Inverse Trig Functions

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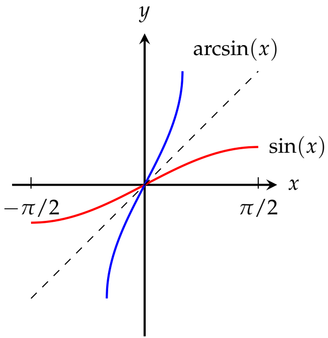
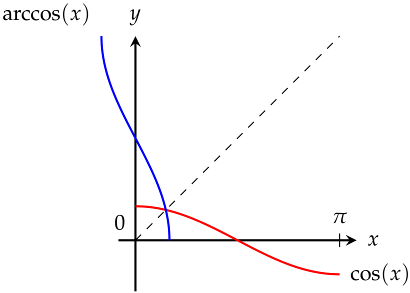
October 7, 2024


y
arcsin(x)
sin(x)
x
−π/2
π/2


arccos(x)
y
0
π
x
cos(x)

Election Info

Today, October 7, is the last day you may register to vote in the November 5, 2024 election. You must be registered by 4 PM today if you wish to vote in the election.

For more information please see here (this is a Commonwealth of Kentucky government website)

Unit II

- September 25 - Derivatives of Exponential Functions (§2.7)
- September 27 - Derivatives of Trig Functions (§2.8)
- September 30 - The Chain Rule (§2.9)
- October 2 - The Natural Logarithm (§2.10)
- October 4 - Implicit Differentiation (§2.11)
- October 7 - Inverse Trig Functions (§2.12)
- October 9 - The Mean Value Theorem (§2.13)
- October 11 - Higher-Order Derivatives (§2.14)
- October 14 - Velocity and Acceleration (§3.1)
- October 16 - Linear and Quadratic Approximation (§3.4.1–3.4.3)
- October 18 - Exam II Review
- October 21 - Exam II Review
- October 22 - Exam II, 5:00-7:00 PM

Reminders for the Week of October 7–11

- Webwork 2.10 is due on Monday, October 7 (tonight!)
- Quiz 5 on 2.7–2.9 is due on Thursday, October 10
- Webworks 2.11 and 2.12 are due on Friday, October 11
- There is no written assignment for the week of October 7–11

Goals of the Day

Today we'll cover the following topics:

- Definitions of inverse trig functions arcsin, arccos, arctan, arcsec including domains and ranges
- Derivatives of arcsin, arccos, arctan, and arcsec
- Practice

The arcsin function

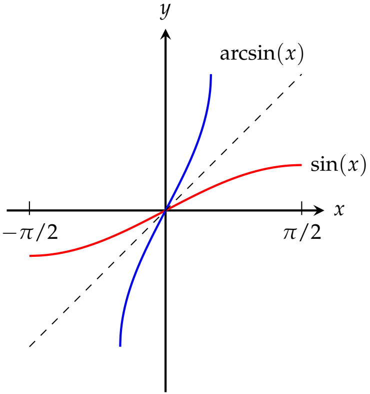
To make the sine function one-to-one we restrict its domain:

Function:	$\sin(x)$
Domain:	$[-\pi/2, \pi/2]$
Range:	$[-1, 1]$

Its inverse function is the arcsin function:

Function:	$\arcsin(x)$
Domain:	$[-1, 1]$
Range:	$[-\pi/2, \pi/2]$

The function $\arcsin(x)$ is sometimes denoted $\sin^{-1}(x)$



The arcsin function: How it Works

From this table…

x	$\sin x$
$-\pi/2$	-1
$-\pi/3$	$-\sqrt{3}/2$
$-\pi/4$	$-\sqrt{2}/2$
$-\pi/6$	$-1/2$
0	0
$\pi/6$	$1/2$
$\pi/4$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$
$\pi/2$	1

Can you fill in this table?

x	$\arcsin(x)$
-1	$-\pi/2$
$-\sqrt{3}/2$	$-\pi/3$
$-\sqrt{2}/2$	$-\pi/4$
$-1/2$	$-\pi/6$
0	0
$1/2$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$
$\sqrt{3}/2$	$\pi/3$
1	1

IClicker Interlude

Recall that arcsin (also known as $\sin^{-1}$) is the inverse function for $\sin(x)$ on the domain $[-\pi/2, \pi/2]$.

So: $\arcsin(x)$ has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$

Which of the following statements are correct?

