Higher-Order Derivatives

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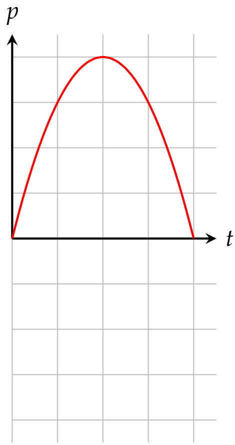
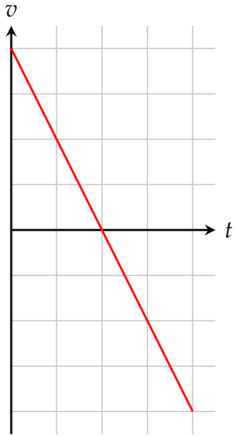
October 11, 2024

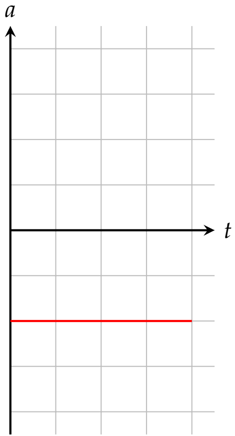
Preview



Position $p(t)$



Velocity $v(t) = p'(t)$



Acceleration $a(t) = v'(t)$

No Office Hours Today

I will not be able to hold office hours this afternoon. Please contact me at pperr0@uky.edu if you need help.

Unit II

- September 25 - Derivatives of Exponential Functions (§2.7)
- September 27 - Derivatives of Trig Functions (§2.8)
- September 30 - The Chain Rule (§2.9)
- October 2 - The Natural Logarithm (§2.10)
- October 4 - Implicit Differentiation (§2.11)
- October 7 - Inverse Trig Functions (§2.12)
- October 9 - The Mean Value Theorem (§2.13)
- October 11 - Higher-Order Derivatives (§2.14)
- October 14 - Velocity and Acceleration (§3.1)
- October 16 - Linear and Quadratic Approximation (§3.4.1–3.4.3)
- October 18 - Exam II Review
- October 21 - Exam II Review
- October 22 - Exam II, 5:00-7:00 PM

Reminders for the Week of October 7–11 and October 14–18

- Webworks 2.11 and 2.12 are due on Friday, October 11
- Webworks 2.13 and 2.14 are due on Tuesday, October 15
- Written Assignment 4 is due on Wednesday, October 16
- Quiz 6 is due on Thursday, October 17

Goals of the Day

Today we'll cover the following topics:

- Higher-order derivatives
- Derivatives of polynomials
- Velocity and Acceleration

Preview: if $s(t)$ gives the position of a moving object as a function of time:

$v(t) = s'(t)$ gives the *velocity* of the moving object
$a(t) = v'(t) = s''(t)$ gives the *acceleration* of the moving object

Higher Derivatives

If $f(x)$ is a function, $f'(x)$ is a new function. Its derivative is

$$\frac{d}{dx}f'(x) = f''(x)$$

Example:

$$f(x) = e^{2x}$$
$$f'(x) = 2e^{2x} \qquad \text{first derivative}$$
$$f''(x) = 4e^{2x} \qquad \text{second derivative}$$

In general, $f^{(n)}(x)$ is the nth derivative of f. Another notation is $\dfrac{d^n f}{dx^n}$

In the example, what is $f^{(4)}(x)$?

$f^{(4)}(x) = 8e^{2x}$ and, more generally

$$f^{(n)}(x) = 2^n e^{2x}$$

Mathematics, the Science of Patterns

Work out the following derivatives and see if you can see a pattern

$f(x)$	$\sin(x)$
$f'(x)$	$\cos(x)$
$f''(x)$	$-\sin(x)$
$f^{(3)}(x)$	$\cos(x)$
$f^{(4)}(x)$	$\sin(x)$
$f^{(8)}(x)$	$\sin(x)$
$f^{(64)}(x)$	$\sin(x)$

