Large-Time Asymptotics for the KP I Equation

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17 March 2025

ArXiV 2409.14480

The Kadomtsev-Petviashvili Equation

The KP I and KP II equations describe nonlinear waves of long wavelength propagating in the x-direction with transverse oscillation.

$$\begin{cases} (u_t + 6uu_x + u_{xxx})_x = 3\lambda u_{yy} \\ u(0,x,y) = u_0(x,y) \end{cases} \tag{1}$$

The KP I equation corresponds to $\lambda = 1$ and the KP II equation corresponds to $\lambda = -1$. Both equations are completely integrable.

- The inverse scattering map for KP I involves a *nonlocal Riemann-Hilbert problem*
- The inverse scattering map for KP II involves a $\bar{\partial}$ *problem*

Our goal is to obtain large-time asymptotics of solutions for KP I with small initial data. Our work builds on Xin Zhou's analysis of inverse scattering for KP I (*Comm. Math. Phys.* 1990). We will obtain different spatial asymptotics in different space-time régimes whose origin may be understood through the linearized KP I equation.

Inverse Scattering

The KP I equation is the consistency condition for the system (Dryuma 1974)

$$i\psi_y + \psi_{xx} + u\psi = 0$$

$$\psi_t + 4\psi_{xxx} + 6u\psi_x + 3\left(u_x - i\int_{-\infty}^{x} u_y\,dx'\right)\psi = 0$$

The first equation defines a scattering problem, and the second gives the time evolution of scattering data.

The *direct scattering map* takes a function $u \in \mathcal{S}(\mathbb{R}^2)$ to scattering data $T^{\pm}(k,l)$. If $u = u(t,x,y)$ solves KP I, then the scattering data evolves linearly in time:

$$T^{\pm}(k,l,t) = e^{4i(l^3-k^3)t}T^{\pm}(k,l)$$

The *inverse scattering map* takes time-evolved scattering data to the solution $u = u(t,x,y)$ in two steps:

(1) The time-evolved scattering data define a nonlocal Riemann-Hilbert problem for a function $\mu^l(k,x;y,t)$

(2) The solution $u(t,x,y)$ is recovered from the scattering data and μ^l via a reconstruction formula

Reconstruction Formula

Given the scattering data $T^{\pm}(k,l)$ and the scattering solution $\mu^l(k,x;y,t)$, we recover $u(t,x,y)$ via the small-data reconstruction formula

$$u(t,x,y) = \frac{1}{\pi}\frac{\partial}{\partial x}\left(\int e^{itS(k,l;\zeta,\eta)} f(k,l)\mu^l(l,x;y,t)\,dl\,dk\right) \tag{2}$$

where

$$\begin{aligned} \zeta = x/t, \quad \eta = y/t & \qquad \text{“slow” variables} \\ f(k,l) = T^+(k,l) + T^-(k,l), & \qquad \text{scattering data} \\ S(k,l;\zeta,\eta) = (l-k)\zeta - (l^2-k^2)\eta + 4(l^3-k^3) & \qquad \text{oscillatory phase} \end{aligned}$$

and $\mu^l(k,x;y,t)$ solves a nonlocal Riemann-Hilbert problem

Remark: The functions $T^{\pm}$ are “triangular”:

$$T^+(k,l) = 0, \quad l < k \qquad\qquad T^-(k,l) = 0, \quad l > k$$

which means the amplitude for (2) is not smooth. The map $u \mapsto T^{\pm}$ are continuous maps into $L^2(\mathbb{R}^2)$ for u of “small norm”

Long-Time Asymptotics

For $u \in \mathcal{S}(\mathbb{R}^2)$, let

$$\tilde{u}(l,y) = \frac{1}{\sqrt{2\pi}} \int e^{-ilx} u(x,y)\, dx$$

Theorem

Let $u_0 \in \mathcal{S}(\mathbb{R}^2)$ with $\|\tilde{u}\|_{L^1} = c < \sqrt{2\pi}$, and $\|\tilde{u}\|_{L^2_y L^{2,-1}_l} < \frac{1}{4}(1-c)$. Fix $\delta > 0$. Then

$$u(t,x,y) \underset{t\to\infty}{\lesssim} \begin{cases} t^{-1}, & 12\zeta - \eta^2 < -\delta, \\ t^{-2/3} & |12\zeta - \eta^2| < \delta, \\ o(t^{-1}), & 12\zeta - \eta^2 > \delta. \end{cases}$$

