3.26pt

Math 213 - Exam II Review

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Reminders

- Exam II takes place Tonight, October 16, 5:00-7:00 PM

Unit II: Functions of Several Variables

Learning Goals

- Find out how to ace Exam II

Acknowledgement:

Most of the sample problems in this lecture were taken from Paul's Online Notes at Lamar University. You can find solutions to these problems in the Calculus III notes there.

Overview

- Arc length, velocity, acceleration
- Partial derivatives, chain rule
- Linear Approximation
- Directional derivatives, gradient
- Second derivative test for local extrema
- Closed interval method for global maxima and minima on a closed, bounded set
- Lagrange Multiplier Method
- Double integrals and Iterated Integrals (Section 15.1 *only*)

Arc Length, Velocity, Acceleration

If $\mathbf{r}(t)$ is a vector function:

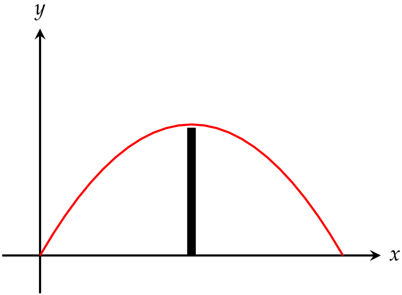
- $\mathbf{r}'(t)$ is the tangent vector to the space curve at the point $\mathbf{r}(t)$
- $\mathbf{r}''(t)$ is the acceleration of the particle at time t
- $|\mathbf{r}'(t)|$ is the speed of a particle moving along the space curve at time t

The *arc length* of a space curve $\mathbf{r}(t)$, $a \leq t \leq b$ is

$$L = \int_a^b |\mathbf{r}'(t)| \, dt.$$

By Popular Demand

A ball is thrown at 60° with a velocity of 20m/sec to clear a wall 2m high. How far away is the wall?



The Chain Rule

1. Find dz/dt if $z = 4x^2 + 3y^2$, $x(t) = \sin(t)$, $y(t) = \cos(t)$.

2. Suppose that $z = f(x, y) = 3x^2 - 2xy + y^2$, $x = 3u + 2v$, $y = 4u - v$. Find $\partial z/\partial u$ and $\partial z/\partial v$.

3. The equation $x^2 + y^3 + xyz = 1$ defines z implicitly as a function of x and y. Find $\partial z/\partial y$ in terms of x, y, and z.

Tangent Planes, Linear Approximation

Find the tangent plane to the graph of $f(x, y) = x^2 + 4y^2$ at the point $(2, 1, 8)$.

Using the linear approximation, estimate $f(0.1, 1.9)$ if $f(x, y) = \sqrt{8 - x^2 - y^2}$.

The Gradient

How to Compute It

If f is a function of *two variables*, $\nabla f(x,y) = \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \right\rangle$

If f ia function of *three variables*, $\nabla f(x,y,z) = \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z} \right\rangle$

What it Means

The magnitude of $\nabla f(a,b)$ (or $\nabla f(a,b,c)$) is the maximum rate of change of f at (a,b) (or (a,b,c))

The direction of $\nabla f(a,b)$ (or $\nabla f(a,b,c)$) is the direction of the maximum rate of change of f at (a,b) (or (a,b,c))

The Gradient

What it Does

The directional derivative of $f(x, y)$ at (a, b) in the direction $\mathbf{u}$ (where $\mathbf{u}$ is a *unit vector* is

$$D_{\mathbf{u}} f(a, b) = \nabla f(a, b) \cdot \mathbf{u}.$$

The gradient of a function of two variables is perpendicular to level curves of f

The gradient of a function of three variables is perpendicular to level surfaces of f

1. Find the maximum rate of change of $f(x, y) = 3x^2 + 4y^2$ at $(1, 2)$, and find the direction $\mathbf{u}$ of that maximum rate of change
2. Find the directional derivative of $f(x, y) = e^{xy}$ at the point $(1, 2)$ in the direction $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$
3. Find the equation of the tangent plane to the level surface of $x^2 + 4y^2 + 9z^2 = 17$ at the point $(2, 1, 1)$.

Second Derivative Test

- Local extrema occur at critical points, i.e., points (a, b) where $f_x(a, b) = f_y(a, b) = 0$
- A critical point corresponds to a local maximum or minimum if

$$D = f_{xx}(a, b) f_{yy}(a, b) - f_{xy}(a, b)^2$$

  is positive
- A critical point with $D > 0$ is a local minimum if $f_{xx}(a, b) > 0$, and a local maximum if $f_{xx}(a, b) < 0$

1. Find the local maxima and minima for the function $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$
2. Find the point on the plane $4x - 2y + z = 1$ closest to the point $(-2, -1, 5)$

Closed Set Method

To find the maximum and minimum of a function f on a bounded closed set D:

1. Find all local maxima and minima of f in D using the second derivative test
2. Find the maximum and minimum of f on the boundary of D using the Closed Interval Method from Calculus I

Find the absolute maximum and minimum of $f(x, y) = 2x^2 - y^2 + 6y$ in the region D with $x^2 + y^2 \leq 16$.

Lagrange Multipliers

A constrained optimization problem consists of:

- An *objective function* f to be maximized or minimized
- One or more *constraint equations* which must also be satisfied

For a constrained optimization problem with one constraint, two variables, solve:

$$\nabla f = \lambda \nabla g \qquad \text{(two equations)}$$
$$g(x,y) = c \qquad \text{(one equation)}$$

For two constraints, three variables, solve:

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \qquad \text{(three equations)}$$
$$g_1(x,y,z) = c_1 \qquad \text{(one equation)}$$
$$g_2(x,y,z) = c_2 \qquad \text{(one equation)}$$

Lagrange Multipliers

1. Find the maximum and minimum of the function $f(x, y) = 5x - 3y$ on the circle $x^2 + y^2 = 136$

2. Find the maximum of $f(x, y, z) = 4y - 2z$ subject to the constraints $2x - y - z = 2$ and $x^2 + y^2 = 1$

Double Integrals

The *double integral* of a function f over a rectangle $R = [a, b] \times [c, d]$ is denoted

$$\iint_R f(x, y)\, dA.$$

To compute it, we can compute the iterated integral

$$\int_a^b \left(\int_c^d f(x, y)\, dy \right) dx$$

or the iterated integral

$$\int_c^d \left(\int_a^b f(x, y)\, dx \right) dy.$$

1. Find $\iint_R 6xy^2\, dA$ if $R = [2, 4] \times [1, 2]$
2. Find $\iint_R xe^{xy}\, dA$ if $R = [-1, 2] \times [0, 1]$