If n is any multiple of four, $f^n(x) = \sin x$

iClicker Interlude

Let $f(x) = \sin(x)$. Find $f^{(65)}(x)$

(S) $\sin(x)$

(I) $-\sin(x)$

(N) $\cos(x)$

(E) $-\cos(x)$

Answer: **N.**

Since 64 is a multiple of four, $f^{(64)}(x) = \sin x$. If we differentiate once more we get

$$f^{(65)}(x) = \cos x$$

Mathematics, the Science of Patterns (Encore)

Let $f(x) = x^5$

Find the derivatives of $f(x)$

$f(x)$	x^5
$\frac{d}{dx}f(x)$	$5x^4$
$\frac{d^2}{dx^2}f(x)$	$5 \cdot 4 \cdot x^3$
$\frac{d^3}{dx^3}f(x)$	$5 \cdot 4 \cdot 3x^2$
$\frac{d^4}{dx^4}f(x)$	$5 \cdot 4 \cdot 3 \cdot 2x$
$\frac{d^5}{dx^5}f(x)$	$5 \cdot 5 \cdot 3 \cdot 2 \cdot 1$

What is the pattern in these derivatives?

Guess

$$\frac{d^4}{dx^4}\left(x^4\right) = 4 \cdot 3 \cdot 2$$

Guess

$$\frac{d^6}{dx^6}\left(x^6\right) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$$

Guess

$$\frac{d^8}{dx^8}\left(x^6\right) = 0$$

Factorials!

For $n = 0, 1, 2, 3, \ldots$ we introduce

$$0! = 1 \quad \text{and } n! = n(n-1)!$$

(1) Fill in the following table:

0!	1
1!	1
2!	2
3!	6
4!	24
5!	120
6!	720
7!	5040

Factorials!

For $n = 0, 1, 2, 3, \ldots$ we introduce

$$0! = 1 \quad \text{and } n! = n(n-1)!$$

(1) Fill in the following table:

0!	1
1!	1
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5!	120
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7!	5040

(2) Find $\frac{8!}{5!}$

$$\frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$
$$= 8 \cdot 7 \cdot 6$$
$$= 336$$

Derivative of x^n

Problem: Find $\frac{d^n}{dx^n}(x^n)$.

Examples:

$$\frac{d}{dx}(x) = 1, \quad \frac{d^2}{dx^2}(x^2) = 2, \quad \frac{d^3}{dx^3}(x^3) = 6, \quad \frac{d^4}{dx^4}(x^4) = 24$$

Guess: $\frac{d^n}{dx^n}(x^n) = n!$

To check this, suppose we know that $\frac{d^n}{dx^n}(x^n) = n!$

Now compute

$$\begin{aligned}\frac{d^{n+1}}{dx^{n+1}}(x^{n+1}) &= \frac{d^n}{dx^n}\left((n+1)x^n\right) \\ &= (n+1)\frac{d^n}{dx^n}x^n \\ &= (n+1)n! = (n+1)!\end{aligned}$$

iClicker Interlude

Suppose that f has two derivatives on the real line, $f''(x) = 2$ for all x, $f'(0) = 3$, and $f(0) = 1$. What is f?

(F) $x^2 + 2x + 3$

(A) $x^2 + 3x + 2$

(C) $x^2 + 3x + 1$

(T) $x^2 + 2x + 1$

Answer: **C**

See the next slide for details!

