Math 213 - Parametric Surfaces and Surface Integrals Supplement

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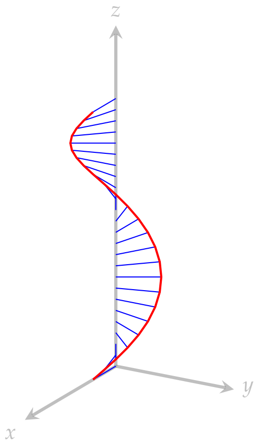
Parameterized Surface Examples

The *helicoid* is given by

$$\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$$

$$0 \le u \le 1,\ 0 \le v \le 2\pi$$

Notice that $\mathbf{r}(1,v) = \langle \cos v, \sin v, v\rangle$ is a helix curve



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$$\mathbf{r}_u(u,v) = \langle \cos v, \sin v, 0\rangle$$
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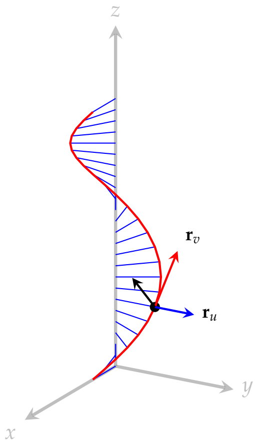
$$\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u\rangle$$



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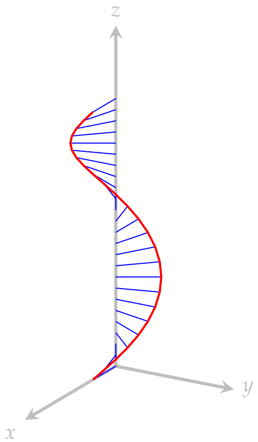
$$\begin{aligned} dS &= \sqrt{1+u^2}\, du\, dv \\ \mathbf{dS} &= \langle \sin v, -\cos v, u\rangle\, du\, dv \end{aligned}$$



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$$\mathbf{r}(u,v) = \langle (2+\cos u)\cos v, (2+\cos u)\sin v, \sin u \rangle$$

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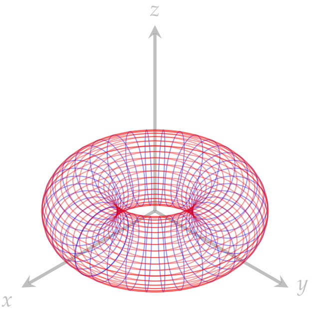

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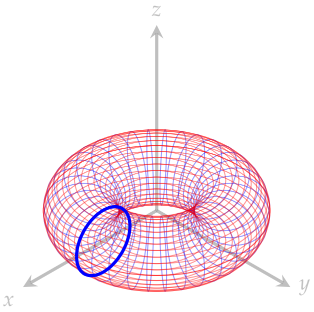

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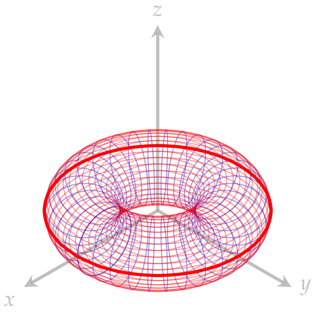

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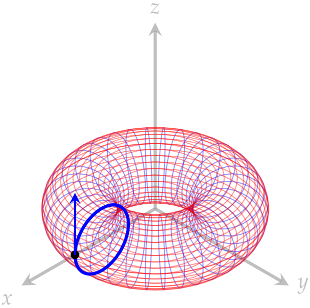

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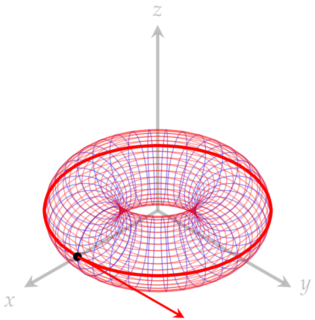

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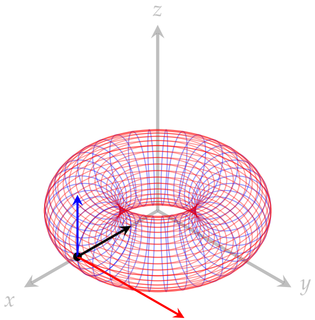

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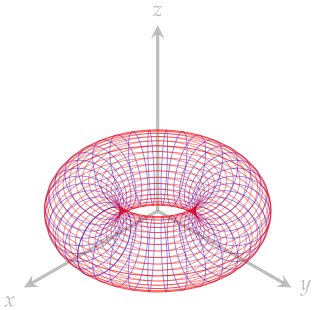
\end{aligned}$$

$$|\mathbf{r}_u \times \mathbf{r}_v| = |2+\cos u|$$



Vector Equations of Curves and Surfaces

Vector Equation of a Space Curve

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$
$$a \leq t \leq b$$

Tangent Vector

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

Element of Arc Length

$$ds = |\mathbf{r}'(t)|\, dt$$

Vector element of Arc Length

$$\mathbf{dr} = \mathbf{r}'(t)\, dt$$

Vector Equation of a Surface

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

$(u,v) \in D$, a domain in the uv plane

Tangent Vectors

$$\mathbf{r}_u(u,v) = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$

$$\mathbf{r}_v(u,v) = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

Element of Surface Area

$$dS = |\mathbf{r}_u \times \mathbf{r}_v|\; du\, dv$$

Vector Element of Surface Area

$$\mathbf{dS} = (\mathbf{r}_u \times \mathbf{r}_v)\; du\, dv$$

Line Integrals and Surface Integrals

Vector Equation of a Line

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle,$$

$$a \leq t \leq b$$

Line Integral of a Scalar Function

$$\int_C f\, ds = \int_a^b f(\mathbf{r}(t))\, \left|\mathbf{r}'(t)\right|\, dt$$

Line Integral of a Vector Function

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

also called "circulation of $\mathbf{F}$ around C"

Vector Equation of a Surface

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

$(u,v) \in D$, a domain in the uv plane

Surface Integral of a Scalar Function

$$\int_S f\, dS = \iint_D f(\mathbf{r}(u,v)) |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$$

Surface Integral of a Vector Function

$$\int_S \mathbf{F} \cdot \mathbf{dS} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv$$

also called the "flux of $\mathbf{F}$ out of the surface S" provided $\mathbf{r}_u \times \mathbf{r}_v$ points outward