We can considerably relax regularity and obtain results for u_0 in certain weighted spaces. Our assumptions imply that $T^{\pm} \in L^2(\mathbb{R}^2)$

Regularity Assumptions

Molinet, Saut, and Tzvetkov (2002) proved global well-posedness of KPI for initial data in the space

$$Z = \{u \in L^2(\mathbb{R}^2) : \|u\|_Z < \infty\}$$

where

$$\|u\|_Z = \|u\|_{L^2(\mathbb{R}^2)} + \|u_{xxx}\|_{L^2(\mathbb{R}^2)} + \|u_y\|_{L^2(\mathbb{R}^2)} + \|u_{xy}\|_{L^2(\mathbb{R}^2)} + \left\|\partial_x^{-1} u_y\right\|_{L^2(\mathbb{R}^2)} + \left\|\partial_x^{-2} u_{yy}\right\|_{L^2(\mathbb{R}^2)} \tag{3}$$

We can prove our results for a space Z_w, continuously embedded in Z, which imposes additional regularity and decay constraints

Remarks

There is extensive discussion in the PDE literature on large-time asymptotics of u_x, but ours appears to be the first result on pointwise asymptotics of u for small data and large times. Papers that have influenced our work include:

- Hayashi, Naumkin, Saut: Asymptotics for large time of global solution to the generalized KP equation, *Comm. Math. Phys.*, 1999
- Hayashi, Naumkin: Large-time asymptotics for the KP equation, *Comm. Math. Phys.*, 2014
- Harrop-Griffiths, Ifrim, Tataru, The Lifespan of Small solutions to the KP-I, *International Math. Research Notices*, 2017

There are several key papers using inverse scattering techniques to find large-time asymptotics for KP I and KP II. These are

- Manakov, Santini, Takhtajan, Asymptotic behavior of the solutions of the KP equation (two-dimensional KdV equation), Physics Letters **75A** (6), 1980
- O. M. Kiselev, Asymptotic behavior of a solution for KP II equation, *Proc. Steklov Inst. Math.* (Approximation Theory, Asymptotic Expansions, 2001)

Previous Inverse Scattering Results - I

Manakov, Santini, and Takhtajan (1980) studied asymptotics of KPI using stationary phase and asymptotics on the solution to a Gelfand-Levitan-Marchenko integral equation. In the region $12\zeta - \eta^2 < 0$, they claimed

$$u(t,x,y) \underset{t\to\infty}{\sim} \frac{4}{t}\psi_\zeta(\zeta,\eta)\,\mathrm{Re}\left(K(\zeta,\zeta,\eta)e^{it\varphi(\zeta,\eta)} + \text{c.c.}\right)$$

where

$$\varphi(\xi,\eta) = \frac{1}{108}(\eta^2 - 12\zeta)^{\frac{3}{2}}$$

and K is derived from the solution to the Gelfand-Levitan-Marchenko equation

These authors only consider the region $\eta^2 - 12\zeta > 0$ and do not treat the case of degenerate stationary phase

Previous Inverse Scattering Results - II

Kiselev determined asymptotics of solutions to the KP II equation for small data (constraints are imposed on the scattering data) in three different spatial regions:

$$u(t,x,y) \underset{t\to\infty}{\sim} \begin{cases} -8t^{-1}\,\mathrm{Re}\left(e^{-11itr}\dfrac{\pi}{12ir}f(r+i\eta/12)+\text{c.c.}\right) & \\ \qquad +o(t^{-1}), & -(12\xi+\eta^2)t^{\frac{1}{3}} \gg 1 \\ o(t^{-1}), & (12\zeta+\eta^2)t^{\frac{1}{3}} \gg 1 \\ 8it^{-1}\sqrt{\pi}f(i\eta/12)F(z)+o(t^{-1}), m & |12\xi+12\eta^2| \ll 1 \end{cases}$$

where

$$r=\sqrt{-\eta^2-12\zeta}, \quad z=8t^{\frac{2}{3}}\left(\eta^2/12+\zeta\right)$$

