of projective $A_{\mathrm{cris}}(R)$-modules M together with $A_{\mathrm{cris}}(R)$-linear maps $F \colon M \otimes_{A_{\mathrm{cris}}(R),\phi} A_{\mathrm{cris}}(R) \to M$ and $V \colon M \to M \otimes_{A_{\mathrm{cris}}(R),\phi} A_{\mathrm{cris}}(R)$ satisfying $VF = p$ and $FV = p$. We let $\mathrm{DMod}^0_R = \mathrm{DMod}_R \otimes \mathbb{Q}$ be the corresponding isogeny category.

We write $G \mapsto M(G)$ (resp., $M^0(G)$) for the Dieudonné module functor $\mathrm{BT}_R \to \mathrm{DMod}_R$ (resp., $\mathrm{BT}^0_R \to \mathrm{DMod}^0_R$).

Remark 3.4.3. In fact $F \colon G \to G^{(p)}$ induces the map V on $M(G)$ and vice versa. This notational awkwardness is avoided when one uses the contravariant Dieudonné module. For $G = (\mathbb{Q}_p/\mathbb{Z}_p)_R$ the constant p-divisible group, the relative Frobenius on G is induced from the identity on G, and so $M(G) = A_{\mathrm{cris}} e$ for a basis element e satisfying $F(e \otimes 1) = pe$ and $V(e) = e \otimes 1$.

Example 3.4.4. In the special case that $R = k$ is a perfect field of characteristic p, we have $A_{\mathrm{cris}}(k) = W(k)$, since the kernel of $W(k) \to k$ is (p) and $p^n/n! \in \mathbb{Z}_p$ for all $n \geq 0$. In this case ϕ is an automorphism of $A_{\mathrm{cris}}(k)$. A classical result of Fontaine [Fon77] states that the Dieudonné module functor $\mathrm{BT}_k \to \mathrm{DMod}_k$ is an equivalence. In retrospect we may define the p-divisible group $G_n \in \mathrm{BT}_{\overline{\mathbb{F}}_p}$ by its Dieudonné module: $M(G_n) = \mathbb{Z}_p^{\oplus n} \otimes_{\mathbb{Z}_p} W(\overline{\mathbb{F}}_p)$, and $F = F_0 \otimes \phi$, where $F_0 \in \operatorname{End} \mathbb{Z}_p^{\oplus n}$ is the matrix

$$\begin{pmatrix} & & & & 1 \\ p & & & & \\ & p & & & \\ & & \ddots & & \\ & & & p & \end{pmatrix}.$$

For more general semiperfect rings R, the Dieudonné module functor can fail to be an equivalence, but under a weak assumption on R the functor is at least fully faithful up to isogeny.

Theorem 3.4.5 ([SW13])**.** *Assume R is the quotient of a perfect ring of characteristic p by a finitely generated ideal. Then the Dieudonné module functor*

$$M^0 \colon \mathrm{BT}^0_R \to \mathrm{DMod}^0_R$$

is fully faithful.

3.5. **The Fargues–Fontaine curve.** Let R be a perfectoid algebra over $\mathbb{Q}_p$. This means that R is a Banach $\mathbb{Q}_p$-algebra whose subring of power-bounded elements R° is p-adically complete, and there exists an element $p^{1/p} \in R^\circ$ which is (up to a unit) a pth root of p, such that the pth power map $R^\circ/p^{1/p} \to R^\circ/p$ is an isomorphism of rings. It follows from this that R°/p is semiperfect. For instance, if C is a complete algebraically closed field containing $\mathbb{Q}_p$, then C is a perfectoid algebra.

If a p-divisible group G over R° is given, its Dieudonné module $M(G)$ is a projective $A_{\mathrm{cris}}(R^\circ)$-module equipped with a ϕ-semilinear action of Frobenius. Therefore it ought to descend to a vector bundle over the quotient of $\operatorname{Spec} A_{\mathrm{cris}}(R^\circ)$ by ϕ. The Fargues–Fontaine curve is morally something like this quotient, although one must invert p.

Define a graded $\mathbb{Q}_p$-algebra by

$$P_R = \bigoplus_{d\geq 0} B^+_{\mathrm{cris}}(R^\circ)^{\phi=p^d},$$

and a scheme X_R by

$$X_R = \operatorname{Proj} P_R.$$

The quotient map $B^+_{\mathrm{cris}}(R^\circ) \to R$ induces a closed immersion $i\colon \operatorname{Spec} R \to X_R$.

Remark 3.5.1. The foundational work [FF18] is concerned with the "absolute curve" X_C appearing in the case that C is an algebraically closed perfectoid field. Despite not being finitely generated over a field, X_C was discovered to share properties with the Riemann sphere: it is a Noetherian scheme of dimension 1, and furthermore the complement in X_C of any closed point is the spectrum of a PID.

The construction of the Fargues–Fontaine curve globalizes, so that we have a scheme X_S for any perfectoid space S. The reader is warned that there is no morphism $X_S \to S$. Nonetheless, it is still something like a family of curves, in that one gets an absolute curve for each rank 1 geometric point of S.

What we have presented here is the "schematic" curve; there is also an "adic" version which is more suitable to applications in p-adic geometry. The two curves have the same category of quasi-coherent sheaves, by "GAGA for the curve" [Far15].

There is a functor $M \mapsto \mathcal{E}(M)$ from rational Dieudonné modules over R to quasi-coherent sheaves on X_R. For an object M of DMod^0_R, think of F as a ϕ-semilinear map $\phi\colon M \to M$, and then define $\mathcal{E}(M)$ as the quasi-coherent sheaf on X_R associated to the graded P_R-module

$$\bigoplus_{d\geq 0} M^{\phi=p^{d+1}}.$$

The definition has been normalized so that if $M = M^0(\mathbb{Q}_p/\mathbb{Z}_p)$ is the rational Dieudonné module of the constant p-divisible group, then $\mathcal{E}(M)$ is the trivial line bundle $\mathcal{O}_{X_R}$. If M is the rational Dieudonné module of our fixed p-divisible group G_n over $\overline{\mathbb{F}}_p$, and if a ring homomoprhism $\overline{\mathbb{F}}_p \to R^\circ/p$ is given, we obtain a vector bundle $\mathcal{E}(M)$ of rank n over X_R, and we denote this $\mathcal{O}_{X_R}(1/n)$, or simply $\mathcal{O}(1/n)$ if the base curve X_R is understood.

A major accomplishment of [FF18] is the complete classification of vector bundles on the absolute curve X_C, by means of a Harder–Narasimhan formalism. Every vector bundle is isomorphic to a direct sum of vector bundles $\mathcal{O}(\lambda)$ parameterized by rational numbers λ ("*slopes*"). For each λ, the degree and rank of $\mathcal{O}(\lambda)$ are the numerator and denominator of λ in lowest terms, and the automorphism group of $\mathcal{O}(\lambda)$ is the group of units in a central division algebra over $\mathbb{Q}_p$ of invariant λ. In particular, every automorphism of $\mathcal{O}(1/n)$ arises from an automophism of G_n in the isogeny category: $\operatorname{Aut}\mathcal{O}(1/n) \cong D_n^*$.

The study of families of vector bundles on the Fargues–Fontaine curve (that is, vector bundles over X_S for a general perfectoid space S) is very rich, and has been shockingly fruitful in attacking the local Langlands correspondence [FS24]. An important first result [FS24, Theorem I.4.1(i)] is that vector bundles form a stack for the pro-étale topology (in fact this is true for the finer v-topology). Another result concerns families whose geometric fibers are all "*isoclinic*" (admitting only one slope); these are vastly simpler.

Proposition 3.5.2 ([FS24, Theorem I.4.1(v)]). *Let S be a perfectoid space over $\operatorname{Spa} W(\overline{\mathbb{F}}_p)$. Let λ be a rational number whose denominator $d \geq 1$ (in lowest terms) is a divisor of n. The following groupoids are equivalent:*

1. *Vector bundles of rank n over X_S which are everywhere on S isoclinic of slope λ, which is to say isomorphic to $\mathcal{E}_0 = \mathcal{O}(\lambda)^{\oplus(n/d)}$.*
2. *Pro-étale J_λ-torsors over S, where J_λ is the group of units in the unique central simple algebra over $\mathbb{Q}_p$ of degree n and invariant λ.*

The equivalence sends a vector bundle $\mathcal{E}$ to the sheaf $\underline{\mathrm{Isom}}(\mathcal{E}_0, \mathcal{E})$ whose R'-points (for a perfectoid R-algebra R') parameterize isomorphisms between the vector bundles after pullback to $X_{R'}$.

We spell out the consequences of Proposition 3.5.2 in the two cases relevant to us:

In the case $\lambda = 0$, we have $J_\lambda = \mathrm{GL}_n(\mathbb{Q}_p)$. Note that a pro-étale $\mathrm{GL}_n(\mathbb{Q}_p)$-torsor is the same thing as a $\mathbb{Q}_p$-local system of rank n. If T is a $\mathbb{Z}_p$-local system of rank n, let us write $T \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_S}$ for the vector bundle on X_S corresponding to the $\mathbb{Q}_p$-local system $T[1/p]$.

In the case $\lambda = 1/n$, we have $J_\lambda = D_n^*$. As a corollary of the proposition, suppose S is a general perfectoid space, with no structure morphism to $\operatorname{Spa} W(\overline{\mathbb{F}}_p)$. Then there is an equivalence of groupoids between (a) vector bundles of rank n on X_S which are everywhere of slope $1/n$, and (b) pro-étale $\mathbb{G}_n^0$-torsors over S admitting a map to $\operatorname{Spa}(K, W)$; or all the same, sections of the stack $[\operatorname{Spa}(K, W)^\diamond/\mathbb{G}_n^0]$ over S.

3.6. p-divisible groups over a perfectoid ring.

Keep the assumption that R is a perfectoid $\mathbb{Q}_p$-algebra, and let $S = \operatorname{Spa}(R, R^\circ)$. Let G be a p-divisible group of height n over R°. We gather here some of the structures associated with G which live over the Fargues–Fontaine curve X_S.

First we have the *Tate module*

$$TG = \varprojlim_{\nu} G[p^\nu](R^\circ) \cong \mathrm{Hom}_{\mathrm{BT}_R}(\mathbb{Q}_p/\mathbb{Z}_p, G).$$

Here we write $\mathbb{Q}_p/\mathbb{Z}_p$ for the constant p-divisible group over R°. Since R° is integrally closed in R and each $G[p^\nu]$ is finite over $\operatorname{Spec} R^\circ$, the natural map $G[p^\nu](R^\circ) \to G[p^\nu](R)$ is an isomorphism. On the other hand $G[p^\nu]_R$ is an étale group scheme of rank $p^{n\nu}$ over R. This shows that TG can be given the structure of a $\mathbb{Z}_p$-local system of rank n over $S_{\text{proét}}$. Let $TG \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_S}$ be the corresponding vector bundle on X_S which is everywhere on S isomorphic to $\mathcal{O}^{\oplus n}$.

On the other hand we have the rational Dieudonné module $M^0(G)$, which gives rise to a quasi-coherent sheaf on X_S, which we notate $\mathcal{E}(G)$. If G arises from an R°-point of $\widehat{\mathcal{M}}^n_{\mathrm{FG}}$, it means there exists pro-étale locally on $\operatorname{Spec} R^\circ/p$ a quasi-isogeny $G_n \to G$, and thus an isomorphism $M^0(G_n) \to M^0(G)$. Translating to vector bundles, we find that $\mathcal{E}(G)$ is a vector bundle on X_S which is everywhere on S isomorphic to $\mathcal{O}(1/n)$.

A morphism $\mathbb{Q}_p/\mathbb{Z}_p \to G$ representing a section of TG induces a map of vector bundles $\mathcal{E}(\mathbb{Q}_p/\mathbb{Z}_p) \to \mathcal{E}(G)$, which is just a global section of $\mathcal{E}(G)$ since $\mathcal{E}(\mathbb{Q}_p/\mathbb{Z}_p)$ is the trivial line bundle. We have defined a map $TG \to H^0(\mathcal{E}(\mathbb{Q}_p/\mathbb{Z}_p))$ of sheaves of groups on the pro-étale site of S; this is adjoint to a map of vector bundles

$$\alpha_G \colon TG \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_S} \to \mathcal{E}(G). \tag{3.6.1}$$

Proposition 3.6.2. *Let R be a perfectoid algebra over $\mathbb{Q}_p$, and let $S = \mathrm{Spa}(R, R^\circ)$. Let G be a p-divisible group over R°. The morphism α_G fits into an exact sequence of $\mathcal{O}_{X_S}$-modules:*

$$0 \to TG \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_S} \stackrel{\alpha_G}{\to} \mathcal{E}(G) \to i_* \operatorname{Lie} G[1/p] \to 0, \tag{3.6.3}$$

where $i\colon \operatorname{Spec} R \to X_S$ is the closed immersion defined above.

Proof. The injectivity of the map α_G comes from the faithfulness part of Theorem 3.4.5.

The surjective map $\mathcal{E}(G) \to i_* \operatorname{Lie} G[1/p]$ arises via adjunction from the surjective map $i^*\mathcal{E}(G) = M^0(G) \otimes_{B^+_{\mathrm{cris}}(R^\circ)} R \to \operatorname{Lie} G[1/p]$ from (3.4.1).

Next we claim that the sequence in (3.6.3) is actually a complex. A section s of $TG \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_R}$ is represented by a morphism $\mathbb{Q}_p/\mathbb{Z}_p \to G$. The image of s in $\mathcal{E}(G)$ is the image of the canonical generator of $\mathcal{E}(\mathbb{Q}_p/\mathbb{Z}_p) = \mathcal{O}$ under $\mathcal{E}(\mathbb{Q}_p/\mathbb{Z}_p) \to \mathcal{E}(G)$. We now use the commutativity of

$$\begin{tikzcd}
\mathcal{E}(\mathbb{Q}_p/\mathbb{Z}_p) \arrow[r] \arrow[d] & i_* \operatorname{Lie} \mathbb{Q}_p/\mathbb{Z}_p \arrow[d] \\
\mathcal{E}(G) \arrow[r] & i_* \operatorname{Lie} G
\end{tikzcd}$$

together with $\operatorname{Lie} \mathbb{Q}_p/\mathbb{Z}_p = 0$ to conclude that the image of s in $i_* \operatorname{Lie} G$ is zero.