(A) $\sin(\arcsin(1/2)) = 1/2$

(R) $\arcsin(1/2) = \pi/6$

(C) $\arcsin(\sin(8\pi)) = 8\pi$

(S) $\sin(\arcsin(4)) = 4$

(I) $\sin(\arcsin(-1/2)) = -1/2$

(N) $\sin(\arcsin(\sqrt{2}/2)) = -\sqrt{2}/2$

Answer: **ARI**. In (C), 8π is not in the range of arcsin, in (S), 4 is not in the domain of arcsin, and in (N), the right-hand side should be $\sqrt{2}/2$, not $-\sqrt{2}/2$.

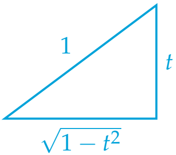
Practice with arcsin

Remember: $\arcsin(x)$ has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$ Find:

$\arcsin(-\sqrt{3}/2) = -\pi/3$

$\cos(\arcsin(t)) = \sqrt{1 - t^2}$

Use the triangle:

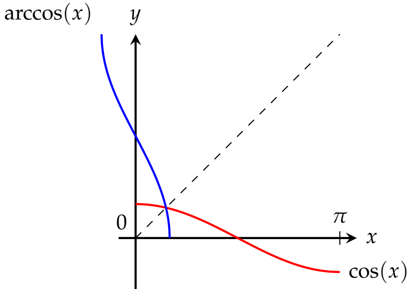


$\arcsin(\sin(5\pi/4)) = -\pi/4$

The arccos story

The cosine function is $1:1$ on $[0,\pi]$ and has range $[-1,1]$

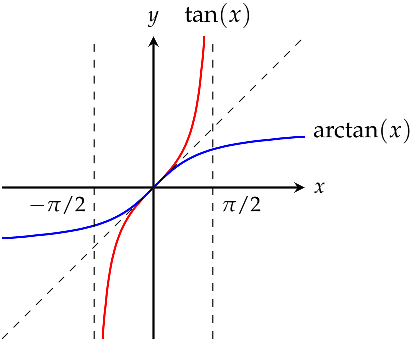
We define its inverse function $\arccos(x)$ with domain $[-1,1]$ and range $[0,\pi]$



The arctan story

The tangent function is 1 : 1 on $(-\pi/2, \pi/2)$ and has range $(-\infty, \infty)$

We define its inverse function $\arctan(x)$ with domain $(-\infty, \infty)$ and range $(-\pi/2, \pi/2)$



The arcsec story

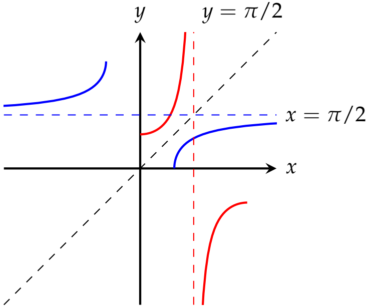
The function $\sec(x) = 1/\cos(x)$ is $1:1$ on $[0, \pi/2) \cup (\pi/2, \pi]$
We restrict $\sec(x)$ so that:

Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$

(what happens when $x = \pi/2$?)

We define $\text{arcsec}(x)$ to be the inverse function for $\sec(x)$ with:

Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[0, \pi/2) \cup (\pi/2, \pi]$

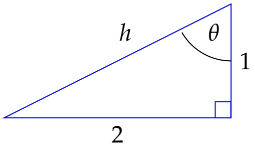


The Story Arc

	Domain	Range		Domain	Range
$\sin(x)$	$[-\pi/2, \pi/2]$	$[-1, 1]$	$\arcsin(x)$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos(x)$	$[0, \pi]$	$[-1, 1]$	$\arccos(x)$	$[-1, 1]$	$[0, \pi]$
$\tan(x)$	$(-\pi/2, \pi/2)$	$(-\infty, \infty)$	$\arctan(x)$	$(-\infty, \infty)$	$(\pi/2, \pi/2)$
$\sec(x)$	$[0, \pi/2) \cup (\pi/2, \pi]$	$(-\infty, -1] \cup [1, \infty)$	$\text{arcsec}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$

Triangles Revisited

Find the angle θ in the triangle at right using:



(1) The arctangent function

$\theta = \arctan(2)$

(2) The arcsecant function

$\theta = \text{arcsec}(h)$

The derivative of arcsin

We'll use implicit differentiation to solve for the derivative of $\arcsin(x)$:

Let $u(t) = \arcsin(t)$. Then

$$\sin(u(t)) = t$$

so

$$\cos(u(t))u'(t) = 1$$

or

$$u'(t) = \frac{1}{\cos(u(t))}$$

Since

$$\cos(\arcsin(t)) = \sqrt{1 - t^2}$$

we get

$$\frac{d}{dt}\arcsin(t) = \frac{1}{1 - t^2}$$

The derivative of arctan

We'll again use implicit differentiation.