See the review paper by Kiselev, *Journal of Math. Sciences* **138** (6), 2006, §3.3

A One-Dimensional Model Problem

Consider the initial value problem

$$\begin{cases} u_t(x,t) = u_{xxx}(x,t) & (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ u(x,0) = f(x) \end{cases}$$

whose solution by Fourier analysis is

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int e^{it\varphi(\xi;v)} \widehat{f}(\xi)\, d\xi, \quad \varphi(\xi;v) = \xi v - \xi^3$$

where $v = x/t$. The phase function $\varphi(\xi;v)$ has

- Nondegenerate critical points at $\xi = \pm(v/3)^{\frac{1}{2}}$ if $v > 0$
- No critical points if $v < 0$
- A degenerate critical point at $\xi = 0$ if $v = 0$

A One-Dimensional Model Problem

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int e^{it\varphi(\xi;v)} \widehat{f}(\xi)\, d\xi, \quad \varphi_\xi(\xi;v) = v - 3\xi^2$$

- For $v > 0$, we can use stationary phase methods to obtain (with $\xi_0 = (v/3)^{\frac{1}{2}}$)

$$u(vt,t) = 2\sqrt{\frac{2\pi}{6\xi_0 t}}\, \mathrm{Re}\left(\widehat{f}(\xi_0) e^{i\varphi(\xi_0;v) - i\pi/4}\right)$$

- For $v < 0$ we can use integration by parts to obtain

$$u(vt,t) \sim o(t^{-n}) \text{ for any } n \in \mathbb{N}$$

A One-Dimensional Model Problem

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int e^{it\varphi(\xi;v)} \widehat{f}(\xi)\, d\xi, \quad \varphi(\xi;v) = \xi v - \xi^3$$

- For $v \sim 0$,

$$u(vt,t) = (3t)^{-\frac{1}{3}} \int \widehat{f}(\xi/(3t)^{\frac{1}{3}}) e^{-i(\xi w + \xi^3/3)}\, d\xi$$

  where $w = -t^{\frac{2}{3}} v / 3^{\frac{1}{3}}$.

Recalling that

$$\mathrm{Ai}(z) = \frac{1}{2\pi} \int e^{i(zs + s^3/3)}\, ds$$

we have

$$u(vt,v) \sim \frac{2\pi}{(3t)^{\frac{1}{3}}} \widehat{u_0}(0)\, \mathrm{Ai}\left(-\frac{t^{\frac{2}{3}} v}{3^{\frac{1}{3}}} \right)$$

as $t \to \infty$ with $t^{\frac{2}{3}} v$ fixed

A Two-Dimensional Model Problem

$$\begin{cases} (v_t + v_{xxx})_x = 3\lambda v_{yy} \\ v(0,x,y) = v_0(x,y) \end{cases} \tag{4}$$

The linear problem has a solution by Fourier analysis:

$$v(t,x,y) = \frac{1}{2\pi} \int e^{i(p_1 x + p_2 y)} e^{it(p_1^3 + 3\lambda p_1^{-1} p_2^2)} \widehat{v}_0(p_1,p_2)\, dp_1\, dp_2$$

For $\lambda = 1$ (KP I) introduce "slow" variables $\zeta = x/t, \eta = y/t$ and let

$$p_1 = l - k, \quad p_2 = -(l^2 - k^2)$$

we get

$$v(t,x,y) = \frac{1}{\pi} \int e^{itS(k,l;\zeta,\eta)} \widehat{v}_0(l-k, k^2 - l^2)|l-k|\, dl\, dk$$

where

$$S(k,l;\zeta,\eta) = (l-k)\zeta - (l^2-k^2)\eta + 4(l^3 - k^3)$$

is a phase function with four stationary points

$$(k,l) = \frac{1}{12}(\eta \pm r, \eta \pm r), \qquad r = \sqrt{\eta^2 - 12\zeta}$$

A Two-Dimensional Model Problem

$$v(t,x,y) = \frac{1}{\pi} \int e^{itS(k,l;\zeta,\eta)} \widehat{v}_0(l-k, k^2-l^2)|l-k| \, dl \, dk$$

$$S(k,l;\zeta,\eta) = (l-k)\zeta - (l^2-k^2)\eta + 4(l^3-k^3)$$

Critical points:

$$(k,l) = \frac{1}{12}(\eta \pm r, \eta \pm r), \qquad r = \sqrt{\eta^2 - 12\zeta}$$