The exactness of (3.6.3) can be checked on every geometric point of S, so let us assume $R = C$ is an algebraically closed field. The result follows from a numerical argument: since α_G is an injective map of vector bundles, its cokernel is torsion of degree equal to $\dim G$; for this to surject onto $i_* \operatorname{Lie} G$ must mean that the cokernel is exactly $i_* \operatorname{Lie} G$. □

3.7. **The diamond stack $\mathfrak{M}_n$.** We return now to the study of the stack $\widehat{\mathcal{M}}^n_{\mathrm{FG}}$ over $\operatorname{Spf} \mathbb{Z}_p$. Our next order of business is to pass from $\widehat{\mathcal{M}}^n_{\mathrm{FG}}$ to its generic fiber to obtain an analytic object living over $\mathbb{Q}_p$.

Definition 3.7.1. Suppose X is a presheaf of sets or groupoids on the category of admissible $\mathbb{Z}_p$-algebras. The *diamond generic fiber* $X^\diamond_{\mathbb{Q}_p}$ of X is the sheafification of the presheaf

$$(R, R^+) \mapsto X(R^+)$$

on perfectoid $(\mathbb{Q}_p, \mathbb{Z}_p)$-algebras with respect to the pro-étale topology.

Thus for a perfectoid space S over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, an object of $X^\diamond_{\mathbb{Q}_p}(S)$ is given by a pro-étale cover of S by affinoids $\mathrm{Spa}(R_i, R_i^+)$, and for each i an object of $X(R_i^+)$, together with a descent datum to S.

Definition 3.7.2. Let

$$\mathfrak{M}_n = \left(\widehat{\mathcal{M}}^n_{FG}\right)^\diamond_{\mathbb{Q}_p},$$

and call this the *diamond stack of 1-dimensional formal groups of height n.*

Suppose S is a perfectoid space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, and let H be an S-point of $\mathfrak{M}_n$. To specify H is to give the data of a covering of $S_{\text{proét}}$ by affinoid perfectoid spaces $\mathrm{Spa}(R, R^+)$ and a formal group over each R^+ representing an R^+-point of $\widehat{\mathcal{M}}^n_{\mathrm{FG}}$. We shall refer to H as a *formal group of height n over $S_{\text{proét}}$*.

We remark here on the following “partial properness” property of $\mathfrak{M}_n$. Suppose (R, R^+) is an affinoid perfectoid space, so that R^+ is an open integrally closed subring of the p-adically complete ring R°. We claim that the natural map $\widehat{\mathcal{M}}^n_{\mathrm{FG}}(R^+) \to \widehat{\mathcal{M}}^n_{\mathrm{FG}}(R^\circ)$ is an equivalence. By Proposition 3.1.2, it is enough to check that $\mathrm{LT}_n(R^+) \to \mathrm{LT}_n(R^\circ)$ is a bijection. This is true because $\mathrm{LT}_n(R^\circ)$ is the $(n-1)$-fold cartesian product of the set of topologically nilpotent elements of R°, and these all lie in R^+ because R^+ is integrally closed in R°. We conclude that

$\mathfrak{M}_n$ is partially proper, meaning that its groupoid of $\mathrm{Spa}(R, R^+)$-points depends only on R and not on the subring $R^+ \subset R$.

Suppose H is a formal group of height n over $S_{\text{proét}}$. Since quasi-coherent $\mathcal{O}$-modules satisfy pro-étale descent, $\operatorname{Lie} H[1/p]$ defines an invertible $\mathcal{O}$-module on the analytic site S_{an}.

Theorem 3.7.3. *The following stacks are equivalent:*

(1) $\mathfrak{M}_n$*;*

(2) the stack Modif_n *whose points over* $S = \mathrm{Spa}(R, R^\circ)$ *classify modifications*

$$0 \to T \otimes \mathcal{O}_{X_S} \to \mathcal{E} \to i_* L \to 0,$$

where:

(a) T *is a* $\mathbb{Z}_p$*-local system on* $S_{\text{proét}}$*,*

(b) $\mathcal{E}$ *is a vector bundle on* X_S *which is isomorphic to* $\mathcal{O}(1/n)$ *everywhere on* S*,*

(c) L *is a locally free* $\mathcal{O}_S$*-module of rank 1;*

(3) the quotient stack $[\mathrm{LT}_{n,K}^\diamond / \mathbb{G}_n]$*;*

(4) the quotient stack $[\mathcal{H}^{n-1,\diamond} / \mathrm{GL}_n(\mathbb{Z}_p)]$*, where* $\mathcal{H}^{n-1}$ *is Drinfeld's symmetric space.*

The equivalence between $\mathfrak{M}_n$ *and* Modif_n *sends a formal group* H *to the exact sequence associated in Proposition 3.6.2 to* $G = H[p^\infty]$*. Furthermore, let* H^{univ} *be the universal formal group over* $(\widehat{\mathcal{M}}^n_{\mathrm{FG}})^\diamond_{\mathbb{Q}_p}$*, so that* $\operatorname{Lie} H^{\text{univ}}[1/p]$ *is an invertible* $\mathcal{O}$*-module on this stack. With respect to the isomorphism to* $[\mathcal{H}^{n-1,\diamond} / \mathrm{GL}_n(\mathbb{Z}_p)]$*,* $\operatorname{Lie} H^{\text{univ}}[1/p]$ *corresponds to the restriction of the* $\mathrm{GL}_n(\mathbb{Z}_p)$*-equivariant tautological line bundle* $\mathcal{O}(1)$ *from* $\mathbb{P}^{n-1}_{\mathbb{Q}_p}$ *to* $\mathcal{H}^{n-1}$*.*

Proof. The equivalence between $[\mathrm{LT}_{n,K}^\diamond / \mathbb{G}_n]$ and $[\mathcal{H}^{n-1,\diamond} / \mathrm{GL}_n(\mathbb{Z}_p)]$ is the "isomorphism between the Lubin–Tate and Drinfeld towers". We phrase it here through the medium of modifications of vector bundles on the Fargues–Fontaine curve. Detailed proofs can be found in [SW20].

The equivalence between (1) and (3) is a consequence of Proposition 3.1.2.

Let $W = W(\overline{\mathbb{F}}_p)$ and $K = W[1/p]$. Let Sht_n be the sheaf whose value on an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ over $\mathrm{Spa}(K, W)$ is the set of modifications

$$0 \to T \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_S} \to \mathcal{O}_{X_S}(1/n) \to i_* L \to 0,$$

where T is a pro-étale $\mathbb{Z}_p$-local system on S, and L is a locally free $\mathcal{O}_S$-module of rank 1. Then Sht_n is an example of a local shtuka space as introduced in [SW20]. There is a morphism $\mathrm{RZ}_n^\diamond \to \mathrm{Sht}_n$ lying over $\mathrm{Spa}(K, W)$, described as follows: if R is a perfectoid K-algebra, and if (G, ρ) represents a $\mathrm{Spa}(R, R^\circ)$-point of $\mathrm{RZ}_n^\diamond$, then ρ induces an isomorphism $\mathcal{O}_{X_S}(1/n) \to \mathcal{E}(G)$, and then Proposition 3.6.2 produces a $\mathrm{Spa}(R, R^\circ)$-point of Sht_n. The map $\mathrm{RZ}_n^\diamond \to \mathrm{Sht}_n$ is an equivalence: this is a special case of [SW20, Theorem 24.2.5].

There is an obvious morphism $\mathrm{Sht}_n \to \mathrm{Modif}_n$, which is a torsor for the automorphism group of the pair $(\overline{\mathbb{F}}_p, \mathcal{O}(1/n))$, which is exactly $\mathbb{G}_n^0$. Therefore

$$\mathrm{Modif}_n \cong [\mathrm{Sht}_n / \mathbb{G}_n^0] \cong [\mathrm{RZ}_n^\diamond / \mathbb{G}_n^0] \cong [\mathrm{LT}_n^\diamond / \mathbb{G}_n],$$

where the last isomorphism derives from Proposition 3.3.2. This is the equivalence between (2) and (3).

For the equivalence between (2) Modif_n and (4) $[\mathcal{H}^{n-1} / \mathrm{GL}_n(\mathbb{Z}_p)]$: We interpret S-points of projective space $\mathbb{P}^{n-1}$ to mean invertible $\mathcal{O}_S$-modules ℓ which are direct summands of $\mathcal{O}_S^{\oplus n}$. Observe that such an ℓ determines a vector bundle $\mathcal{E}$ containing $\mathcal{O}_{X_S}^{\oplus n}$, whose sections are the same except they are allowed to have a simple pole along the hyperplane determined by ℓ. Thus ℓ determines a modification

$$0 \to \mathcal{O}_{X_S}^{\oplus n} \to \mathcal{E} \to i_* L \to 0,$$

where L is an invertible $\mathcal{O}_S$-module. (In fact L is a twist of ℓ by the tangent line to the curve X_C along the point i; this tangent line is canonically a Tate twist of C.) The kernel of $\mathcal{O}_S^{\oplus n} = i^*\mathcal{O}_{X_S}^{\oplus n} \to i^*\mathcal{E}$ is exactly ℓ.

Let $\mathrm{Spa}(C, C^\circ)$ be a geometric point of S, and let $\ell_C \subset C^n$ and $\mathcal{E}_C$ be the pullbacks of those objects to C. The following statements are all equivalent:

(1) The line ℓ_C is contained in a $\mathbb{Q}_p$-rational hyperplane.
(2) There exists a decomposition of $\mathbb{Q}_p^n$ as the direct sum of two nonzero $\mathbb{Q}_p$-linear subspaces, such that ℓ is contained in one of them.
(3) There exists a decomposition of $\mathcal{O}_{X_C}^{\oplus n}$ as the direct sum of two nonzero sub-bundles of slope 0 (i.e., they are both direct sums of $\mathcal{O}_{X_C}$), such that ℓ is contained in the fiber of one of them. (The equivalence of this statement with the last one boils down to $H^0(X_C, \mathcal{O}_{X_C}) = \mathbb{Q}_p$.)
(4) The modified vector bundle $\mathcal{E}$ admits a trivial summand.
(5) The modified vector bundle $\mathcal{E}$ is not isomorphic to $\mathcal{O}(1/n)$.

For the equivalence of the last two statements: a priori $\mathcal{E}$ has degree 1 and rank n, and all its slopes are nonnegative because it contains $\mathcal{O}^{\oplus n}$ as a maximal rank sub-bundle. Therefore $\mathcal{E}$ must be isomorphic to $\mathcal{O}(1/i) \oplus \mathcal{O}^{\oplus(n-i)}$ for some $i = 1, \dots, n$. This contains a nontrivial $\mathcal{O}$-summand if and only if $i < n$.

A morphism $S \to \mathbb{P}^{n-1}$ determines a line ℓ, which determines in turn a modification; the above discussion shows that the modified vector bundle $\mathcal{E}$ is everywhere isomorphic to $\mathcal{O}(1/n)$ if and only if $S \to \mathbb{P}^{n-1}$ factors through Drinfeld symmetric space $\mathcal{H}^{n-1}$, defined as the complement in $\mathbb{P}^{n-1}$ of all $\mathbb{Q}_p$-rational hyperplanes.

Thus we find a morphism $\mathcal{H}^{n-1} \to \mathrm{Modif}_n$ carrying a line ℓ to its corresponding modification and forgetting the trivialization of the $\mathbb{Z}_p$-local system T of rank n. This morphism a torsor for the group $\mathrm{GL}_n(\mathbb{Z}_p)$. $\square$

3.8. **The determinant map $\mathfrak{M}_n \to \mathfrak{M}_1$.** To access the determinant sphere $\mathbb{S}_{K(n)}[\det]$ as a Picard class on $\mathfrak{M}_n$, we will need to understand the role of the determinant in the study of 1-dimensional formal groups.

Theorem 3.8.1. *There exists a determinant morphism*

$$\det\colon \mathfrak{M}_n \to \mathfrak{M}_1$$

making the following diagram commute:

$$\begin{tikzcd}
{[\mathrm{LT}^\diamond_{n,K}/\mathbb{G}_n]} \arrow[r, "\cong"] \arrow[d, "\det_{\mathrm{LT}}"'] & \mathfrak{M}_n \arrow[r, "\cong"] \arrow[d, "\det"'] & {[\mathcal{H}^{n-1,\diamond}/\mathrm{GL}_n(\mathbb{Z}_p)]} \arrow[d, "\det_{\mathcal{H}}"] \\
{[\mathrm{LT}^\diamond_{1,K}/\mathbb{G}_1]} \arrow[r, "\cong"'] & \mathfrak{M}_1 \arrow[r, "\cong"'] & {[\mathcal{H}^{0,\diamond}/\mathbb{Z}_p^*].}
\end{tikzcd}$$

In the diagram:

(1) *the map labeled* $\det_{\mathrm{LT}}$ *is induced from:*
  (a) *the structure map* $\mathrm{LT}_{n,K} \to \mathrm{LT}_{1,K} \cong \mathrm{Spa}(K, W)$*;*
  (b) *the determinant map* $\mathbb{G}_n \to \mathbb{G}_1$*, explained in the remarks below,*
(2) *the map labeled* $\det_{\mathcal{H}}$ *is induced from:*
  (a) *the structure map* $\mathcal{H}^{n-1} \to \mathcal{H}^0 \cong \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$*;*
  (b) *the determinant map* $\mathrm{GL}_n(\mathbb{Z}_p) \to \mathbb{Z}_p^*$*.*

Before beginning the proof, we offer some remarks. For a perfectoid space S over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, the S-points of the morphism $\det\colon \mathfrak{M}_n \to \mathfrak{M}_1$ should input a formal group H of dimension 1 and height n over $S_{\text{proét}}$, and output a formal group $\bigwedge^n H$ over $S_{\text{proét}}$ of dimension 1 and height 1. The crucial computation that makes this work comes from the case of Dieudonné modules over $\overline{\mathbb{F}}_p$: if $M(G_n)$ is the Dieudonné module of G_n as described in Example 3.4.4, then the top exterior power $\bigwedge^n_{W(\overline{\mathbb{F}}_p)} M(G_n)$ is isomorphic to $M(G_1)$. Consequently we have an isomorphism of line bundles on X_S for any perfectoid space S: $\bigwedge^n \mathcal{O}_{X_S}(1/n) \cong \mathcal{O}_{X_S}(1)$.