Systematic Solution

$$f''(x) = 2, \quad f'(0) = 3, \quad f(0) = 1$$

First, $f''(x) = 2$ so

$$\frac{d}{dx}(f'(x) - 2x) = 0$$

By MVT,

$$f'(x) = 2x + C$$

Next, $f'(0) = 3$ so

$$C = 3$$

Since $f'(x) = 2x + 3$, we have

$$\frac{d}{dx}(f(x) - (x^2 + 3x)) = 0$$

By MVT

$$f(x) - (x^2 + 3x) = D$$

Since $f(0) = 1$ we get

$$D = 1$$

so

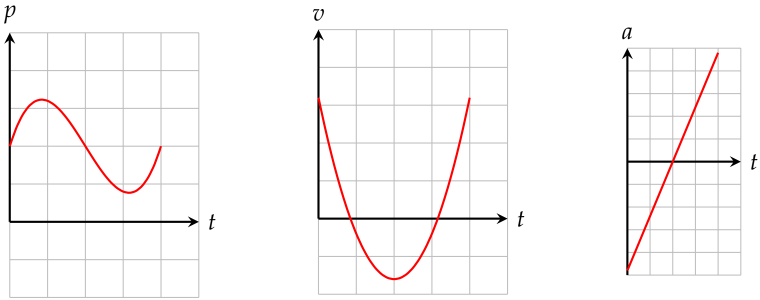
$$f(x) = x^2 + 3x + 1$$

Velocity and Acceleration

Suppose $p(t)$ denotes the position of a particle moving along a line at time t

$v(t) = p'(t)$ is the instantaneous *velocity*.If $v(t) > 0$ then $p(t)$ is increasing, while if $v(t) < 0$ then $p(t)$ is decreasing. We'll call $|p'(t)|$ the instantaneous *speed*, which tells us how fast a particle is moving but ignores the direction.

The function $a(t) = v'(t)$ tells us how the velocity is changing. If $a(t) > 0$, the velocity is increasing, while if $a(t) < 0$, the velocity is decreasing



Constant Acceleration

Problem Suppose that $h(t)$ represents the height of particle at time t. Suppose that

- The particle has height h_0 at time 0
- The particle has velocity v_0 at time 0
- The particle has constant acceleration $-g$

Find $h(t)$.

$$h''(t) = -g$$

so

$$h'(t) = -gt + C \qquad h'(0) = C = v_0$$

$$h(t) = -\frac{1}{2}gt^2 + v_0 t + D \qquad h(0) = D = h_0$$

so finally

$$h(t) = -\frac{1}{2}gt^2 + v_0 t + h_0$$

Oscillatory Motion

Suppose $p(t) = \sin(t)$

(a) Show that $p''(t) + p(t) = 0$

(b) Describe the equation in part (a) in words

- Take derivatives:

$$p'(t) = \cos t$$
$$p''(t) = -\sin t$$
$$p''(t) + p(t) = -\sin t + \sin t = 0$$

See the next slide for an example!

Oscillatory Motion

Suppose $p(t) = \sin(t)$

(a) Show that $p''(t) + p(t) = 0$

(b) Describe the equation in part (a) in words

- Take derivatives:

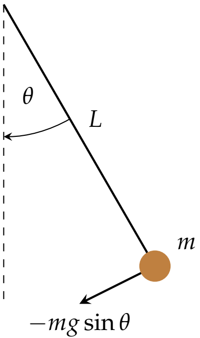
$$p'(t) = \cos t$$
$$p''(t) = -\sin t$$
$$p''(t) + p(t) = -\sin t + \sin t = 0$$

- Acceleration is the negative of position. For what physical situation does this actually happen?

See the next slide for an example!

Oscillatory Motion

A physical system that exhibits this kind of oscillatory motion is a *pendulum*



The equation of motion for a pendulum of mass m and with length L is:

$$mL^2 \frac{d^2\theta}{dt^2} = -(mg\sin\theta)L$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta \simeq -\frac{g}{L}\theta$$

We used $\sin\theta \simeq \theta$ for small angles

(For an amazing video that shows pendulum waves, see here!)

It's an iClicker Moment

If $f'(x) = \cos(x)$ for all x and $f(0) = 10$, what is $f(x)$?

(S) $f(x) = 10 + \sin(x)$

(I) $f(x) = 10 - \sin(x)$

(N) $f(x) = 10\sin(x)$

(E) $f(x) = 10\cos(x)$

Answer: **S.**

If $f'(x) = \cos x$ then $f(x) = \sin x + C$. Since $f(0) = 10$ we get

$$10 = \sin(0) + C$$

so

$$C = 10$$

and

$$f(x) = \sin x + 10$$