Make a change of variables

$$k \to k - \eta/12, \quad l \to l - \eta/12$$

Then

$$v(t,x,y) = \frac{1}{\pi} \int e^{12itS(k,l;a)} b(k,l)|l-k| \, dl \, dk$$

where

$$S(k,l;a) = a(l-k) + \frac{1}{3}(l^3-k^3), \qquad a = 12\xi - \eta^2$$

A Two-Dimensional Model Problem

$$v(t,x,y) = \frac{1}{\pi} \int e^{12itS(k,l;a)} b(k,l) |l-k| \, dl \, dk$$

where

$$S(k,l;a) = a(l-k) + \frac{1}{3}(l^3 - k^3), \qquad a = 12\xi - \eta^2$$

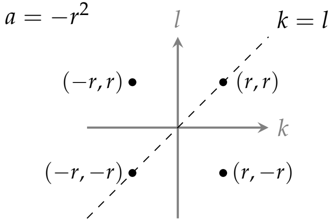
Note that

$$\frac{\partial S}{\partial k} = -(a+k^2) \qquad \frac{\partial S}{\partial l} = a + l^2$$

so, if $a = -r^2$, S has critical points at $(\pm r, \pm r)$

We have three regimes of asymptotic behavior:

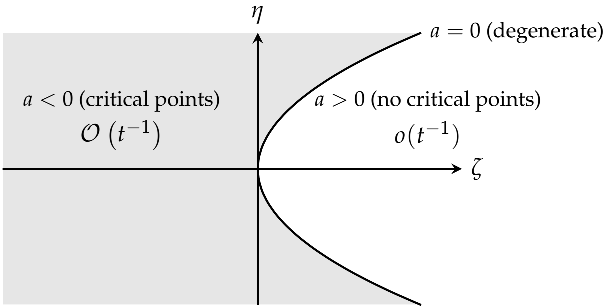
- $a < 0$: nondegenerate critical points
- $a = 0$: degenerate critical point
- $a > 0$: no critical points



A Two-Dimensional Model Problem

Spatial Asymptotics: Set $a = 12\zeta - \eta^2$ and recall that $\xi = x/t, \quad \eta = y/t$.

We will find that asymptotic behavior of the solution is as follows:



Asymptotics with Ai

Perhaps special functions provide an economical and shared culture analogous to books: places to keep knowledge in, so that we can use our heads for better things.

Malcolm Berry, "Why are special functions special?"
*Physics Today, **54**, 2001*

As we have seen, the Airy function

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(zs+s^3/3)}\,ds$$

arises naturally in the study of oscillatory integrals with degenerate stationary phase.

The Airy function satisfies the relations

$$|\,\mathrm{Ai}(x)| \leq \frac{C}{(1+|x|)^{\frac{1}{4}}}, \qquad -\infty < x < \infty$$

$$\mathrm{Ai}(-x) \underset{x\to\infty}{\sim} \frac{1}{\sqrt{\pi}}\left(\cos\left(\frac{2}{3}x^{\frac{3}{2}} - \pi/4\right) + \mathcal{O}\left(x^{-\frac{3}{2}}\right)\right)$$

Asymptotics - Stationary Phase

Consider the integral (for $a < 0$)

$$v(t,x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{k}^{\infty} e^{12itS(k,l;a)} b(k,l)(l-k)\psi(k,l)\, dl\, dk.$$

(here $\psi(k,l) = \varphi(k)\varphi(l)$ localizes to the neighborhood of a critical point). If $S(k;a) = ak + k^3/3, \quad S(l;a) = al + l^3/3$ then

$$v(t,x,y) = \frac{1}{\pi} \int f(k)\, g(k)\, dk = \frac{1}{\pi} \int \widehat{f}(-\xi)\widehat{g}(\xi)\, d\xi$$

where

$$f(k) = e^{-12itS(k,a)}, \quad g(k) = \int_{k}^{\infty} e^{12itS(l;a)} b(k,l)\psi(k,l)(l-k)\, dl$$

so that

$$\widehat{f}(\xi) = \frac{\sqrt{2}}{(12t)^{1/3}} \operatorname{Ai}\left((12t)^{\frac{2}{3}}(a - \xi/12t)\right)$$

We obtain an $\mathcal{O}\left(t^{-1}\right)$ estimate by combining the time-decay of f with careful estimates on the time decay of $\widehat{g}$.