Considering the action of $\mathbb{G}_n = \operatorname{Aut}(G_n, \overline{\mathbb{F}}_p)$ on $\bigwedge^n_{W(\overline{\mathbb{F}}_p)} M(G_n) \cong M(G_1)$, we find a determinant map

$$\det\colon \mathbb{G}_n \to \mathbb{G}_1.$$

This map is isomorphic to the profinite completion of the reduced norm map $D_n^* \to \mathbb{Q}_p^*$, where $D_n = \operatorname{End} G_n \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Remark 3.8.2. One might wonder whether general exterior powers exist beyond the determinant. For a general p-divisible group G of height n over a perfect field k of characteristic p, the exterior powers $\bigwedge^i_{W(k)} M(G)$ are Dieudonné modules if and only if $\dim G \leq 1$, so one can only expect exterior powers of G itself to exist in this case. A construction of exterior powers of p-divisible groups of dimension ≤ 1 over the ring of integers in a nonarchimedean local field appeared previously in [Hed15].

Proof of Theorem 3.8.1. Let S be a perfectoid space over $\operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, and let H be a formal group of dimension 1 and height n over $S_{\text{proét}}$, representing an S-point of $\mathfrak{M}_n$. Let

$$0 \to T \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_S} \overset{\alpha_H}{\to} \mathcal{E}(H) \to i_* L \to 0$$

represent the corresponding S-point of Modif_n.

The determinant of α_H is an injective morphism of line bundles:

$$\det \alpha_H \colon \bigwedge^n_{\mathbb{Z}_p} T \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_S} \to \bigwedge^n \mathcal{E}(H),$$

and $\bigwedge^n \mathcal{E}(H)$ is locally isomorphic to $\mathcal{O}_{X_S}(1)$. The cokernel of $\det \alpha_H$ has degree 1 at every geometric point of S and is supported on the image of i, so it can be none other than the line bundle $i_* i^* \bigwedge^n \mathcal{E}$. Therefore $\det \alpha_H$ determines an S-point of Modif_1, corresponding to a formal group $\bigwedge^n H$ of height 1 over $S_{\text{proét}}$.

For any $n \geq 1$, the stack $[\operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)^\diamond / \operatorname{GL}_n(\mathbb{Z}_p)]$ classifies $\mathbb{Z}_p$-local systems of rank n. The commutativity of the right square of the diagram is exactly the statement that there is a functorial isomorphism of $\operatorname{GL}_1(\mathbb{Z}_p) = \mathbb{Z}_p^*$-torsors

$$T\left(\bigwedge^n H\right) \cong \bigwedge^n_{\mathbb{Z}_p} TH,$$

and this is in turn an artifact of our construction of $\bigwedge^n H$.

Similarly, the commutativity of the left square of the diagram traces to the statement that there is a functorial isomorphism of vector bundles

$$\mathcal{E}\left(\bigwedge^n H\right) \cong \bigwedge^n \mathcal{E}(H);$$

once again this is an artifact of our construction. □

4. The fundamental exact sequence

The goal of this section is to construct what we dub the *fundamental exact sequence*, describing the group $H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**})$ we are interested in in terms of $H^1_{\mathrm{cts}}(\mathbb{G}_1, A_1^{**})$ and $H^1_{\mathrm{pro\acute{e}t}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})^{\Pi_n}$. The key inputs are the isomorphism of towers Theorem 3.7.3, the determinant map, as well as a logarithm exact sequences for the sheaf of principal units (Lemma 4.2.1) to compute the terms. Throughout, special care is needed for the "anomalous" case $n = p = 2$.

The objects of study in this section are closely related to those considered by Ertl, Gilles, and Nizioł in [EGN23], whose methods inspired the approach in this section.

4.1. **The sheaf of principal units $\mathcal{O}^{**}$.** As a general reference for the material we recall in this subsection, we refer the interested reader to [Hub94] and [BCKW19].

For a topological ring A, let A° be the subset of power-bounded elements, and let $A^{\circ\circ} \subset A^\circ$ be the subset of topologically nilpotent elements. Let K be a *nonarchimedean field* of characteristic $(0,p)$. This means that K is complete with respect to a real-valued ultrametric $|\ |$ which satisfies $0 < |p| < 1$. In particular K contains $\mathbb{Q}_p$ as a subfield. The subset K° (also called K^+ or $\mathcal{O}_K$) is a valuation subring of K, equal to the set of all $f \in K$ with $|f| \leq 1$.

Recall the notion of a *Huber pair* (A, A^+) over (K, K^+). In this situation A is a topological K-algebra containing an open subring A_0 (a ring of definition) which carries the p-adic topology, and $A^+ \subset A$ is an open and integrally closed K^+-subalgebra contained in A°. The pair (A, A^+) is *complete* if A contains a p-adically complete ring of definition.

Lemma 4.1.1. *Let (A, A^+) be a complete Huber pair over (K, K^+).*

(1) For $f \in A$, there exists $n \geq 0$ such that $p^n f \in A^+$.
(2) $pA^\circ \subset A^+$.
(3) $A^{\circ\circ} = \mathrm{rad}(pA^+)$, the radical of pA^+.
(4) $1 + A^{\circ\circ}$ is an open subgroup of A^*.

Proof. (1) Let $f \in A$. Since $p^n \to 0$ in $A_0 \subset A$, and multiplication in A is continuous, we have $p^n f \to 0$. Since A^+ is open we have $p^n f \in A^+$ for n large enough.

(2) If $f \in A^\circ$, the sequence f^n is bounded, so that $(pf)^n \to 0$. For n large enough $(pf)^n \in A^+$, and then $pf \in A^+$ since A^+ is integrally closed.

(3) If $f \in A^{\circ\circ}$, then $f^n \to 0$, so for n large enough $f^n \in pA^+$. (Note that $pA^+ \subset A$ is open since multiplication by p^{-1} is continuous on A.) This implies $f \in A^+$ (once again since A^+ is integrally closed) and so $f \in \mathrm{rad}(pA^+)$. For the reverse inclusion, it is enough to show $pA^+ \subset A^{\circ\circ}$, and indeed $pA^+ \subset pA^\circ \subset A^{\circ\circ}$ since p is topologically nilpotent.

(4) The subset $1 + A^{\circ\circ}$ is clearly closed under multiplication. To see that it contains inverses, let $x \in A^{\circ\circ}$, and note that $(1+x)^{-1} = 1 - x + x^2 - \cdots$ converges to an element of $1 + A^{\circ\circ}$. It is open because it contains the neighborhood $1 + pA_0$ of 1. □

In the context of the lemma we write

$$A^{**} = 1 + A^{\circ\circ},$$

and we call this the group of *principal units* of A.

Let X be an adic space over $\mathrm{Spa}(K, K^+)$. Thus X is covered by affinoid adic spaces $\mathrm{Spa}(A, A^+)$, where (A, A^+) is a complete Huber pair over (K, K^+). For $x \in X$, let K_x denote the completed residue field of x, and let K_x^+ denote its valuation ring. Then (K_x, K_x^+) is an affinoid field. If f is an analytic function defined in a neighborhood of x, let $f(x)$ denote the image of f in K_x.

Recall the topologies $X_{\mathrm{an}}, X_{\mathrm{\acute{e}t}}, X_{\mathrm{pro\acute{e}t}}$ on X, listed here in increasing order of fineness. Each topology has a basis consisting of affinoid objects $U \to X$ which are open immersions in the case of X_{an}, étale morphisms in the case of $X_{\mathrm{\acute{e}t}}$, and pro-étale morphisms from perfectoid affinoids in the case of $X_{\mathrm{pro\acute{e}t}}$.

Each of the three topologies carries a structure sheaf, written $\mathcal{O}_X$ or simply $\mathcal{O}$. In the case of $X_{\text{proét}}$, what we mean is the completed structure sheaf (written $\widehat{\mathcal{O}}$ in [BMS18]). Let $\mathcal{O}^* \subset \mathcal{O}$ be the sheaf of invertible elements. For each topology $t \in \{\text{an}, \text{ét}, \text{proét}\}$, the first cohomology group $\operatorname{Pic}_t X := H^1_t(X, \mathcal{O}^*)$ classifies invertible $\mathcal{O}$-modules on X_t. As one has descent of $\mathcal{O}$-modules in the étale topology, the natural map $\operatorname{Pic}_{\text{an}}(X) \to \operatorname{Pic}_{\text{ét}}(X)$ is an isomorphism. However, $\operatorname{Pic}_{\text{proét}}(X)$ is generally strictly larger than $\operatorname{Pic}_{\text{an}}(X)$. A systematic treatment of Picard groups in finer topologies is given in [Heu22]; see also [EGN23] for the case that X is a Stein space.

Let $\mathcal{O}^+ \subset \mathcal{O}$ be the subsheaf of integral elements; that is, for every U as above we have

$$\mathcal{O}^+(U) = \left\{ f \in \mathcal{O}(U) \;\middle|\; |f(x)| \leq 1, \text{ all } x \in U \right\}.$$

Finally, define the sheaf of *principal units* $\mathcal{O}^{**} \subset \mathcal{O}^{+*}$ by

$$\mathcal{O}^{**}(U) = \left\{ f \in \mathcal{O}(U) \;\middle|\; |f(x) - 1| < 1, \text{ all } x \in U \right\}.$$

Lemma 4.1.2. *If $U = \operatorname{Spa}(A, A^+)$ is an affinoid object of X_{an}, $X_{\text{ét}}$, or $X_{\text{proét}}$, then $\mathcal{O}^{**}(U) = A^{**}$.*

Proof. The lemma can be restated as follows: Given $f \in A$, we have that $|f(x)| < 1$ for all $x \in U$ if and only if $f \in A^{\circ\circ}$. Consider the rational subset

$$U(1/f) = \left\{ x \in U \;\middle|\; |f(x)| \geq 1 \right\}.$$

We have $U(1/f) = \operatorname{Spa}(R, R^+)$, where $R = R_0[1/p]$, and R_0 is the completion of $A_0[1/f]$. We have that $U(1/f)$ is empty if and only this completion is the zero ring, which is true if and only if f is topologically nilpotent. □

This lemma allows us to use the notation $\mathcal{O}^{**}(U)$ and A^{**} interchangeably, and we will do so below. In particular, we note:

Example 4.1.3. Let $A = \mathcal{O}_K[\![T_1, \dots, T_d]\!]$, and let $D = \operatorname{Spf} A$. Let D_K be the adic generic fiber of D, so that D_K is the nonvanishing locus of $|p|$ in $\operatorname{Spa}(A, A)$. Then (in whatever topology) $H^0(D_K, \mathcal{O}^{**}) \cong 1 + M$, where $M \subset A$ is the maximal ideal. Applied to Lubin–Tate space, we find $H^0(\mathrm{LT}_{n,K}, \mathcal{O}^{**}) \cong A_n^{**} = 1 + \mathfrak{m}$.

4.2. **The logarithm exact sequence.** The following *logarithm exact sequence* will allow us to access the cohomology of $\mathcal{O}^{**}$ on the pro-étale site.

Since $\mathcal{O}^{**}(U)$ is a topological group for all U in a basis for the topology, we can think of $\mathcal{O}^{**}$ as a sheaf of condensed abelian groups.

Lemma 4.2.1. *Let X be a rigid-analytic variety. There is an exact sequence of sheaves of condensed abelian groups on $X_{\text{ét}}$ (and hence on $X_{\text{proét}}$):*

$$0 \to \mu_{p^\infty} \to \mathcal{O}^{**} \xrightarrow{\log} \mathcal{O} \to 0. \tag{4.2.2}$$

Here μ_{p^∞} is the sheaf of pth power roots of 1.

Proof. Suppose $U \to X$ is an affinoid object of $X_{\text{ét}}$, with $U = \operatorname{Spa}(A, A^+)$. The description of $\mathcal{O}^{**}$ in the previous lemma shows that the usual logarithm series converges on $\mathcal{O}^{**}(U)$ with values in $\mathcal{O}(U)$.

We claim first that the kernel of $\log\colon \mathcal{O}^{**}(U) \to \mathcal{O}(U)$ is $\mu_{p^\infty}(U)$. Given $f \in \mathcal{O}^{**}(U)$, there exists $n \geq 0$ large enough so that $f^{p^n} - 1 \in p^2 A_0$. Then

$$p^n \log f = \log f^{p^n} = \sum_{i \geq 1} (-1)^{i-1} \frac{(f^{p^n} - 1)^i}{i} \in (f^{p^n} - 1)(1 + pA_0).$$

(Here we have used $2(i-1) \geq v_p(i)+1$ for all $i \geq 2$, where v_p is the p-adic valuation on integers.) Thus if $\log f = 0$, we must have $f^{p^n} = 1$.

Now we claim that $\log\colon \mathcal{O}^{**} \to \mathcal{O}$ is surjective. Let $U = \mathrm{Spa}(A, A^+)$ as above. For $f \in \mathcal{O}(U) = A$, let $n \geq 0$ be large enough so that $p^n f \in p^2 A^+$. Then $\exp(p^n f)$ converges to an element $e \in A^{**}$ such that $\log e = p^n f$. Let $B = A[e^{1/p^n}]$; since $e \in A^*$ and p is invertible in A, we have that B is a finite étale A-algebra. We find that if $V = \mathrm{Spa}(B, B^+)$ (with $B^+ =$ integral closure of A^+ in B), then $V \to U$ is an étale cover for which $\mathcal{O}^{**}(V)$ contains a preimage of f under log, namely e^{1/p^n}. $\square$

4.3. **The fundamental exact sequence.** In this section we leverage the isomorphisms between stacks in Theorem 3.7.3 to control the $H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**})$. By Example 4.1.3, $A_n^{**} \cong \mathcal{O}^{**}(\mathrm{LT}_{n,K})$.

We repeatedly apply the following general formalism: Suppose a stack $\mathfrak{X}$ on the pro-étale site of $\mathrm{Perf}_{\mathbb{Q}_p}$ admits a presentation $\mathfrak{X} \cong [Y/G]$ for a diamond Y and a profinite group G which acts on Y. Let $\mathcal{F}$ be a sheaf of condensed abelian groups on $\mathfrak{X}$. Then we have a low-degree exact sequence

$$0 \to H^1_{\mathrm{cts}}(G, \mathcal{F}(Y)) \to H^1_{\mathrm{proét}}(\mathfrak{X}, \mathcal{F}) \to H^1_{\mathrm{proét}}(Y, \mathcal{F})^G \tag{4.3.1}$$

arising from the descent spectral sequence of E_2-signature

$$H^i_{\mathrm{cts}}(G, H^j_{\mathrm{proét}}(Y, \mathcal{F})) \implies H^{i+j}_{\mathrm{proét}}(\mathfrak{X}, \mathcal{F}).$$

(For background on sheaves of condensed abelian groups on the pro-étale site, see [BSSW24, §3], and especially Proposition 3.6.3 for an example of how continuous G-cohomology appears naturally in the context of a pro-étale G-torsor.)

