The Mean Value Theorem

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Unit II

- September 25 - Derivatives of Exponential Functions (§2.7)
- September 27 - Derivatives of Trig Functions (§2.8)
- September 30 - The Chain Rule (§2.9)
- October 2 - The Natural Logarithm (§2.10)
- October 4 - Implicit Differentiation (§2.11)
- October 7 - Inverse Trig Functions (§2.12)
- October 9 - The Mean Value Theorem (§2.13)
- October 11 - Higher-Order Derivatives (§2.14)
- October 14 - Velocity and Acceleration (§3.1)
- October 16 - Linear and Quadratic Approximation (§3.4.1–3.4.3)
- October 18 - Exam II Review
- October 21 - Exam II Review
- October 22 - Exam II, 5:00-7:00 PM

Reminders for the Week of October 7–11

- Quiz 5 on 2.7–2.9 is due on Thursday, October 10
- Webworks 2.11 and 2.12 are due on Friday, October 11
- There is no written assignment for the week of October 7–11

Goals of the Day

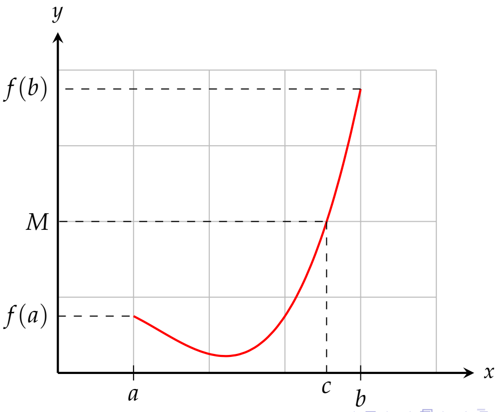
Today we'll cover the following topics:

- Statement of the Mean Value Theorem
- Examples illustrating hypotheses
- Using the derivative to show that a function is constant, increasing, or decreasing

Total Recall

Remember the last theorem we discussed?

Theorem (Intermediate Value Theorem) Suppose that f is continuous on $[a, b]$. Then, for any number M between $f(a)$ and $f(b)$, there is a c so that $f(c) = M$.

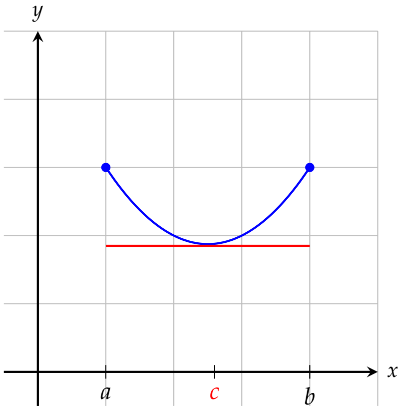


The Warm-Up: Rolle's Theorem

Theorem Suppose f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b), and $f(a) = f(b)$. Then there is a number c in the interval (a, b) so that $f'(c) = 0$.

This is true because every continuous function must have a maximum and a minimum value in $[a, b]$. So, either:

- One these is in the interval (a, b), or
- Both are at the endpoints and $f(a) = f(b)$ so the function is constant on the interval



IClicker Moment

Let $f(x) = |x|$ and let the domain of f be the interval $[-2, 2]$.

(R) There is no c in $(-2, 2)$ where $f'(c) = 0$

(O) Since $f(2) = f(-2)$, Rolle's Theorem tells us that $f'(c) = 0$ for some c in $(-2, 2)$

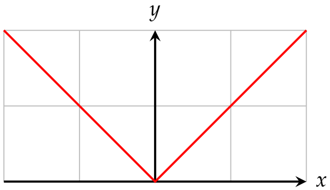
(L) Rolle's Theorem does not apply since f is not differentiable on $(-2, 2)$

(L) Rolle's Theorem does not apply since f is not differentiable on $(-2, 2)$

(E) Rolle's Theorem does not apply since f is not continuous on $[-2, 2]$

Answer: **RLL**.

f is continuous on $[-2, 2]$ but not differentiable at $x = 0$.