Asymptotics - No Stationary Phase

Consider again (now for $a > 0$)

$$v(t,x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{k}^{\infty} e^{12itS(k,l;a)} b(k,l)(l-k)\, dl\, dk.$$

In the absence of stationary phase points, we may integrate by parts to obtain

$$v(t,x,y) = \frac{1}{\pi t} \left(\int_{-\infty}^{\infty} e^{-12itS(k;a)} \int_{k}^{\infty} e^{12itS(l;a)} A(k,l;a)\, dl \right)$$

for

$$A(k,l;a) = \frac{\partial}{\partial l} \left(\frac{(l-k)b(k,l)}{12(a+l^2)} \right)$$

and use a density argument to show that the integral is $o(1)$ as $t \to \infty$.

Asymptotics - Degenerate Stationary Phase

Suppose now that $a \sim 0$. We repeat the argument used for nondegenerate stationary phase with some modifications. As before we write

$$v(t,x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{k}^{\infty} e^{12itS(k,l;a)} b(k,l)(l-k)\varphi(k,l)\, dl\, dk$$

as

$$v(t,x,y) = \frac{1}{\pi} \int f(k) g(k)\, dk = \frac{1}{\pi} \int \widehat{f}(-\xi)\widehat{g}(\xi)\, d\xi$$

with

$$f(k) = e^{-12itS(k,a)}, \quad g(k) = \int_{k}^{\infty} e^{12itS(l;a)} \varphi(k,l) b(k,l)(l-k)\, dl$$

but we now only have

$$|\widehat{f}(\xi)| \lesssim t^{-\frac{1}{3}}$$

owing to degenerate stationary phase. We obtain an $\mathcal{O}\left(t^{-\frac{2}{3}}\right)$ estimate on $I(t,x,y)$ in this case.

Connections

(1) The map

$$v_0 \to \widehat{v}_0(l-k, k^2 - l^2)$$

is precisely the linearization of the KP I scattering transform at the 0 potential

(2) The oscillatory integral

$$v(t,x,y) = \frac{1}{\pi} \int e^{itS(k,l;\zeta,\eta)} \widehat{v}_0(l-k, k^2 - l^2)|l-k| \, dl \, dk$$

is precisely the linearization of the KP I reconstruction formula at the 0 potential, with the correct phase function

Nonlinear Problem - Reconstruction Formula

$$u(t,x,y) = \frac{1}{\pi}\int e^{itS(k,l;\zeta,\eta)} i(l-k)(T^+(k,l)+T^-(k,l))\mu^l(l,x;y,t)\,dl\,dk$$
$$+\frac{1}{\pi}\int e^{itS(k,l;\zeta,\eta)}(T^+(k,l)+T^-(k,l))\frac{\partial\mu^l}{\partial x}(l,x;y,t)\,dl\,dk$$

where

$$S(k,l;\zeta,\eta) = (l-k)\zeta - (l^2-k^2)\eta + 4(l^3-k^3)$$

and $\mu^l = \mu^l(k,x;y,t)$ solves the following nonlocal Riemann-Hilbert problem: let

$$(\mathcal{T}^{\pm}_{t,x,y}f)(k) = \int T^{\pm}(k,l)e^{itS(k,l;\zeta,\eta)}f(l)\,dl$$

and let $C_\pm$ be Cauchy projections on $L^2(\mathbb{R},dk)$ (recall that $\|C_\pm\|_{L^2\to L^2} = 1$ and $C_+ - C_- = I$). Then