The formation of the exact sequence in (4.3.1) is functorial in the following sense: If a morphism of pairs $(Y_1, G_1) \to (Y_2, G_2)$ is interpreted to mean a homomorphism $\phi\colon G_1 \to G_2$ and $\psi\colon Y_1 \to Y_2$ such that $\psi(gy) = \phi(g)\psi(y)$ for $g \in G_1$, $y \in Y_1$, then such a morphism induces a morphism of descent spectral sequences and hence a morphism between the corresponding low-degree exact sequences.

Now suppose there are two such presentations for $\mathfrak{X}$:

$$\mathfrak{X} \cong [Y_1/G_1] \cong [Y_2/G_2].$$

The corresponding low-degree exact sequences assemble to give a diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & P \arrow[r] \arrow[d] & H^1_{\mathrm{cts}}(G_1, \mathcal{F}(Y_1)) \arrow[d] & \\
0 \arrow[r] & H^1_{\mathrm{cts}}(G_2, \mathcal{F}(Y_2)) \arrow[r] & H^1_{\mathrm{proét}}(\mathfrak{X}, \mathcal{F}) \arrow[r] \arrow[d] & H^1_{\mathrm{proét}}(Y_2, \mathcal{F})^{G_2} \\
 & & H^1_{\mathrm{proét}}(Y_1, \mathcal{F})^{G_1}, &
\end{tikzcd} \tag{4.3.2}$$

where P is the indicated pullback in the diagram. In particular, there is an exact sequence

$$0 \to P \to H^1_{\mathrm{cts}}(G_1, \mathcal{F}(Y_1)) \to H^1_{\mathrm{proét}}(Y_2, \mathcal{F})^{G_2} \tag{4.3.3}$$

as well as a similar one which swaps the roles of the two presentations. The formation of the diagrams in (4.3.2) and (4.3.3) are functorial on the category of data consisting of (Y_1, G_1),(Y_2, G_2), and an isomorphism $[Y_1/G_1] \xrightarrow{\sim} [Y_2/G_2]$ and with morphisms consisting of a pair of maps of pairs and a 2-commuting square of stacks.

Let C be the completion of an algebraic closure of $\mathbb{Q}_p$, and let

$$\Gamma_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p),$$

so that $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \cong \mathrm{Spa}(C, C^\circ)/\Gamma_{\mathbb{Q}_p}$. Theorem 3.7.3 shows that for each $n \geq 1$ there is a pair of presentations of the stack $\mathfrak{M}_n$:

$$\mathfrak{M}_n \cong [\mathrm{LT}^\diamond_{n,K}/\mathbb{G}_n] \cong [\mathcal{H}^{n-1,\diamond}_C/(\Gamma_{\mathbb{Q}_p} \times \mathrm{GL}_n(\mathbb{Z}_p))]. \tag{4.3.4}$$

Here, we have replaced $\mathcal{H}^{n-1,\diamond}$ with its equivalent presentation $\mathcal{H}^{n-1,\diamond}_C/\Gamma_{\mathbb{Q}_p}$. It will be convenient to use the notation

$$\Pi_n = \Gamma_{\mathbb{Q}_p} \times \mathrm{GL}_n(\mathbb{Z}_p). \tag{4.3.5}$$

Recall from Theorem 3.8.1 that the determinant morphism $\det\colon \mathfrak{M}_n \to \mathfrak{M}_1$ is compatible with the two presentations of $\mathfrak{M}_n$ given in (4.3.4). Applying (4.3.2) functorially to the determinant morphism, we obtain a commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & & \\
 & P_1 \arrow[rr] \arrow[dr, "q"] \arrow[dd, "r"] & & H^1_{\mathrm{cts}}(\Pi_1, \mathcal{O}^{**}(\mathcal{H}^0_C)) \arrow[dr, "\det^*_{\mathcal{H}}"] \arrow[dd] & 0 \arrow[d] & & \\
 & & P_n \arrow[rr] \arrow[dd] & & H^1_{\mathrm{cts}}(\Pi_n, \mathcal{O}^{**}(\mathcal{H}^{n-1}_C)) \arrow[dd] & & \\
0 \arrow[r] & H^1_{\mathrm{cts}}(\mathbb{G}_1, \mathcal{O}^{**}(\mathrm{LT}_{1,K})) \arrow[rr] \arrow[dr, "\det^*_{\mathrm{LT}}"] & & H^1_{\mathrm{pro\acute{e}t}}(\mathfrak{M}_1, \mathcal{O}^{**}) \arrow[rr] \arrow[dr] \arrow[dd] & & H^1_{\mathrm{pro\acute{e}t}}(\mathrm{LT}_{1,K}, \mathcal{O}^{**})^{\mathbb{G}_1} \arrow[dr, "\det^*_{\mathrm{LT}}"] & \\
 & 0 \arrow[r] & H^1_{\mathrm{cts}}(\mathbb{G}_n, \mathcal{O}^{**}(\mathrm{LT}_{n,K})) \arrow[rr] & & H^1_{\mathrm{pro\acute{e}t}}(\mathfrak{M}_n, \mathcal{O}^{**}) \arrow[rr] \arrow[dd] & & H^1_{\mathrm{pro\acute{e}t}}(\mathrm{LT}_{n,K}, \mathcal{O}^{**})^{\mathbb{G}_n} \\
 & & & H^1_{\mathrm{pro\acute{e}t}}(\mathcal{H}^0_C, \mathcal{O}^{**})^{\Pi_1} \arrow[dr] & & & \\
 & & & & H^1_{\mathrm{pro\acute{e}t}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**})^{\Pi_n}. & &
\end{tikzcd} \tag{4.3.6}$$

We proceed with six lemmas concerning objects in the diagram (4.3.6).

Lemma 4.3.7. *Considering $\mathbb{Q}_p^{**} = 1 + p\mathbb{Z}_p$ with its trivial $\mathrm{SL}_n(\mathbb{Z}_p)$-action, we have:*

$$H^1_{\mathrm{cts}}(\mathrm{SL}_n(\mathbb{Z}_p), \mathbb{Q}_p^{**}) \cong \begin{cases} 0, & (n,p) \neq (2,2); \\ \mathbb{Z}/2 & (n,p) = (2,2). \end{cases}$$

In the case $(n,p) = (2,2)$, the group $H^1_{\mathrm{cts}}(\mathrm{SL}_n(\mathbb{Z}_p), \mathbb{Q}_p^{**})$ is generated by the homomorphism given by the composition

$$\mathrm{SL}_2(\mathbb{Z}_2) \xrightarrow{\text{mod } 2} \mathrm{SL}_2(\mathbb{Z}/2) \cong S_3 \xrightarrow{\text{sign}} \{\pm 1\} \subset \mathbb{Q}_2^{**}.$$

Proof. Since $\mathbb{Q}_p^{**}$ is abelian, $H^1_{\mathrm{cts}}(\mathrm{SL}_n(\mathbb{Z}_p), \mathbb{Q}_p^{**})$ is isomorphic to the group of continuous homomorphisms $\mathrm{SL}_n(\mathbb{Z}_p)^{\overline{\mathrm{ab}}} \to \mathbb{Q}_p^{**}$, where

$$\mathrm{SL}_n(\mathbb{Z}_p)^{\overline{\mathrm{ab}}} = \mathrm{SL}_n(\mathbb{Z}_p)/\overline{[\mathrm{SL}_n(\mathbb{Z}_p), \mathrm{SL}_n(\mathbb{Z}_p)]}.$$

Since $\mathrm{SL}_n(\mathbb{Z})$ is dense in $\mathrm{SL}_n(\mathbb{Z}_p)$, the image of $\mathrm{SL}_n(\mathbb{Z})^{\mathrm{ab}} \to \mathrm{SL}_n(\mathbb{Z}_p)^{\overline{\mathrm{ab}}}$ is also dense. We apply the well-known result [HO13, 1.2.11, 1.2.15, Theorem 4.3.22]:

$$\mathrm{SL}_n(\mathbb{Z})^{\mathrm{ab}} = \begin{cases} 0, & n \neq 2; \\ \mathbb{Z}/12 & n = 2. \end{cases}$$

This settles the claim for $n \neq 2$, and shows that in all cases $\mathrm{SL}_n(\mathbb{Z}_p)^{\overline{\mathrm{ab}}}$ is a (possibly trivial) torsion cyclic group. For $p \neq 2$, $\mathbb{Q}_p^{**} \cong \mathbb{Z}_p$ is torsion free, so this settles the claim for $p \neq 2$.

We are left with the case $(n,p)=(2,2)$. As $\mathbb{Q}_2^{**}\cong\mathbb{Z}_2\oplus\mathbb{Z}/2$ and $\mathrm{SL}_2(\mathbb{Z}_2)^{\overline{\mathrm{ab}}}$ is torsion cyclic we see that $H^1_{\mathrm{cts}}(\mathrm{SL}_2(\mathbb{Z}_2),\mathbb{Q}_2^{**})$ is of order at most 2. To conclude we observe that the map $\mathrm{SL}_2(\mathbb{Z}_2)\to\mathbb{Q}_2^{**}$ described above determines a non-trivial element in $H^1_{\mathrm{cts}}(\mathrm{SL}_2(\mathbb{Z}_2),\mathbb{Q}_2^{**})$. □

Lemma 4.3.8. *For $(n,p)\neq(2,2)$, the map*

$$\det{}^*_{\mathcal{H}}\colon H^1_{\mathrm{cts}}(\Pi_1,\mathcal{O}^{**}(\mathcal{H}^0_C))\to H^1_{\mathrm{cts}}(\Pi_n,\mathcal{O}^{**}(\mathcal{H}^{n-1}_C))$$

is an isomorphism. For $(n,p)=(2,2)$, $\det^*_{\mathcal{H}}$ is an inclusion of a direct summand with complement $\mathbb{Z}/2$ generated by the class $\alpha\in H^1_{\mathrm{cts}}(\Pi_2,C^{**})$ defined as the composite

$$\Pi_2\to\mathrm{GL}_2(\mathbb{Z}_2)\xrightarrow{\text{mod }2}\mathrm{GL}_2(\mathbb{Z}/2)\simeq S_3\xrightarrow{\text{sign}}\{\pm1\}\subset C^{**},$$

where the first map is the canonical projection.

Proof. We first recall from [BSSW24, Theorem 6.3.1] that every bounded-by-1 function on $\mathcal{H}^{n-1}_{\mathbb{Q}_p}$ is constant: $H^0(\mathcal{H}^{n-1}_{\mathbb{Q}_p},\mathcal{O}^+)\cong\mathbb{Z}_p$. This follows from the existence of an integral model $\mathcal{H}^{n-1}_{\mathbb{Z}_p}$ for $\mathcal{H}^{n-1}$, whose special fiber $\mathcal{H}^{n-1}_{\mathbb{F}_p}$ is a connected scheme each of whose irreducible components is projective (see [GK05, §6]). Indeed, this shows that $H^0(\mathcal{H}^{n-1}_{\mathbb{F}_p},\mathcal{O})\cong\mathbb{F}_p$, which suffices because $H^0(\mathcal{H}^{n-1}_{\mathbb{Q}_p},\mathcal{O}^+)\cong H^0(\mathcal{H}^{n-1}_{\mathbb{Z}_p},\mathcal{O})$ is p-adically complete. Consequently $H^0(\mathcal{H}^{n-1}_C,\mathcal{O}^{**})\cong C^{**}$.

The map $\det^*_{\mathcal{H}}$ agrees with the one appearing in the inflation-restriction sequence for C^{**} and the map $\Pi_n\to\Pi_1$ induced by the determinant map $\mathrm{GL}_n(\mathbb{Z}_p)\to\mathbb{Z}_p^*$ with kernel $\mathrm{SL}_n(\mathbb{Z}_p)$:

$$0\to H^1_{\mathrm{cts}}(\Pi_1,(C^{**})^{\mathrm{SL}_n(\mathbb{Z}_p)})\xrightarrow{\det^*_{\mathcal{H}}}H^1_{\mathrm{cts}}(\Pi_n,C^{**})\to H^1_{\mathrm{cts}}(\mathrm{SL}_n(\mathbb{Z}_p),C^{**})^{\Pi_1}.$$

Using that $\mathrm{SL}_n(\mathbb{Z}_p)$ acts trivially on C^{**}, Tate's theorem [Tat67, §3, Theorem 1] which shows that $C^{\Gamma_{\mathbb{Q}_p}}=\mathbb{Q}_p$, and postcomposing with $H^1_{\mathrm{cts}}(\mathrm{SL}_n(\mathbb{Z}_p),C^{**})^{\mathbb{Z}_p^*}\subseteq H^1_{\mathrm{cts}}(\mathrm{SL}_n(\mathbb{Z}_p),C^{**})$, we obtain an exact sequence

$$0\to H^1_{\mathrm{cts}}(\Pi_1,C^{**})\xrightarrow{\det^*_{\mathcal{H}}}H^1_{\mathrm{cts}}(\Pi_n,C^{**})\to H^1_{\mathrm{cts}}(\mathrm{SL}_n(\mathbb{Z}_p),\mathbb{Q}_p^{**}).$$

For $(n,p)\neq(2,2)$, the claim follows from Lemma 4.3.7. In the case $(n,p)=(2,2)$, it follows from Lemma 4.3.7 and the fact that the class $\alpha\in H^1_{\mathrm{cts}}(\Pi_2,C^{**})$ is a lift of the nontrivial class in $H^1_{\mathrm{cts}}(\mathrm{SL}_2(\mathbb{Z}_2),\mathbb{Q}_2^{**})$ with $2\alpha=0$, that the above sequence is split short exact. □

Lemma 4.3.9. *The map*

$$r\colon P_1\to H^1_{\mathrm{cts}}(\mathbb{G}_1,\mathcal{O}^{**}(\mathrm{LT}_{1,K}))$$

is an isomorphism.

Proof. Consider the following exact sequence which is an instance of (4.3.3) applied to (4.3.6):

$$0\to P_1\xrightarrow{r}H^1_{\mathrm{cts}}(\mathbb{G}_1,\mathcal{O}^{**}(\mathrm{LT}_{1,K}))\to H^1_{\text{proét}}(\mathcal{H}^0_C,\mathcal{O}^{**})^{\Pi_1}.$$

Since $\mathcal{H}^0_C=\mathrm{Spa}(C,\mathcal{O}_C)$, we have $H^1_{\text{proét}}(\mathcal{H}^0_C,\mathcal{O}^{**})^{\Pi_1}=0$. □

Lemma 4.3.10. *The map*

$$\det{}^*_{\mathrm{LT}}\colon H^1_{\text{proét}}(\mathrm{LT}_{1,K},\mathcal{O}^{**})^{\mathbb{G}_1}\to H^1_{\text{proét}}(\mathrm{LT}_{n,K},\mathcal{O}^{**})^{\mathbb{G}_n}$$

is injective.