Let $u(t) = \arctan(t)$. Then

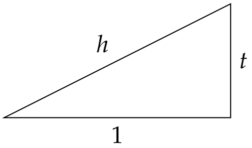
$$\tan(u(t)) = t$$

so

$$\sec^2(u(t))u'(t) = 1$$

or

$$u'(t) = \cos^2(u(t))$$



The derivative of arctan

We'll again use implicit differentiation.

Let $u(t) = \arctan(t)$. Then

$$\tan(u(t)) = t$$

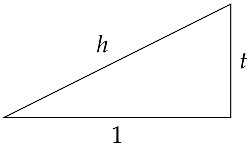
so

$$\sec^2(u(t))u'(t) = 1$$

or

$$u'(t) = \cos^2(u(t))$$

From the triangle at right, we get



$$h = \sqrt{1+t^2}$$

$$\cos(u(t)) = \frac{1}{\sqrt{1+t^2}}$$

so

$$\frac{d}{dt}\arctan(t) = \frac{1}{1+t^2}$$

The derivative of arcsec

Let $u = \text{arcsec}(t)$. Then

$$t = \sec(u)$$

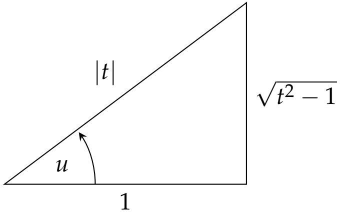
so

$$1 = \sec(u)\tan(u)\frac{du}{dt}$$

Hence

$$\frac{du}{dt} = \cos(u)\cot(u)$$

From the triangle at right we get



$$\frac{du}{dt} = \frac{1}{|t|}\cdot\frac{1}{\sqrt{t^2-1}}$$

Conclusion:

$$\frac{d}{dt}\text{arcsec}(t) = \frac{1}{|t|\sqrt{t^2-1}}$$

All the Inverse Trig Function Derivatives You'll Ever Need

$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$	$\frac{d}{dx}\text{arccot}(x) = -\frac{1}{1+x^2}$
$\frac{d}{dx}\text{arcsec}(x) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}\text{arccsc}(x) = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$

The Incredible Expanding Triangle

(1) Find an expression for θ in terms of a and x

$$\theta = \arcsin\left(\frac{x}{a}\right)$$

(2) Find an expression for the length of the side adjacent to θ

$$l = \sqrt{a^2 - x^2}$$

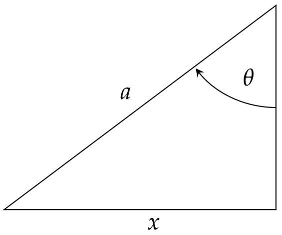
from the Pythagorean Theorem

(3) Find the rate of change of θ with respect to x

From $\theta = \arcsin(x/a)$ we get

$$\frac{d\theta}{dx} = \frac{1}{\sqrt{1 - x^2/a^2}} \cdot \frac{1}{a} \quad \text{chain rule}$$

$$= \frac{1}{\sqrt{a^2 - x^2}} = \frac{1}{l}$$



Practice with Derivatives

Find the derivative of $f(x) = x^2 \arccos x$. Recall that

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$$

(A) $2x\arccos(x) + x^2\arcsin(x)$

(R) $2x\arccos(x) - \dfrac{x^2}{\sqrt{1-x^2}}$

(C) $2x\arccos(x) + \dfrac{x^2}{\sqrt{1-x^2}}$

Answer: **R**. Use the product rule to get

$$\begin{aligned}\frac{d}{dx}\left(x^2\arccos(x)\right) &= 2x\arccos(x) + x^2\frac{d}{dx}\arccos(x)\\ &= 2x\arccos(x) - \frac{x^2}{\sqrt{1-x^2}}\end{aligned}$$

Practice with Derivatives

Find the derivative of

$$f(x) = \sqrt{4 - x^2} + \arcsin(6x)$$

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{4 - x^2}} \cdot (-2x) + \frac{1}{\sqrt{1 - (6x)^2}} \cdot 6 \qquad \text{(chain rule)} \\ &= -\frac{x}{\sqrt{4 - x^2}} + \frac{6}{\sqrt{1 - 36x^2}} \end{aligned}$$