$$\mu^l = 1 + \mathcal{C}_T\mu^l$$

where

$$\mathcal{C}_T f = C_+(\mathcal{T}^- f) + C_-(\mathcal{T}^+ f)$$

Nonlinear Problem - Reconstruction Formula

It will be useful to divide

$$u(t,x,y) = u_1(t,x,y) + u_2(t,x,y)$$

where

$$u_1(t,x,y) = \frac{1}{\pi}\int e^{itS(k,l;\xi,\eta)} i(l-k) f(k,l)\, dl\, dk$$

$$u_2(t,x,y) = \frac{1}{\pi}\int e^{itS(k,l;\xi,\eta)} i(l-k) f(k,l) (\mu^l(l,x;y,t) - 1)\, dl\, dk + \frac{1}{\pi}\int e^{itS(k,l;\xi,\eta)} f(k,l) \frac{\partial \mu^l}{\partial x}(l,x;y,t)\, dl\, dk$$

where

$$f(k,l) = T^+(k,l) + T^-(k,l).$$

Nonlocal Riemann-Hilbert Problem

Let

$$(\mathcal{T}^{\pm}_{t,x,y} f)(k) = \int T^{\pm}(k,l) e^{itS(k,l;\zeta,\eta)} f(l)\, dl$$

and let $C_{\pm}$ be Cauchy projections on $L^2(\mathbb{R}, dk)$. Then

$$\mu^l = 1 + \mathcal{C}_T \mu^l \tag{5}$$

where

$$\mathcal{C}_T f = C_+(\mathcal{T}^- f) + C_-(\mathcal{T}^+ f)$$

is a Hilbert-Schmidt integral operator with norm

$$\|C_T\|_{H.S.} \lesssim \|T^+\|_{L^2} + \|T^-\|_{L^2}.$$

Proposition Suppose that $T^{\pm} \in L^2(\mathbb{R}^2)$ has small norm. Then $(I - \mathcal{C}_T)^{-1}$ exists as a map from $L^2(\mathbb{R}, dk)$ to itself with bounds uniform in x, y, t.

Theorem Suppose that $T^{\pm} \in L^2(\mathbb{R}^2)$ of small norm. There is a unique solution $\mu^l = \mu^l(k, x; y, t)$ of (5) with $\mu^l - 1 \in L^2(\mathbb{R}, dk)$.

Large-Time Asymptotics: Scattering Solution

From

$$\mu^l - 1 = \mathcal{C}_T(1) + \mathcal{C}_T(\mu^l - 1)$$

we get the solution formula

$$\mu^l - 1 = (I - \mathcal{C}_T)^{-1}(\mathcal{C}_T 1)$$

Note that

$$\mathcal{C}_T 1 = C_+(\mathcal{T}^-_{t,x,y} 1) + C_-(\mathcal{T}^+_{t,x,y} 1)$$

where

$$\mathcal{T}^\pm_{t,x,y} 1 = \int T^\pm(k,l) e^{itS(k,l;\zeta,\eta)} \, dl$$

Similarly

$$\frac{\partial \mu^l}{\partial x} = (I - \mathcal{C}_T)^{-1} \left[\frac{\partial}{\partial x}(C_T(1)) + \mathcal{C}_{\partial T/\partial x}(\mu^l - 1) \right]$$

Large-time asymptotics in $L^2(\mathbb{R})$ (with parameters x, y) are determined by

$$\mathcal{C}_T(1), \quad \frac{\partial}{\partial x}(\mathcal{C}_T(1))$$

Large-Time Asymptotics: Scattering Solution

$\mathcal{C}_T(1)$ and $\frac{\partial}{\partial x}(\mathcal{C}_T(1))$ have L^2 norms bounded by the L^2 norms of

$$\mathcal{T}^{\pm}_{t,x,y}(1) = \int T^{\pm}(k,l)e^{itS(k,l;\zeta,\eta)}\,dl$$

$$\frac{\partial}{\partial x}\mathcal{T}^{\pm}_{t,x,y}(1) = \int i(l-k)T^{\pm}(k,l)e^{itS(k,l;\zeta,\eta)}\,dl$$

Lemma Let $\delta > 0$. The following estimates hold (recall $a = 12\xi - \zeta^2$)

$$\|\mathcal{C}_T(1)\|_{L^2} \lesssim \begin{cases} t^{-\frac{1}{2}}, & a < -\delta \\ t^{-\frac{1}{3}}, & |a| < \delta \\ t^{-1}, & a > \delta \end{cases}$$

$$\left\|\frac{\partial}{\partial x}\mathcal{C}_T(1)\right\|_{L^2} \lesssim \begin{cases} t^{-\frac{1}{2}}, & a < -\delta < 0 \\ t^{-\frac{1}{3}}, & |a| < \delta \\ t^{-1}, & a > \delta \end{cases}$$