Proof. It is enough to show that the map

$$\det{}^*_{\mathrm{LT}}\colon H^1_{\text{proét}}(\mathrm{LT}_{1,K},\mathcal{O}^{**})\to H^1_{\text{proét}}(\mathrm{LT}_{n,K},\mathcal{O}^{**})$$

is injective. This follows from the observation that any point $\mathrm{Spa}(K,W)\to\mathrm{LT}_{n,K}$ induces a retract for $\det^*$. □

Lemma 4.3.11. *Assume that* $(n,p)=(2,2)$. *The map*

$$H^1_{\text{proét}}(\Pi_2, \mathcal{O}^{**}(\mathcal{H}^1_C)) \to H^1_{\text{proét}}(\mathrm{LT}_{2,K}, \mathcal{O}^{**})$$

of the low degree exact sequence (4.3.1) *sends the class* α *from Lemma 4.3.8 to* 0. *Therefore, the class* α *lifts to* P_2.

Proof. Consider the short exact sequence of sheaves:

$$0 \to \mu_2 \to \mathcal{O}^{**} \xrightarrow{(\,)^2} \mathcal{O}^{**} \to 0.$$

This short exact sequence gives rise to the following diagram with exact right vertical column:

$$\begin{tikzcd}
 & & \mathcal{O}^{**}(\mathrm{LT}_{2,K}) \arrow[d, "(\,)^2"'] \\
 & & \mathcal{O}^{**}(\mathrm{LT}_{2,K}) \arrow[d, "\partial"'] \\
H^1_{\text{cts}}(\Pi_2, \mu_2(\mathcal{H}^1_C)) \arrow[r] \arrow[d] & H^1_{\text{proét}}(\mathfrak{M}_2, \mu_2) \arrow[r] \arrow[d] & H^1_{\text{proét}}(\mathrm{LT}_{2,K}, \mu_2) \arrow[d] \\
H^1_{\text{cts}}(\Pi_2, \mathcal{O}^{**}(\mathcal{H}^1_C)) \arrow[r] & H^1_{\text{proét}}(\mathfrak{M}_2, \mathcal{O}^{**}) \arrow[r] & H^1_{\text{proét}}(\mathrm{LT}_{2,K}, \mathcal{O}^{**}).
\end{tikzcd}$$

Let $\tilde{\alpha} \in H^1_{\text{cts}}(\Pi_2, \mu_2(\mathcal{H}^1_C))$ be the class defined as the composite

$$\Pi_2 \to \mathrm{GL}_2(\mathbb{Z}_2) \xrightarrow{\text{mod } 2} \mathrm{GL}_2(\mathbb{Z}/2) \simeq S_3 \xrightarrow{\text{sign}} \{\pm 1\} \simeq \mu_2(\mathcal{H}^1_C).$$

The image of $\tilde{\alpha}$ under the map $H^1(\Pi_2, \mu_2(\mathcal{H}^1_C)) \to H^1(\Pi_2, \mathcal{O}^{**}(\mathcal{H}^1_C))$ is α. Let $\overline{\alpha} \in H^1(\mathrm{LT}_{2,K}, \mu_2)$ be the image of $\tilde{\alpha}$ under the composition of the top horizontal maps. It now suffices to show that $\overline{\alpha}$ lies in the image of $\partial\colon \mathcal{O}^{**}(\mathrm{LT}_{2,K}) \to H^1(\mathrm{LT}_{2,K}, \mu_2)$.

The group $H^1(\mathrm{LT}_{2,K}, \mu_2)$ classifies étale μ_2-torsors (i.e., double covers) of the rigid-analytic space $\mathrm{LT}_{2,K}$, and the map ∂ sends a principal unit in A_2^{**} to the double cover of $\mathrm{LT}_{2,K}$ obtained by adjoining its square root. The μ_2-torsor corresponding to $\overline{\alpha}$ can be described this way: Let H_2^{univ} be the universal deformation of H_2 over A_2. The 2-torsion subgroup $H_2^{\text{univ}}[2]$ is a group scheme of order 4 over $\mathrm{LT}_2 = \operatorname{Spf} A_2$, whose generic fiber $H_2^{\text{univ}}[2]_K$ is étale over $\mathrm{LT}_{2,K}$. Trivializations of $H_2^{\text{univ}}[2]$ over $\mathrm{LT}_{2,K}$ (or, what is the same, markings of its three nonzero sections) form a torsor under the group $\mathrm{GL}_2(\mathbb{Z}/2) \cong S_3$. Finally, $\overline{\alpha}$ is the quotient of this torsor corresponding to the sign homomorphism $S_3 \to \mu_2$, which is to say it is the torsor classifying markings of the three nonzero sections of $H_2^{\text{univ}}[2]$ up to cyclic permutation.

Let E be a supersingular elliptic curve over $\overline{\mathbb{F}}_2$, with universal deformation E^{univ}. By Serre–Tate theory, there is an isomorphism between LT_2 and the formal scheme parameterizing deformations of E, which identifies H_2^{univ} with the completion of E^{univ} along its origin. In particular $H_2^{\text{univ}}[2]$ is identified with the 2-torsion group $E^{\text{univ}}[2]$, and $\overline{\alpha}$ is the double cover of $\mathrm{LT}_{2,K}$ which classifies markings of the three nonzero sections of $E^{\text{univ}}[2]$ up to cyclic permutation.

A choice of nonvanishing differential ω on E^{univ} determines a Weierstrass model for E^{univ} over the local ring A_2. Over $A_2[1/2]$, the model can be brought into reduced form, with affine equation $y^2 = f(x)$, for a monic cubic polynomial $f(x) \in A_2[1/2][x]$. The nonzero 2-torsion sections of E^{univ} correspond to roots e_1, e_2, e_3 of $f(x)$. Therefore $\overline{\alpha}$ is the double cover of $\mathrm{LT}_{2,K}$ obtained by adjoining the square root of $\Delta(E^{\text{univ}}, \omega)$, where $\Delta(E^{\text{univ}}, \omega) = \prod_{i<j}(e_i - e_j)^2 \in A_2^*$ is the discriminant of the Weierstrass model determined by ω. For a scalar $\lambda \in A_2^*$ we have $\Delta(E^{\text{univ}}, \lambda\omega) = \lambda^{12}\Delta(E^{\text{univ}}, \omega)$; since every element is a 12th power in $\overline{\mathbb{F}}_2^*$, we can find an ω

with $\Delta(E^{\mathrm{univ}}, \omega) \in A_2^{**}$. We conclude that $\overline{\alpha}$ lies in the image of ∂; indeed it is the image of $\Delta(E^{\mathrm{univ}}, \omega)$. $\square$

Lemma 4.3.12. *For $(n,p) \neq (2,2)$, the map $q\colon P_1 \to P_n$ is an isomorphism. When $(n,p) = (2,2)$, the map q is an inclusion of a direct summand with complement $\mathbb{Z}/2$ generated by α.*

Proof. Formation of the exact sequence (4.3.3) is functorial, so we find a diagram

$$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r] \arrow[d, "q"] & H^1_{\mathrm{cts}}(\Pi_1, \mathcal{O}^{**}(\mathcal{H}_C^0)) \arrow[r] \arrow[d, "\det^*_{\mathcal{H}}"] & H^1_{\text{proét}}(\mathrm{LT}_{1,K}, \mathcal{O}^{**})^{\mathbb{G}_1} \arrow[d, "\det^*_{\mathrm{LT}}"] \\ 0 \arrow[r] & P_n \arrow[r] & H^1_{\mathrm{cts}}(\Pi_n, \mathcal{O}^{**}(\mathcal{H}_C^{n-1})) \arrow[r] & H^1_{\text{proét}}(\mathrm{LT}_{n,K}, \mathcal{O}^{**})^{\mathbb{G}_n}. \end{tikzcd}$$

When $(n,p) \neq (2,2)$, the map $\det^*_{\mathcal{H}}$ is an isomorphism by Lemma 4.3.8 and $\det^*_{\mathrm{LT}}$ is injective by Lemma 4.3.10, so q is an isomorphism. In the case $(n,p) = (2,2)$, the map q is injective since $\det^*_{\mathcal{H}}$ is injective by Lemma 4.3.8. The induced map from the cokernel of q to the cokernel of $\det^*_{\mathcal{H}}$ is injective so has size at most $\mathbb{Z}/2$, again by Lemma 4.3.8. By Lemma 4.3.11, α is a 2-torsion element of P_n that is not in the image of q. Since $\det^*_{\mathrm{LT}}$ is injective by Lemma 4.3.10, this class thus generates the $\mathbb{Z}/2$ cokernel. $\square$

When $p = 2$, we denote the image of α under the map $P_2 \to H^1(\mathbb{G}_2, A_2^{**})$ by $\hat{\alpha}$. Combining the lemmas above, we deduce the following theorem.

Theorem 4.3.13. *For $(n,p) \neq (2,2)$, there is an exact sequence*

$$0 \to H^1_{\mathrm{cts}}(\mathbb{G}_1, A_1^{**}) \xrightarrow{\det^*_{\mathrm{LT}}} H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**}) \xrightarrow{b} H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})^{\Pi_n},$$

where b is the composite $H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**}) \to H^1_{\text{proét}}(\mathfrak{M}_n, \mathcal{O}^{**}) \to H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})^{\Pi_n}$ in (4.3.6).

When $(n,p) = (2,2)$ there is an exact sequence

$$0 \to H^1_{\mathrm{cts}}(\mathbb{G}_1, A_1^{**}) \oplus \mathbb{Z}/2 \xrightarrow{\det^*_{\mathrm{LT}} \oplus \hat{\alpha}} H^1_{\mathrm{cts}}(\mathbb{G}_2, A_2^{**}) \xrightarrow{b} H^1_{\text{proét}}(\mathcal{H}_C^1, \mathcal{O}^{**})^{\Pi_2},$$

where b is the composite $H^1_{\mathrm{cts}}(\mathbb{G}_2, A_2^{**}) \to H^1_{\text{proét}}(\mathfrak{M}_2, \mathcal{O}^{**}) \to H^1_{\text{proét}}(\mathcal{H}_C^1, \mathcal{O}^{**})^{\Pi_2}$ in (4.3.6).

Proof. Consider the following commutative diagram, which is part of (4.3.6):

$$\begin{tikzcd} P_n \arrow[r] \arrow[d] & H^1_{\mathrm{cts}}(\Pi_n, \mathcal{O}^{**}(\mathcal{H}_C^{n-1})) \arrow[d] \\ H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**}) \arrow[r] \arrow[dr, "b"'] & H^1_{\text{proét}}(\mathfrak{M}_n, \mathcal{O}^{**}) \arrow[d] \\ & H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})^{\Pi_n}. \end{tikzcd}$$

As an instance of (4.3.3), we obtain an exact sequence

$$0 \to P_n \to H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**}) \xrightarrow{b} H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})^{\Pi_n}.$$

Lemma 4.3.9 and Lemma 4.3.12 provide isomorphisms

$$P_n \cong \begin{cases} H^1_{\mathrm{cts}}(\mathbb{G}_1, A_1^{**}) & \text{if } (n,p) \neq (2,2); \\ H^1_{\mathrm{cts}}(\mathbb{G}_1, A_1^{**}) \oplus \mathbb{Z}/2 & \text{if } (n,p) = (2,2). \end{cases}$$

Together with the commutativity of the left square in (4.3.6) and the class α, this gives the two exact sequences of the theorem. $\square$

When $n = 1$, the theorem is uninteresting: $\det_{\mathrm{LT}}^*$ is an isomorphism, and the target of b is 0 for dimension reasons. The next objective is to identify $H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})^{\Pi_n}$ for $n \geq 2$. In the next section we will prove that this group is a free $\mathbb{Z}_p$-module of rank 1 and the map b is surjective, generated by the image of $\varepsilon_p(\Sigma^2\mathbb{S}_{K(n)})$. This claim implies Theorem 2.7.1.

5. The cohomology of Drinfeld symmetric space $\mathcal{H}_C^{n-1}$

Via the fundamental exact sequence in Theorem 4.3.13, we have reduced the crucial Theorem 2.7.1 to a question about $H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})$, the cohomology of $\mathcal{O}^{**}$ on the pro-étale site of Drinfeld's symmetric space $\mathcal{H}_C^{n-1}$. Recall that $\mathcal{H}_C^{n-1}$ is the complement in rigid-analytic $\mathbb{P}_C^{n-1}$ of all $\mathbb{Q}_p$-rational hyperplanes; as such it admits an action of $\mathrm{GL}_n(\mathbb{Q}_p)$. In particular $\mathcal{H}_C^1 = \mathbb{P}_C^1 \backslash \mathbb{P}^1(\mathbb{Q}_p)$ is analogous to the (upper and lower) half-plane $\mathbb{P}^1(\mathbb{C})\backslash\mathbb{P}^1(\mathbb{R})$.

The cohomology of $\mathcal{H}_C^{n-1}$ was studied first by Drinfeld [Dri74] in the case $n = 2$, and then by Schneider–Stuhler [SS91] for general n. The main theorem of [SS91] is that if H is a cohomology theory on rigid-analytic varieties satisfying certain reasonable axioms (for instance, homotopy-invariance relative to the unit disc D), then $H^i(\mathcal{H}_C^{n-1})$ is a generalized Steinberg representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ for $0 \leq i \leq n$, and is 0 otherwise. (The construction of the generalized Steinberg representations are reviewed below.) The result applies to ℓ-adic étale cohomology and to de Rham cohomology.

In contrast, étale cohomology with $\mathbb{Z}_p$-coefficients does not satisfy homotopy-invariance; indeed, $H^1_{\text{ét}}(D, \mathbb{Z}_p)$ is very large. One therefore might expect that there cannot be a meaningful description of $H^i_{\text{ét}}(\mathcal{H}_C^{n-1}, \mathbb{Z}_p)$, let alone $H^i_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathbb{Q}_p)$. Nonetheless, both groups were identified by Colmez–Dospinescu–Niziol [CDN20], [CDN21]. In the latter setting, $H^i_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathbb{Q}_p)$ is an extension of a space of differential forms by a general Steinberg representation.