The Main Event: The Mean Value Theorem

Theorem If f is continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b), then there is a number c in the interval (a,b) so that

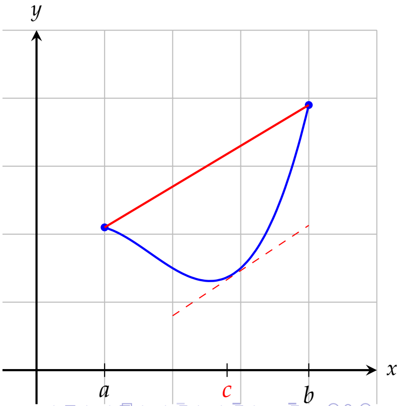
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Here $\dfrac{f(b)-f(a)}{b-a}$ is the slope of the (red) line from $(a, f(a))$ to $(b, f(b))$

$f'(c)$ is the slope of the (dashed red) tangent line to the graph at $(c, f(c))$.

The Mean Value Theorem states that, at some point in (a,b), the instantaneous rate of change $f'(c)$ equals the average rate of change

$$\frac{f(b)-f(a)}{b-a}$$



The Mean Value Theorem and Driving

Theorem If f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is a number c in the interval (a, b) so that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

If you drive for one hour and travel 60 miles, then at some point during that hour your exact speed is 60 miles per hour.

Problem: A trucker travels from one toll station to another, 25 miles from the first, in 15 minutes. At the second toll station, the trucker is pulled over by the police and is given a speeding ticket. Why?

The driver travels 25 miles in 15 minutes for an average speed of 100mi/hr. According to the mean value theorem, his instantaneous speed was 100mi/hr at least once during this time, so the driver was undoubtedly speeding.

Apologies to truck drivers, who are typically professional drivers and don't speed.

The Mean Value Theorem

Theorem If f is continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b), then there is a number c in the interval (a,b) so that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Proof: Let $m = \dfrac{f(b)-f(a)}{b-a}$ and $g(x) = f(x) - m(x-a)$ Then

$$g(a) = f(a) - 0 = f(a)$$
$$g(b) = f(b) - m(b-a) = f(a)$$

By Rolle's Theorem, there is a c in the interval (a,b) so that

$$g'(x) = f'(c) - m = 0$$

which means that $f'(c) = m$.

IClicker Moment

Theorem If f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is a number c in the interval (a, b) so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Let $f(x) = x^2$ and let the domain of f be the interval $[1, 4]$. Find a value c as guaranteed by the Mean Value Theorem.

(M) $c = 2$

(E) $c = 5/2$

(A) $c = 3$

(N) $c = 7/2$

Answer: **E.**

Since $f'(x) = 2x$ and

$$\frac{f(b) - f(a)}{b - a} = \frac{16 - 1}{4 - 1} = 5$$

we solve

$$2c = 5$$

to get $c = 5/2$.

How Many Solutions?

Show that the equation $f(x) = 0$ has exactly one solution, where

$$f(x) = x^3 - 7x^2 + 25x + 8$$

Step 1 There is at least one solution

Note that $f(0) = 8$, $f(-1) = -25$. What theorem guarantees a solution between -1 and 0? The Intermediate Value Theorem

Step 2: Show that there can't be two solutions.

Suppose that $f(a) = f(b) = 0$. Then $f'(c) = 0$ for some c between a and b by Rolle's Theorem.

$$f'(x) = 3x^2 - 14x + 25$$

Is this function ever zero?
No. The equation $3x^2 - 14x + 25 = 0$ has no solutions, as you can check using the quadratic formula

Recovering f from f'

Question: If we know f', do we know f?

Example 1: The functions

$$f_1(x) = x^2, \quad f_2(x) = x^2 + 5$$

both have the same derivative, $2x$.

So $f'(x)$ does not determine $f(x)$ uniquely!