Large-Time Asymptotics: Scattering Solution

Proving the estimates on $\mathcal{C}_T(1)$ and $\frac{\partial}{\partial x}\mathcal{C}_T(1)$–consider $\mathcal{T}^+_{t,x,y}(1)$:

$$\mathcal{T}^+_{t,x,y}(1)(k) = \int_k^\infty T^+(k,l)e^{itS(k,l;\xi,\eta)}\,dl$$

By the change of variables $(k,l) \to (k+\eta/12, l+\eta/12)$, we get

$$\mathcal{T}^+_{t,x,y}(k) = e^{-12it(ak+k^3/3)}\int_k^\infty \widetilde{T}^+(k,l)e^{12it(al+l^3/3)}\,dl$$

Using Fourier transforms we can rewrite the integral as $\int h(k,\xi)g(\xi;t,a)\,d\xi$ where $h(k,\xi)$ is a partial Fourier transform of $\widetilde{T}$ and

$$g(\xi;t,a) = (2\pi)^{-\frac{1}{2}}\int_k^\infty e^{-i\xi l}e^{12it(al+l^3/3)}\,dl$$

is an Airy type integral with

$$|g(\xi;t,a)| \lesssim_r t^{-\frac{1}{2}}(1+|\xi|), \qquad a < -r^2$$

$$|g(\xi;t,a)| \lesssim_c t^{-\frac{1}{3}}, \qquad |a| < c$$

Long-Time Asymptotics: Scattering Solution

From the estimates on $\mathcal{C}_T(1)$ and $\mathcal{C}_{\partial T/\partial x}(1)$ and the formulas

$$\mu^l - 1 = (I - \mathcal{C}_T)^{-1}(\mathcal{C}_T 1),$$

and

$$\frac{\partial \mu^l}{\partial x} = (I - \mathcal{C}_T)^{-1}\left[\frac{\partial}{\partial x}(C_T(1)) + \mathcal{C}_{\partial T/\partial x}(\mu^l - 1)\right],$$

we obtain:

Proposition Let $\delta > 0$. The following estimates hold (recall $a = 12\zeta - \eta^2$):

$$\left\|\mu^l - 1\right\|_{L^2} \lesssim \begin{cases} t^{-\frac{1}{2}}, & a < -\delta, \\ t^{-\frac{1}{3}}, & |a| \leq \delta, \\ t^{-1}, & a > \delta. \end{cases}$$

$$\left\|\frac{\partial \mu^l}{\partial x}\right\|_{L^2} \lesssim \begin{cases} t^{-\frac{1}{2}}, & a < -\delta \\ t^{-\frac{1}{3}}, & |a| \leq \delta \\ t^{-1}, & a > \delta \end{cases}$$

The Return of Ai

The reconstruction formula may be written

$$u(t,x,y) = u_1(t,x,y) + u_2(t,x,y)$$

where

$$u_1(t,x,y) = \frac{1}{\pi}\int e^{12itS(k,l;a)} i(l-k) f(k,l)\,dk\,dl$$

$$u_2(t,x,y) = \frac{1}{\pi}\int e^{12itS(k,l;a)} i(l-k) f(k,l)\left(\mu^l(l+\eta/12, x; y, t) - 1\right) dl\,dk + \frac{1}{\pi}\int e^{12itS(k,l;a)} f(k,l)\frac{\partial \mu^l}{\partial x}(l+\eta/12, x; y, t)\,dl\,dk$$

where

$$f(k,l) = T^+(k,l) + T^-(k,l)$$

We will analyze u_1 using Airy asymptotics as in the linear problem, and u_2 by a combination of Airy asymptotics and L^2 estimates on the solutions of the nonlocal Riemann-Hilbert problem

Local Term: Ai and Asymptotics

$$u_1(t,x,y) = \frac{1}{\pi} \int e^{12itS(k,l;a)} i(l-k) f(k,l)\, dk\, dl$$

Exactly as in the linear case, we obtain

$$u_1(t,x,y) = \begin{cases} \mathcal{O}\left(t^{-1}\right), & a < -\delta, \\ \mathcal{O}\left(t^{-2/3}\right), & |a| < \delta, \\ o(t^{-1}), & a > \delta. \end{cases}$$