Combining these results with the logarithm exact sequence (Lemma 4.2.1), we gain access to $H^i_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})$, see Theorem 5.3.1 below. Specializing to the case $i = 1$, we find that $H^1_{\text{proét}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})^{\Pi_n}$ is (for $n \geq 2$) isomorphic to the space of $\mathrm{GL}_n(\mathbb{Z}_p)$-invariants in a generalized Steinberg representation with coefficients in $\mathbb{Q}_p/\mathbb{Z}_p$. This space of invariants is shown to be a free rank 1 $\mathbb{Z}_p$-module in Lemma 5.1.2.

5.1. **Generalized Steinberg representations: definitions.** The following is a rephrasing of the constructions appearing in [CDN20]. For $0 \leq r \leq n$, let X_r be the set of flags

$$V_1 \subset V_2 \subset \cdots \subset V_r \subseteq \mathbb{Q}_p^{\oplus n},$$

where V_i is a $\mathbb{Q}_p$-vector space of dimension i. Note that we in particular allow the empty flag. For $1 \leq j \leq r$, let $X_{r,j}$ be set of flags as above with the jth vector space omitted. The set X_r (resp., $X_{r,j}$) is a quotient of $\mathrm{GL}_n(\mathbb{Q}_p)$ by a parabolic subgroup. As such, it is a profinite set admitting a continuous action of $\mathrm{GL}_n(\mathbb{Q}_p)$; note also the quotient map $X_r \to X_{r,j}$.

For a profinite set $S = \varprojlim S_i$ and a ring A, let $\mathrm{LC}(S, A)$ denote the ring of locally constant functions on S valued in A. If A carries a topology, we give $\mathrm{LC}(S, A) = \varinjlim \mathrm{LC}(S_i, A)$ the colimit topology, where each $\mathrm{LC}(S_i, A)$ (a finite free A-module) has the natural topology. Define the *generalized Steinberg (or special)* representation by

$$\mathrm{St}_r(A) = \frac{\mathrm{LC}(X_r, A)}{\sum_{j=1}^r \mathrm{LC}(X_{r,j}, A)}.$$

Each St_r is a topological group admitting a continuous action of $\mathrm{GL}_n(\mathbb{Q}_p)$. As a special case,

$$\mathrm{St}_1(A) = \frac{\mathrm{LC}(\mathbb{P}^{n-1}(\mathbb{Q}_p), A)}{A}.$$

For any topological A-module M, let

$$M^* = \mathrm{Hom}_{\mathrm{cts}}(M, A),$$

meaning continuous A-module homomorphisms $M \to A$. One can think of $\mathrm{St}_r(A)^*$ as the module of distributions on X_r whose value is 0 on functions pulled back from any $X_{r,j}$.

In the case $r = 1$, identify $\mathbb{P}^{n-1}(\mathbb{Q}_p)$ with the set of linear forms in n variables up to scaling by $\mathbb{Q}_p^*$. Let δ_ℓ denote the Dirac distribution at $\ell \in \mathbb{P}^{n-1}(\mathbb{Q}_p)$. Then any difference $\delta_\ell - \delta_{\ell'}$ determines an element of $\mathrm{St}_1(A)^*$.

We give an ad hoc definition of the $\mathrm{GL}_n(\mathbb{Q}_p)$-module $\mathrm{St}_r(\mathbb{Q}_p/\mathbb{Z}_p)^*$, defining this to be the quotient:

$$0 \to \mathrm{St}_r(\mathbb{Z}_p)^* \to \mathrm{St}_r(\mathbb{Q}_p)^* \to \mathrm{St}_r(\mathbb{Q}_p/\mathbb{Z}_p)^* \to 0 \tag{5.1.1}$$

Thus for instance elements of $\mathrm{St}_1(\mathbb{Q}_p/\mathbb{Z}_p)^*$ are $\mathbb{Q}_p/\mathbb{Z}_p$-valued distributions on $\mathbb{P}^{n-1}(\mathbb{Q}_p)$ with total measure 0.

Lemma 5.1.2. *Assume $n \geq 2$. The subset of $\mathrm{St}_1(\mathbb{Q}_p/\mathbb{Z}_p)^*$ fixed by $\mathrm{GL}_n(\mathbb{Z}_p)$ is a free $\mathbb{Z}_p$-module of rank 1.*

Proof. Let μ be a $\mathbb{Q}_p/\mathbb{Z}_p$-valued distribution on $\mathbb{P}^{n-1}(\mathbb{Q}_p) = \mathbb{P}^{n-1}(\mathbb{Z}_p)$. For each $m \geq 1$, we have a $\mathrm{GL}_n(\mathbb{Z}_p)$-equivariant surjective map

$$\pi_m \colon \mathbb{P}^{n-1}(\mathbb{Z}_p) \to \mathbb{P}^{n-1}(\mathbb{Z}/p^m\mathbb{Z}).$$

As m varies, the fibers of the π_m form a basis of open subsets of $\mathbb{P}^{n-1}(\mathbb{Z}_p)$, so that μ is determined by its values on the fibers. Now suppose μ is fixed by $\mathrm{GL}_n(\mathbb{Z}_p)$. Since $\mathrm{GL}_n(\mathbb{Z}_p)$ acts transitively on $\mathbb{P}^{n-1}(\mathbb{Z}/p^m\mathbb{Z})$, the value of μ is the same on all fibers of π_m; let a_m denote the common value. Then by the additivity of μ:

$$\mu(\mathbb{P}^{n-1}(\mathbb{Z}_p)) = \frac{p^n-1}{p-1}a_1 = \frac{p^n-1}{p-1}p^{n-1}a_2 = \frac{p^n-1}{p-1}p^{2(n-1)}a_3 = \cdots$$

Via $\mu \mapsto (a_1, a_2, \dots)$, the subgroup of $\mathrm{St}_1(\mathbb{Q}_p/\mathbb{Z}_p)^*$ fixed by $\mathrm{GL}_n(\mathbb{Z}_p)$ may be identified with the subgroup of $\prod_{m\geq 1} \mathbb{Q}_p/\mathbb{Z}_p$ satisfying

$$0 = a_1 = p^{n-1}a_2 = p^{2(n-1)}a_3 = \cdots.$$

This group is $\varprojlim_m \mathbb{Z}/p^{(n-1)m}\mathbb{Z} \cong \mathbb{Z}_p$. □

Taking invariants under $\mathrm{GL}_n(\mathbb{Z}_p)$ in the exact sequence (5.1.1), we find an injective connecting map

$$\partial \colon H^0(\mathrm{GL}_n(\mathbb{Z}_p), \mathrm{St}_1(\mathbb{Q}_p/\mathbb{Z}_p)^*) \to H^1(\mathrm{GL}_n(\mathbb{Z}_p), \mathrm{St}_1(\mathbb{Z}_p)^*). \tag{5.1.3}$$

Let $\mu \in H^0(\mathrm{GL}_n(\mathbb{Z}_p), \mathrm{St}_1(\mathbb{Q}_p/\mathbb{Z}_p)^*)$ be the generator of this group with $a_m = 1/p^{(n-1)(m-1)}$ for all $m \geq 2$.

Lemma 5.1.4. *Let $\ell \in \mathbb{P}^{n-1}(\mathbb{Q}_p)$ be arbitrary. Then $\partial(\mu)$ is represented by the cocycle*

$$g \mapsto \delta_\ell - \delta_{g(\ell)}.$$

Proof. The measure μ can be lifted to a (non-$\mathrm{GL}_n(\mathbb{Z}_p)$-invariant) measure $\tilde{\mu} \in \mathrm{St}_1(\mathbb{Q}_p)^*$ defined by the formula (for $m \geq 2$)

$$\tilde{\mu}(\pi_m^{-1}(x)) = \begin{cases} \frac{1}{p^{(n-1)(m-1)}} - 1, & \pi_m(\ell) = x; \\ \frac{1}{p^{(n-1)(m-1)}}, & \text{otherwise.} \end{cases}$$

Then $\partial(\mu)$ is represented by the cocycle sending $g \in \mathrm{GL}_n(\mathbb{Z}_p)$ to $g(\tilde{\mu}) - \tilde{\mu}$, and this is easily seen to agree with $\delta_\ell - \delta_{g(\ell)}$. □

5.2. **Results of Colmez–Dospinescu–Niziol.** We turn now to some crucial results of Colmez–Dospinescu–Niziol on the étale and pro-étale cohomology of Drinfeld symmetric space $\mathcal{H}_C^{n-1}$ over a complete algebraically closed field $C/\mathbb{Q}_p$.

Recall the difference between étale and pro-étale cohomology: whenever X is a scheme (resp. stack, rigid-analytic space, etc.), one has the constant sheaf $\mathbb{Z}/p^n\mathbb{Z}$ on the étale site of X, and then one *defines*

$$\begin{aligned} H^i_{\text{ét}}(X,\mathbb{Z}_p) &= \varprojlim H^i_{\text{ét}}(X,\mathbb{Z}/p^n\mathbb{Z}) \\ H^i_{\text{ét}}(X,\mathbb{Q}_p) &= H^i_{\text{ét}}(X,\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p. \end{aligned}$$

The latter is not the cohomology of any sheaf "$\mathbb{Q}_p$" on $X_{\text{ét}}$. On the other hand, one has the constant sheaf $\mathbb{Q}_p$ on the pro-étale topology $X_{\text{proét}}$, defined by sending an object $U\in X_{\text{proét}}$ to the ring of continuous maps $|U|\to\mathbb{Q}_p$, and then $H^i_{\text{proét}}(X,\mathbb{Q}_p)$ really is the sheaf cohomology of $\mathbb{Q}_p$ on $X_{\text{proét}}$. The projection $X_{\text{proét}}\to X_{\text{ét}}$ induces a map

$$H^i_{\text{ét}}(X,\mathbb{Q}_p)\to H^i_{\text{proét}}(X,\mathbb{Q}_p)$$

which can fail to be an isomorphism if X is not quasi-compact (for instance, if X is the discrete infinite union of points).

The cohomology groups $H^i_{\text{ét}}(\mathcal{H}_C^{n-1},\mathbb{Z}_p)\simeq H^i_{\text{proét}}(\mathcal{H}_C^{n-1},\mathbb{Z}_p)$ and $H^i_{\text{proét}}(\mathcal{H}_C^{n-1},\mathbb{Q}_p)$ were computed in [CDN20] and [CDN21], respectively. We now review these results, starting with $H^i_{\text{ét}}$. We need one more ingredient before we can give the statements, namely the Kummer map

$$\kappa\colon \mathcal{O}^*(\mathcal{H}_C^{n-1})\to H^1_{\text{ét}}(\mathcal{H}_C^{n-1},\mathbb{Z}_p(1)).$$

This map arises as follows: Consider the sequence of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mu_{p^{n+1}} \arrow[r] \arrow[d] & \mathcal{O}^* \arrow[r, "(-)^{p^{n+1}}"] \arrow[d, "(-)^p"'] & \mathcal{O}^* \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \mu_{p^n} \arrow[r] & \mathcal{O}^* \arrow[r, "(-)^{p^n}"] & \mathcal{O}^* \arrow[r] & 0. \end{tikzcd}$$

Taking limits results in another short exact sequence,

$$0\longrightarrow \mathbb{Z}_p(1)\longrightarrow \varprojlim_{(-)^p}\mathcal{O}^*\longrightarrow\mathcal{O}^*\longrightarrow 0,$$

and κ is the boundary map in cohomology of this short exact sequence.

Theorem 5.2.1. *There is a $\Gamma_{\mathbb{Q}_p}\times\mathrm{GL}_n(\mathbb{Q}_p)$-equivariant isomorphism*

$$r_i\colon \mathrm{St}_i(\mathbb{Z}_p)^*\xrightarrow{\sim} H^i_{\text{ét}}(\mathcal{H}_C^{n-1},\mathbb{Z}_p(i))$$

such that (in the $i=1$ case) the relation

$$r_1(\delta_{\ell_1}-\delta_{\ell_2})=\kappa(\ell_1/\ell_2)$$

holds for all $\ell_1,\ell_2\in\mathbb{P}^{n-1}(\mathbb{Q}_p)$. On the right side of the relation, ℓ_1 and ℓ_2 are interpreted as $\mathbb{Q}_p$-rational linear forms in n variables, so that $\ell_1/\ell_2\in\mathcal{O}^*(\mathcal{H}_C^{n-1})$ is well-defined up to $\mathbb{Q}_p^*$, and $\kappa(\ell_1/\ell_2)$ is a well-defined class in $H^1_{\text{ét}}(\mathcal{H}_C^{n-1},\mathbb{Z}_p(1))$.

Proof. The map r_i is defined in [CDN21, Proposition 4.7]; for $i=1$ it is characterized by the relation $r_1(\delta_{\ell_1}-\delta_{\ell_2})=\kappa(\ell_1/\ell_2)$. (There is a characterization of r_i for general i in terms of a "regulator map.") The isomorphy of r_i is [CDN21, Theorem 5.1]. □

Let us turn now to pro-étale cohomology. For a rigid space X over C, the short exact sequence of abelian sheaves on $X_{\mathrm{pro\acute{e}t}}$ from Lemma 4.2.1,

$$0 \to \mu_{p^\infty} \to \mathcal{O}^{**} \overset{\log}{\to} \mathcal{O} \to 0,$$

gives rise, after taking sequential limits along the multiplication by p map to a short exact sequence of sheaves of $\mathbb{Q}_p$-vector spaces on $X_{\mathrm{pro\acute{e}t}}$

$$0 \to \mathbb{Q}_p(1) \to \varprojlim_{\times p} \mathcal{O}^{**} \overset{\widetilde{\log}}{\to} \mathcal{O} \to 0.$$

We thus get a connecting map $\partial_{\widetilde{\log}} \colon \mathcal{O}[-1] \to \mathbb{Q}_p(1)$ in the derived category of sheaves on $X_{\mathrm{pro\acute{e}t}}$. The pro-étale cohomology of $\mathcal{O}$ was studied in [Sch13, Proposition 3.23]: If $\nu \colon X_{\mathrm{pro\acute{e}t}} \to X_{\mathrm{\acute{e}t}}$ is the projection, then $R^i\nu_*\mathcal{O} \cong \Omega^i(-i)$, where $\Omega^i = \Omega^i_{X/C}$ is the sheaf of differential i-forms. As a result there is a spectral sequence

$$H^i_{\mathrm{\acute{e}t}}(X, \Omega^j(-j)) \implies H^{i+j}_{\mathrm{pro\acute{e}t}}(X, \mathcal{O}).$$

In the special case that X is a Stein space, the coherent sheaves $\Omega^j_{X/C}$ are acyclic, and so

$$H^i_{\mathrm{pro\acute{e}t}}(X, \mathcal{O}) \cong \Omega^i(X)(-i). \tag{5.2.2}$$

Therefore the connecting map $\mathcal{O}[-1] \to \mathbb{Q}_p(1)$ induces a map

$$\exp\colon \Omega^{i-1}(X)(1-i) \to H^i_{\mathrm{pro\acute{e}t}}(X, \mathbb{Q}_p(1)). \tag{5.2.3}$$

The map exp is related to the Bloch–Kato exponential, see [EGN23, §3.2] for a discussion. When X admits a semistable formal model over the ring of integers of a local field, [CDN20] identifies the kernel of exp with the module of closed forms $\ker d$, and identifies the cokernel of exp in terms of the Hyodo–Kato cohomology of the special fiber of the model. We are now ready to state the theorem of Colmez–Dospinescu–Niziol on the pro-étale cohomology of $\mathcal{H}^{n-1}_C$.