Example 2 Suppose that f and g are defined on $(0, 2) \cup (3, 4)$. Let

$$f(x) = 0$$
$$g(x) = \begin{cases} 0, & 0 < x < 2, \\ 2, & 3 < x < 4. \end{cases}$$

Then $f'(x) = g'(x) = 0$ on $(0, 2) \cup (3, 4)$, but $f(x) \neq g(x)$

So, What is f' Good For?

Theorem

(i) If $f'(x) = 0$ for all x in an interval (a, b), f is constant there

(ii) If $f'(x) > 0$ for all x in an interval (a, b), then f is strictly increasing there

(iii) If $f'(x) < 0$ for all x in an interval (a, b), then f is strictly decreasing there

Proof: Choose x and y with $a < x < y < b$. Since f is continuous on $[x, y]$ and differentiable on (x, y), the MVT assures us that there is a point c in (x, y) so that

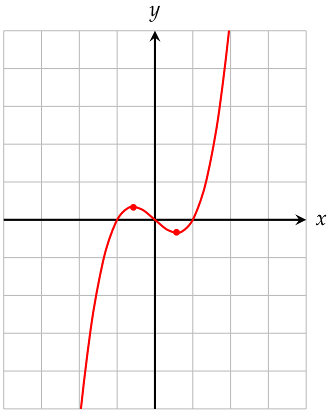
$$f(y) - f(x) = f'(c)(y - x)$$

If $f'(c) = 0$, then $f(y) = f(x)$

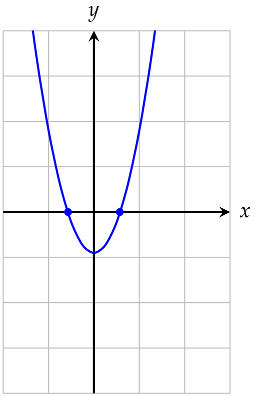
If $f'(c) > 0$, then $f(y) > f(x)$

If $f'(c) < 0$, then $f(y) < f(x)$

Graphical Proof



$f(x) = x^3 - x$



$f'(x) = 3x^2 - 1$

Recovering f from f'

Corollary If f and g are differentiable for x in an interval (a, b) and $f'(x) = g'(x)$ for all $x \in (a, b)$, there is a constant C so that $f(x) = g(x) + C$.

Proof: Consider $h(x) = f(x) - g(x)$. Then $h'(x) = 0$ on (a, b), so, by the preceding theorem, $h(x)$ is constant on (a, b). Hence,

$$f(x) - g(x) = C$$

or

$$f(x) = g(x) + C.$$

iClicker Interlude

Suppose that $f'(x) = 4x$ for $x \in (-\infty, \infty)$ and $f(1) = 3$. Find $f(x)$.

(M) $f(x) = 2x^2 + 1$

(V) $f(x) = 2x^2 - 1$

(T) $f(x) = 2x^2$

Since $f'(x) = 4x$, and the function $g(x) = 2x^2$ also has derivative $4x$, we conclude that $f(x) = g(x) + C = 2x^2 + C$. We can find C using the fact that $f(1) = 3$:

$$\begin{aligned} f(1) &= 3 \\ 2(1)^2 + C &= 3 \\ C &= 1 \end{aligned}$$

so the correct answer is **M**.

Estimating Function Values

Problem Suppose that $f(x)$ is continuous on $[-7,0]$ and differentiable on $(-7,0)$, $f(-7) = -3$, and $f'(x) \leq 2$. What is the largest possible value of $f(0)$?

According to the Mean Value Theorem with $a = -7$, $b = 0$,

$$\frac{f(0) - f(-7)}{0 - (-7)} = f'(c)$$

for some c between -7 and 0. Since $f'(x) \leq 2$ for all $x \in (-7,0)$, we get

$$\frac{f(0) - f(-7)}{7} \leq 2$$

$$\frac{f(0) + 3}{7} \leq 2$$

$$f(0) + 3 \leq 14$$

$$f(0) \leq 11$$