Remark: For the local term, we can obtain a result reminiscent of the asymptotic formula of Manakov, Santini, and Takhtajan:

$$u_1(t,x,y) \underset{t\to\infty}{\sim} \frac{1}{t} \operatorname{Re}\left(e^{i(16tr^3-\pi/2)} \widetilde{T}^{+}(-r,r) + e^{-i(16tr^3-\pi/2))} \widetilde{T}^{-}(r,-r) + o(1)\right)$$

Nonlocal Term

$$u_2(t,x,y) = \frac{1}{\pi}\int e^{12itS(k,l;a)} i(l-k) f(k,l)(\mu^l(l+\eta/12,x;y,t)-1)\,dl\,dk$$
$$+\frac{1}{\pi}\int e^{12itS(k,l;a)} f(k,l)\frac{\partial\mu^l}{\partial x}(l+\eta/12,x;y,t)\,dl\,dk$$

Our strategy will be to make:

(1) L^2 estimates on μ^l and $\partial\mu^l/\partial x$ together with L^2 estimates on scattering data for the integration over l

(2) Stationary phase estimates for the integration over k

Nondegenerate stationary phase:

We have $\left\|\mu^l - 1\right\|_{L^2_l} \lesssim t^{-\frac{1}{2}}$ and $\left\|\frac{\partial\mu^l}{\partial x}\right\|_{L^2_l} \lesssim t^{-\frac{1}{2}}$

We gain an additional $\mathcal{O}\left(t^{-1/2}\right)$ from nondegenerate stationary phase in the k integration

Nonlocal Term

$$u_2(t,x,y) = \frac{1}{\pi}\int e^{12itS(k,l;a)} i(l-k) f(k,l)(\mu^l(l+\eta/12, x; y, t) - 1)\, dl\, dk$$
$$+ \frac{1}{\pi}\int e^{12itS(k,l;a)} f(k,l) \frac{\partial \mu^l}{\partial x}(l+\eta/12, x; y, t)\, dl\, dk$$

Our strategy will be to make:

(1) L^2 estimates on μ^l and $\partial\mu^l/\partial x$ together with L^2 estimates on scattering data for the integration over l

(2) Stationary phase estimates for the integration over k

Degenerate stationary phase:

We have $\left\|\mu^l - 1\right\|_{L^2_l} \lesssim t^{-\frac{1}{3}}$ and $\left\|\frac{\partial\mu^l}{\partial x}\right\|_{L^2_l} \lesssim t^{-\frac{1}{3}}$

We gain an additional $\mathcal{O}\left(t^{-\frac{1}{3}}\right)$ from degenerate stationary phase in the k integration

Nonlocal Term

$$u_2(t,x,y) = \frac{1}{\pi}\int e^{12itS(k,l;a)} i(l-k) f(k,l)(\mu^l(l+\eta/12,x;y,t)-1)\,dl\,dk$$
$$+\frac{1}{\pi}\int e^{12itS(k,l;a)} f(k,l)\frac{\partial \mu^l}{\partial x}(l+\eta/12,x;y,t)\,dl\,dk$$

Our strategy will be to make:

(1) L^2 estimates on μ^l and $\partial\mu^l/\partial x$ together with L^2 estimates on scattering data for the integration over l

(2) Stationary phase estimates for the integration over k

No Stationary Phase:

We have $\left\|\mu^l - 1\right\|_{L^2_l} = \mathcal{O}\left(t^{-1}\right)$ and $\left\|\frac{\partial \mu^l}{\partial x}\right\|_{L^2_l} = \mathcal{O}\left(t^{-1}\right)$

We gain an additional $\mathcal{O}\left(t^{-1}\right)$ decay through integration by parts in the k variable

Next steps

- Obtain sharp(er) asymptotics for $\mu^l(l, x; y, t)$, the solution of the nonlocal RHP
- Obtain sharp(er) estimates for $u(t, x, y)$ near the critical region
- Obtain complete asymptotics in the sense of Kiselev's work on KP II

Thank you for your attention!