Theorem 5.2.4. *There is a $\Gamma_{\mathbb{Q}_p} \times \mathrm{GL}_n(\mathbb{Q}_p)$-equivariant exact sequence of $\mathbb{Q}_p$-vector spaces*

$$0 \to \frac{\Omega^{i-1}(\mathcal{H}^{n-1}_C)}{\ker d} \overset{\exp}{\to} H^i_{\mathrm{pro\acute{e}t}}(\mathcal{H}^{n-1}_C, \mathbb{Q}_p(i)) \to \mathrm{St}_i(\mathbb{Q}_p)^* \to 0,$$

compatible with Theorem 5.2.1, in the sense that the evident diagram involving $H^i_{\mathrm{\acute{e}t}}(\mathcal{H}^{n-1}_C, \mathbb{Z}_p(i)) \to H^i_{\mathrm{pro\acute{e}t}}(\mathcal{H}^{n-1}_C, \mathbb{Q}_p(i))$ commutes.

We readily deduce the pro-étale cohomology of μ_{p^∞} on $\mathcal{H}^{n-1}_C$:

Corollary 5.2.5. *There is a $\Gamma_{\mathbb{Q}_p} \times \mathrm{GL}_n(\mathbb{Q}_p)$-equivariant exact sequence*

$$0 \to \frac{\Omega^{i-1}(\mathcal{H}^{n-1}_C)}{\ker d} \overset{\exp}{\to} H^i_{\mathrm{pro\acute{e}t}}(\mathcal{H}^{n-1}_C, \mu_{p^\infty}(i-1)) \to \mathrm{St}_i(\mathbb{Q}_p/\mathbb{Z}_p)^* \to 0.$$

Proof. Theorem 5.2.1 and Theorem 5.2.4 together supply a map of exact sequences

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & H^i_{\mathrm{pro\acute{e}t}}(\mathcal{H}^{n-1}_C, \mathbb{Z}_p(i)) \arrow[r, "\cong"] \arrow[d] & \mathrm{St}_i(\mathbb{Z}_p)^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{\Omega^{i-1}(\mathcal{H}^{n-1}_C)}{\ker d} \arrow[r, "\exp"] & H^i_{\mathrm{pro\acute{e}t}}(\mathcal{H}^{n-1}_C, \mathbb{Q}_p(i)) \arrow[r] & \mathrm{St}_i(\mathbb{Q}_p)^* \arrow[r] & 0.
\end{tikzcd}$$

Note that the right vertical map is injective with quotient $\mathrm{St}_i(\mathbb{Q}_p/\mathbb{Z}_p)^*$, see (5.1.1), while the middle vertical map comes from $(i-1)$-fold twist of the short exact sequence

$$0 \to \mathbb{Z}_p(1) \to \mathbb{Q}_p(1) \to \mu_{p^\infty} \to 0. \tag{5.2.6}$$

The snake lemma then furnishes the short exact sequence of the corollary. $\square$

5.3. **Consequences for $H^i_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**})$.** We gather here some consequences of Theorem 5.2.1 and Theorem 5.2.4. The next results concern the $\mathcal{O}^{**}$-cohomology of $\mathcal{H}^{n-1}_C$.

Theorem 5.3.1. *For $i \geq 0$ there is a short exact sequence*

$$0 \to \mathrm{St}_i(\mathbb{Q}_p/\mathbb{Z}_p)^*(1-i) \to H^i_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**}) \to \Omega^{i,\mathrm{cl}}(\mathcal{H}^{n-1}_C)(-i) \to 0,$$

where $\Omega^{i,\mathrm{cl}}$ denotes the sheaf of closed differential i-forms on $\mathcal{H}^{n-1}_C$.

Proof. Let $\partial_{\log}\colon \mathcal{O} \to \mu_{p^\infty}[1]$ be the connecting homomorphism of the logarithm exact sequence (4.2.2), and write

$$\partial^i_{\log}\colon H^i_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}) \to H^{i+1}_{\text{proét}}(\mathcal{H}^{n-1}_C, \mu_{p^\infty})$$

for the induced map on cohomology. It follows from the long exact sequence associatd to the logarithm sequence that there is a short exact sequence

$$0 \to \mathrm{coker}(\partial^{i-1}_{\log}) \to H^i_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**}) \to \ker(\partial^i_{\log}) \to 0.$$

Using Corollary 5.2.5, we can identify the kernel and cokernel of $\partial^i_{\log}$. To this end, first note that there is a lift

$$\begin{tikzcd} & & \mathcal{O} \arrow[d, "\partial_{\log}"] \arrow[dll, dashed] & \\ \mathbb{Q}_p(1)[1] \arrow[rr] & & \mu_{p^\infty}[1] \arrow[r] & \mathbb{Z}_p(1)[2], \end{tikzcd}$$

where the bottom exact sequence is a rotation of the triangle coming from (5.2.6). Indeed, this lift exists because the composite $\mathcal{O} \to \mu_{p^\infty}[1] \to \mathbb{Z}_p(1)[2]$ is trival, since the target is p-complete while the source has p acting invertibly. On cohomology, this lift is precisely the map (5.2.3).

Combining this observation with Corollary 5.2.5 and (5.2.2), we thus obtain a map of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \Omega^i(\mathcal{H}^{n-1}_C)(-i) \arrow[r, "\cong"] \arrow[d, two heads] & H^i_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}) \arrow[rr] \arrow[d, "\partial^i_{\log}"] & & 0 \arrow[d] \\ 0 \arrow[r] & \frac{\Omega^i(\mathcal{H}^{n-1}_C)}{\ker d}(-i) \arrow[r, "\exp"] & H^{i+1}_{\text{proét}}(\mathcal{H}^{n-1}_C, \mu_{p^\infty}) \arrow[r] & \mathrm{St}_{i+1}(\mathbb{Q}_p/\mathbb{Z}_p)^*(-i) \arrow[r] & 0 \end{tikzcd}$$

By our discussion above, the left vertical map is the canonical quotient map. The snake lemma then implies that

$$\ker(\partial^i_{\log}) \cong \Omega^{i,\mathrm{cl}}(\mathcal{H}^{n-1}_C)(-i) \qquad \text{and} \qquad \mathrm{coker}(\partial^i_{\log}) \cong \mathrm{St}_{i+1}(\mathbb{Q}_p/\mathbb{Z}_p)^*(-i),$$

as desired. □

Corollary 5.3.2. *Consider the short exact sequence of pro-étale sheaves on $\mathcal{H}^{n-1}_C$:*

$$0 \to \mathbb{Z}_p(1) \to \lim_p \mathcal{O}^{**} \to \mathcal{O}^{**} \to 0.$$

The associated long exact sequence for cohomology has zero boundary maps. Therefore for every $m \geq 0$ we obtain a short exact sequence of Π_n-modules

$$0 \to H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathbb{Z}_p(1)) \to H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \lim_p \mathcal{O}^{**}) \to H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**}) \to 0.$$

Proof. The claim is equivalent to showing that the map

$$H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \lim_p \mathcal{O}^{**}) \to H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**})$$

is surjective for all m. Note that this map factors as

$$H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \lim_p \mathcal{O}^{**}) \to \lim_p H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**}) \to H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**}).$$

By the Milnor sequence, the first map

$$H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \lim_p \mathcal{O}^{**}) \to \lim_p H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**})$$

is a surjection. The second map is surjective as well, since $H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**})$ is p-divisible. This follows from Theorem 5.3.1, which exhibits $H^m_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**})$ as an extension of p-divisible abelian groups. □

Finally, we identify the Π_n-invariants in $H^1_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**})$.

Corollary 5.3.3. *There is a canonical isomorphism*

$$H^1_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**})^{\Pi_n} \cong (\mathrm{St}_1(\mathbb{Q}_p/\mathbb{Z}_p)^*)^{\mathrm{GL}_n(\mathbb{Z}_p)}.$$

Thus if $n \geq 2$, then by Lemma 5.1.2, $H^1_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**})^{\Pi_n}$ is a free $\mathbb{Z}_p$-module of rank 1 with distinguished generator. When $n = 1$ we have $H^1_{\text{proét}}(\mathcal{H}^0_C, \mathcal{O}^{**})^{\Pi_1} = 0$.

Proof. Take Π_n-invariants in the exact sequence in the $i = 1$ case of Theorem 5.3.1. Since $\mathcal{H}^{n-1}$ is defined over $\mathbb{Q}_p$ we have an isomorphism

$$\Omega^{1,\mathrm{cl}}(\mathcal{H}^{n-1}_C)(-1) \cong \Omega^{1,\mathrm{cl}}(\mathcal{H}^{n-1})\widehat{\otimes}_{\mathbb{Q}_p} C(-1),$$

so that

$$H^0(\Gamma_{\mathbb{Q}_p}, \Omega^{1,\mathrm{cl}}(\mathcal{H}^{n-1}_C)(-1)) \cong \Omega^{1,\mathrm{cl}}(\mathcal{H}^{n-1})\widehat{\otimes}_{\mathbb{Q}_p} H^0(\Gamma_{\mathbb{Q}_p}, C(-1)) = 0.$$

Therefore we have an isomorphism

$$H^0(\Pi_n, \mathrm{St}_1(\mathbb{Q}_p/\mathbb{Z}_p)^*) \cong H^0(\Pi_n, H^1_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**})).$$

The group Π_n acts on $\mathrm{St}_1(\mathbb{Q}_p/\mathbb{Z}_p)^*$ through its quotient $\mathrm{GL}_n(\mathbb{Z}_p)$, and we have seen in Lemma 5.1.2 that the module of invariants is rank 1 over $\mathbb{Z}_p$ for $n \geq 2$. For $n = 1$, $\mathcal{H}^0_C = \mathrm{Spa}(C, \mathcal{O}_C)$ and thus $H^1_{\text{proét}}(\mathcal{H}^0_C, \mathcal{F})$ is zero for any sheaf $\mathcal{F}$. □

5.4. **A primitivity result.** Assume throughout that $n \geq 2$. Let us recall the fundamental exact sequence from Theorem 4.3.13, applying what we have learned from Corollary 5.3.3:

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\text{cts}}(\mathbb{G}_1, A_1^{**}) \arrow[r] & H^1_{\text{cts}}(\mathbb{G}_n, A_n^{**}) \arrow[r, "b"] & H^1_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**})^{\Pi_n} \arrow[d, "\cong"'] \\ & & & \mathbb{Z}_p \end{tikzcd}$$

Our current goal is to prove that the map labeled b is surjective. To do this, it suffices to show that the image under b of the class $\varepsilon_p(\Sigma^2\mathbb{S}_{K(n)})$ is primitive, i.e. that it is a generator of the $\mathbb{Z}_p$. More precisely, we will prove:

Theorem 5.4.1. *The map b in Theorem 4.3.13 sends $\varepsilon_p(\Sigma^2\mathbb{S}_{K(n)})$ to the distinguished generator of the $\mathbb{Z}_p$-module $H^1_{\text{proét}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**})^{\Pi_n}$.*

The players in the proof of Theorem 5.4.1 fit into the following commutative diagram:

$$
\begin{tikzcd}
H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**}) \arrow[d, "g_0"] \arrow[dd, bend right=60, "b"'] & & H^1_{\mathrm{pro\acute{e}t}}(\mathfrak{M}_n, \mathcal{O}^*) \\
H^1_{\mathrm{pro\acute{e}t}}(\mathfrak{M}_n, \mathcal{O}^{**}) \arrow[urr, "g_1"] \arrow[d, "e_1"] \arrow[drr, "f_1"] & & H^1_{\mathrm{cts}}(\Pi_n, \mathcal{O}^*(\mathcal{H}_C^{n-1})) \arrow[u, "g_2"'] \arrow[d, "f_2"] \\
H^0(\Pi_n, H^1_{\mathrm{pro\acute{e}t}}(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})) \arrow[rr, "\phi_2"'] \arrow[d, "i_1"', "\cong"] & & H^1_{\mathrm{cts}}(\Pi_n, H^1_{\mathrm{pro\acute{e}t}}(\mathcal{H}_C^{n-1}, \mathbb{Z}_p(1))) \arrow[d, "i_2"] \\
H^0(\mathrm{GL}_n(\mathbb{Z}_p), \mathrm{St}_1(\mathbb{Q}_p/\mathbb{Z}_p)^*) \arrow[rr, "\partial"] & & H^1_{\mathrm{cts}}(\mathrm{GL}_n(\mathbb{Z}_p), \mathrm{St}_1(\mathbb{Z}_p)^*).
\end{tikzcd}
\tag{5.4.2}
$$

We explain the maps appearing in (5.4.2). The map labeled g_0 arises from the equivalence $\mathfrak{M}_n \cong [\mathrm{LT}^{\diamond}_{n,K}/\mathbb{G}_n]$, g_1 arises from the inclusion $\mathcal{O}^{**} \to \mathcal{O}^*$, and g_2 arises from the equivalence $\mathfrak{M}_n \cong [\mathcal{H}_C^{n-1}/\Pi_n]$. The map labeled ϕ_2 appears as a connecting map when applying the long exact sequence in continuous Π_n-cohomology to the short exact sequence of Π_n-modules in Corollary 5.3.2. The bottom map labeled ∂ is the connecting map from (5.1.3); it is injective as noted in the discussion around (5.1.3). The map i_1 is the isomorphism from Corollary 5.3.3, and i_2 arises from the identification of $H^1_{\mathrm{pro\acute{e}t}}(\mathcal{H}_C^{n-1}, \mathbb{Z}_p(1))$ with $\mathrm{St}_1(\mathbb{Z}_p)^*$ in Theorem 5.2.1. Finally, e_1 is the map from (4.3.6) arising in the edge sequence, and $b = e_1 \circ g_0$ and $f_1 = \phi_2 \circ e_1$ are the composites.

For the construction of f_2, start with the short exact sequence of Π_n-equivariant sheaves on $\mathcal{H}_C^{n-1}$:

$$0 \to \mathbb{Z}_p(1) \to \lim_p \mathcal{O}^* \to \mathcal{O}^* \to 0.$$

Consider the boundary map

$$\mathcal{O}^*(\mathcal{H}_C^{n-1}) \to H^1_{\mathrm{pro\acute{e}t}}(\mathcal{H}_C^{n-1}, \mathbb{Z}_p(1))$$

which is a map of Π_n-modules. Applying $H^1_{\mathrm{cts}}(\Pi_n, -)$ gives

$$f_2 \colon H^1_{\mathrm{cts}}(\Pi_n, \mathcal{O}^*(\mathcal{H}_C^{n-1})) \to H^1_{\mathrm{cts}}(\Pi_n, H^1_{\mathrm{pro\acute{e}t}}(\mathcal{H}_C^{n}, \mathbb{Z}_p(1))).$$

Lemma 5.4.3. *Let $a_1 = g_0\varepsilon(\Sigma^{2(p^n-1)}\mathbb{S}_{K(n)})$, and let $a_2 \in H^1_{\mathrm{cts}}(\Pi_n, \mathcal{O}^*(\mathcal{H}_C^{n-1}))$ be the class of the cocycle $g \mapsto (g(\ell)/\ell)^{p^n-1}$, where $\ell \in \mathcal{O}^*(\mathcal{H}_C^{n-1})$ is a $\mathbb{Q}_p$-rational linear form. Then $g_1(a_1) = g_2(a_2)$.*

Proof. Recall that H is the universal formal group over $\mathfrak{M}_n$, and $\mathrm{Lie}\, H[1/p]$ is its corresponding line bundle. Also recall that by Theorem 3.7.3, under the isomorphism $\mathfrak{M}_n \cong [\mathcal{H}_C^{n-1}/\Pi_n]$, the line bundle $\mathrm{Lie}\, H[1/p]$ corresponds to $\mathcal{O}(1)|_{\mathcal{H}_C^{n-1}}$. Since ℓ is a nowhere vanishing section for $\mathcal{O}(1)|_{\mathcal{H}_C^{n-1}}$, the image of a_2 in $H^1_{\mathrm{pro\acute{e}t}}(\mathfrak{M}_n, \mathcal{O}^*)$ is $(\mathrm{Lie}\, H[1/p])^{\otimes(p^n-1)}$. This equals $g_1(a_1)$, ultimately because $\varepsilon(\Sigma^2\mathbb{S}_{K(n)}) \in H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^*)$ is the class of $\mathrm{Lie}\, H$ as a $\mathbb{G}_n$-equivariant invertible A_n-module. □

Proposition 5.4.4. *Consider two classes $x_1 \in H^1_{\mathrm{pro\acute{e}t}}(\mathfrak{M}_n, \mathcal{O}^{**})$ and $x_2 \in H^1_{\mathrm{cts}}(\Pi_n, \mathcal{O}^*(\mathcal{H}_C^{n-1}))$. If $g_1(x_1) = g_2(x_2)$, then $f_1(x_1) = f_2(x_2)$.*

The proof of Proposition 5.4.4 is postponed to the next subsection.

Proof of Theorem 5.4.1 assuming Proposition 5.4.4. By Lemma 5.4.3 and Proposition 5.4.4, we have

$$f_1 g_0 \varepsilon(\Sigma^{2(p^n-1)}\mathbb{S}_{K(n)}) = (p^n - 1) f_2(c),$$

where c the class of the cocycle $g \mapsto g(\ell)/\ell$ in $H^1_{\mathrm{cts}}(\Pi_n, \mathcal{O}^*(\mathcal{H}^{n-1}_C))$. The prime-to-$p$ part of $H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^*)$ has order (p^n-1), and so $(p^n-1)\varepsilon_p(\Sigma^2\mathbb{S}_{K(n)}) = \varepsilon(\Sigma^{2(p^n-1)}\mathbb{S}_{K(n)})$. Since (p^n-1) acts invertibly on the $\mathbb{Z}_p$-module $H^1_{\mathrm{cts}}(\Pi_n, H^1_{\mathrm{pro\acute{e}t}}(\mathcal{H}^{n-1}_C, \mathbb{Z}_p(1)))$, we must have $f_1 g_0 \varepsilon_p(\Sigma^2\mathbb{S}_{K(n)}) = f_2(c)$. Apply i_2 and use the commutativity of the diagram in (5.4.2) to find that $\partial i_1 b\varepsilon_p(\Sigma^2\mathbb{S}_{K(n)})$ equals the cocycle $g \mapsto \delta_{g(\ell)} - \delta_\ell$ from Lemma 5.1.4. Since ∂ is injective and i_1 is an isomorphism, we conclude that $b\varepsilon_p(\Sigma^2\mathbb{S}_{K(n)})$ is the distinguished generator of $H^1_{\mathrm{pro\acute{e}t}}(\mathcal{H}^{n-1}_C, \mathcal{O}^{**})^{\Pi_n}$. □

5.5. **Proof of Proposition 5.4.4 and Theorem 2.7.1.** We begin with some preparatory lemmas.

Lemma 5.5.1. *Assume that C and D are complexes in $D(\mathrm{Ab})$ such that $C \simeq \tau^{[0,1]}C$ and $D \simeq \tau^{[0,1]}D$ (i.e., C, D are concentrated in cohomological degrees 0 and 1). Assume that $H^*(C)$ is p-divisible and $H^*(D)$ is derived p-complete. Then every map $f\colon C \to D$ factors as*

$$C \to H^1(C)[-1] \to H^0(D) \to D,$$

where $H^1(C)$ and $H^0(D)$ are viewed as complexes concentrated in degree 0.

Proof. Since the cohomology groups $H^*(D)$ are derived p-complete, D is derived p-complete, so the map f factors as

$$C \to \widehat{C} \to D,$$

where $\widehat{C}$ is the derived p-completion of C. Since $H^*(C)$ is p-divisible, we have $H^i(\widehat{C}) = \widehat{H^{i+1}(C)}$. In particular, we have $\widehat{C} \simeq \tau^{[-1,0]}\widehat{C}$. We thus get a decomposition

$$C \to \widehat{C} \to H^0(\widehat{C}) \to H^0(D) \to D$$

and $H^0(\widehat{C}) \cong \widehat{H^1(C)[-1]}$. We get that the composition $C \to \widehat{C} \to H^0(\widehat{C})$ factors as

$$C \to H^1(C)[-1] \to \widehat{H^1(C)[-1]} \simeq H^0(\widehat{C}).$$

Combining the last two displayed factorizations gives the claim. □

When A is a p-divisible sheaf (for any topology) there is a short exact sequence of sheaves

$$0 \to T_pA \to \varprojlim_p A \to A \to 0,$$

where $T_pA = \varprojlim_m A_{p^m}$. This induces a map $A \to T_pA[1]$. Now, the map $\tau^{[0,1]}R\Gamma(A) \to \tau^{[0,1]}R\Gamma(T_pA[1])$ satisfies the requirements of Lemma 5.5.1. This provides us with a factorization:

$$\begin{tikzcd}
\tau^{[0,1]}R\Gamma(A) \arrow[r] \arrow[d] & H^1(A)[-1] \arrow[d] \\
\tau^{[0,1]}R\Gamma(T_pA[1]) & H^1(T_pA)[0]. \arrow[l]
\end{tikzcd}$$

Lemma 5.5.2. *In the derived category of abelian groups with Π_n-action, the map*

$$H^1_{\mathrm{pro\acute{e}t}}(\mathcal{H}^{n-1}_C, \mathbb{Z}_p(1))[0] \to \tau^{[0,1]}R\Gamma(\mathcal{H}^{n-1}_C, \mathbb{Z}_p(1)[1])$$

is sent to an isomorphism via the functor

$$A \mapsto H^1_{\mathrm{cts}}(\Pi_n, A) := H^1(A^{h\Pi_n}).$$

Proof. Consider the cofiber sequence

$$H^1_{\mathrm{\acute{e}t}}(\mathcal{H}^{n-1}_C, \mathbb{Z}_p(1))[0] \to \tau^{[0,1]}R\Gamma_{\mathrm{\acute{e}t}}(\mathcal{H}^{n-1}_C, \mathbb{Z}_p(1)[1]) \to H^2_{\mathrm{\acute{e}t}}(\mathcal{H}^{n-1}_C, \mathbb{Z}_p(1))[-1].$$

It is enough to show that

$$H^0(\Pi_n, H^2_{\mathrm{\acute{e}t}}(\mathcal{H}^{n-1}_C, \mathbb{Z}_p(1))) = 0.$$

From Theorem 5.2.1 we have $H^2_{\text{ét}}(\mathcal{H}_C^{n-1}, \mathbb{Z}_p(1)) \cong \mathrm{St}(\mathbb{Z}_p)^*(-1)$, which has no nonzero Π_n-fixed vectors because $\mathbb{Z}_p(-1)^{\Gamma_{\mathbb{Q}_p}} = 0$. $\square$

Proof of Proposition 5.4.4. Consider the following diagram, in which all cohomology is computed on the pro-étale site of $\mathcal{H}_C^{n-1}$:

$$\begin{tikzcd}
& \tau^{[0,1]}R\Gamma(\mathcal{O}^{**}) \arrow[rr] \arrow[dl] \arrow[dd] & & H^1(\mathcal{O}^{**})[-1] \arrow[dd] \\
\tau^{[0,1]}R\Gamma(\mathcal{O}^{*}) \arrow[rr, dash] \arrow[dr] & & H^0(\mathcal{O}^*)[0] \arrow[dr] & \\
& \tau^{[0,1]}R\Gamma(\mathbb{Z}_p(1)[1]) & & H^0(\mathbb{Z}_p(1)[1]). \arrow[ll]
\end{tikzcd}$$

The diagram is commutative: the front rectangle commutes by Lemma 5.5.1, the bottom parallelogram commutes by naturality of the edge maps, and the left triangle arises from the composition $\mathcal{O}^{**} \to \mathcal{O}^* \to \mathbb{Z}_p(1)[1]$.

Applying the functor $A \mapsto H^1(A^{h\Pi_n})$ to the above diagram yields:

$$\begin{tikzcd}
& H^1(\mathfrak{M}_n, \mathcal{O}^{**}) \arrow[rr, "e_1"] \arrow[dl, "g_1"'] \arrow[dd, "\phi_1"'] & & H^1(\mathcal{H}_C^{n-1}, \mathcal{O}^{**})^{\Pi_n} \arrow[dd, "\phi_2"] \\
H^1(\mathfrak{M}_n, \mathcal{O}^{*}) \arrow[rr, dash, "g_2"] \arrow[dr, "\delta"'] & & H^1(\Pi_n, \mathcal{O}^*(\mathcal{H}_C^{n-1})) \arrow[dr, "f_2"] & \\
& H^1(\Pi_n, R\Gamma(\mathcal{H}_C^{n-1}, \mathbb{Z}_p(1)[1])) & & H^1(\Pi_n, H^1(\mathcal{H}_C^{n-1}, \mathbb{Z}_p(1)[1])). \arrow[ll, "e_2", "\sim"']
\end{tikzcd}$$

Here e_2 is an isomorphism by Lemma 5.5.2. We can now prove Proposition 5.4.4, in which $f_1 = \phi_2 \circ e_1$. Given that $g_1(a_1) = g_2(a_2)$, we have

$$f_1(a_1) = \phi_2 e_1(a_1) = e_2^{-1}\delta g_1(a_1) = e_2^{-1}\delta g_2(a_2) = f_2(a_2),$$

as desired. $\square$

Theorem 5.4.1 is now proved. We explain how it implies Theorem 2.7.1, which in turn implies our main theorem (Theorem 2.4.3), see the discussion following Theorem 2.7.1.

Proof of Theorem 2.7.1. We first consider the case $(n,p) \neq (2,2)$. Theorem 4.3.13 together with Theorem 5.4.1 state that there is an exact sequence

$$0 \to H^1_{\text{cts}}(\mathbb{G}_1, A_1^{**}) \xrightarrow{\det_{\mathrm{LT}}^*} H^1_{\text{cts}}(\mathbb{G}_n, A_n^{**}) \to \mathbb{Z}_p \to 0$$

in which $\varepsilon_p(\Sigma^2 \mathbb{S}_{K(n)})$ is sent to the generator of the $\mathbb{Z}_p$. Since $\mathbb{Z}_p$ is free, the exact sequence splits:

$$H^1_{\text{cts}}(\mathbb{G}_n, A_n^{**}) \cong \mathbb{Z}_p \oplus H^1_{\text{cts}}(\mathbb{G}_1, A_1^{**}). \tag{5.5.3}$$

Similarly for $(n,p) = (2,2)$, we have an isomorphism

$$H^1_{\text{cts}}(\mathbb{G}_2, A_2^{**}) \cong \mathbb{Z}_2 \oplus H^1_{\text{cts}}(\mathbb{G}_1, A_1^{**}) \oplus \mathbb{Z}/2, \tag{5.5.4}$$

where in both cases the $\mathbb{Z}_p$ summand is generated by $\varepsilon_p(\Sigma^2\mathbb{S}_{K(n)})$. The term $H^1_{\mathrm{cts}}(\mathbb{G}_1, A_1^{**})$ has been computed in Lemma 2.9.1; substituting this into (5.5.3) resp. (5.5.3) yields isomorphisms

$$H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**}) \cong \begin{cases} \mathbb{Z}_p^2 & \text{if } p > 2; \\ \mathbb{Z}_2^2 \oplus (\mathbb{Z}/2)^{\oplus 2} & \text{if } p = 2 \text{ and } n > 2; \\ \mathbb{Z}_2^2 \oplus (\mathbb{Z}/2)^{\oplus 3} & \text{if } p = 2 \text{ and } n = 2. \end{cases}$$

As explained in Theorem 4.3.13, a generator of the torsion-free part of $H^1_{\mathrm{cts}}(\mathbb{G}_1, A_1^{**})$ is sent to the determinant in $H^1_{\mathrm{cts}}(\mathbb{G}_n, A_n^{**})$, thus finishing the proof. □

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