Lax Representations and What to do With Them

Peter Perry

University of Kentucky

March 15, 2025

Acknowledgments

Parts of this lecture are taken from 2018 lecture notes of Peter Miller for a graduate course at the University of Michigan (see especially Topic 3 and Topic 5)

Some key references are:

1. Mark Ablowitz, David Kaup, Alan Newell, Harvey Segur. The inverse scattering transform: Fourier analysis for nonlinear problems. *Studies in Applied Math* **53** (1974), 249-315
2. Percy Deift, Xin Zhou. Long-time asymptotics for solutions of the NLS equation with initial data in a weighted Sobolev space. *Comm. Pure Appl. Math.* **56** (2003), no. 8, 1029-1077.
3. Vladimir Zakharov, A. B. Shabat. Exact theory of two-dimensional self-focussing and one-dimensional self-modulation of waves in nonlinear media. *Soviet Physics JETP* **34** (1972), no. 1, 62–29. Translated from Ž. Èksper. Teoret. Fiz. **71** (1976), no. 1, 118-134.

The KdV Equation

The Korteweg-de Vries (KdV) equation

$$q_t + q_{xxx} - 6qq_x = 0$$

is one of the earliest examples of a completely integrable dispersive PDE. It admits a *Lax Representation*: given $q = q(x,t) \in C^\infty(\mathbb{R} \times \mathbb{R})$, there are differential operators $L(t)$ and $A(t)$ given by

$$\begin{aligned} L(t) &= -\partial_x^2 + q \\ A(t) &= -4\partial_x^3 + 6q\partial_x + 3q_x \end{aligned}$$

so that the Lax equation

$$\frac{\partial}{\partial t} L(t) + [L(t), A(t)] = 0$$

is equivalent the the KdV equation.

Another Point of View

Equivalently, the Lax condition

$$\frac{\partial}{\partial t}L(t) = [A(t), L(t)]$$

is the compatibility condition for the system of equations

$$L(t)\psi = \lambda\psi \tag{1}$$

$$\psi_t = A(t)\psi \tag{2}$$

for $\psi = \psi(x,t)$ to have a solution: From (1),

$$\frac{\partial}{\partial t}(L(t)\psi) = \dot{L}\psi + L\psi_t = \dot{L}\psi + LA\psi$$

while

$$\frac{\partial}{\partial t}(\lambda\psi) = \lambda\psi_t = \lambda A\psi = A(\lambda\psi) = AL\psi$$

which gives

$$\dot{L}\psi + (LA - AL)\psi = 0$$

Zero-Curvature Representation

Let

$$w = \begin{pmatrix} \psi \\ \psi_x \end{pmatrix}$$

Using $\psi_{xx} = (q - \lambda)\psi$ to express higher-order derivatives of ψ in terms of ψ and ψ_x we obtain

$$\frac{\partial}{\partial x} w = \begin{pmatrix} 0 & 1 \\ q - \lambda & 0 \end{pmatrix} w$$

$$\frac{\partial}{\partial t} w = \begin{pmatrix} -q_x & (2q + 4\lambda) \\ -q_{xx} + 2q^2 + 2\lambda q - 4\lambda^2 & q_x \end{pmatrix} w$$

This new form of the Lax equations is called a *zero-curvature representation*

$$\frac{\partial w}{\partial x} = \mathbf{U} w,$$

$$\frac{\partial w}{\partial t} = \mathbf{V} w$$

Zero-Curvature Representation

The compatibility condition for the system

$$\frac{\partial w}{\partial x} = \mathbf{U}w, \quad \frac{\partial w}{\partial t} = \mathbf{V}w$$

is obtained by cross-differentiation:

$$w_{xt} = \frac{\partial \mathbf{U}}{\partial t}w + \mathbf{UV}w, \qquad w_{tx} = \frac{\partial \mathbf{V}}{\partial x}w + \mathbf{VU}w$$

so

$$\frac{\partial \mathbf{U}}{\partial t} - \frac{\partial \mathbf{V}}{\partial x} = [\mathbf{U}, \mathbf{V}]$$

In the case of KdV, the compatibility condition becomes

$$\begin{pmatrix} 0 & 0 \\ q_t + q_{xxx} - 6qq_x & 0 \end{pmatrix} = \mathbf{0}$$

which is exactly the KdV equation

The AKNS-ZS System

We can begin with a linear equation $\frac{\partial w}{\partial x} = \mathbf{U}w$ and attempt to find matrices $\mathbf{V}$ which lead to integrable systems.

The AKNS system is given by

$$\frac{\partial w}{\partial x} = \mathbf{U}w, \quad \mathbf{U} = -i\lambda\sigma_3 + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$$

where q and r are functions of x and t, and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This system, and integrable equations associated to it, were studied in a landmark paper by Ablowitz, Kaup, Newell and Segur (1974).

The case where $r = \pm\overline{q}$ is called the Zakharov-Shabat system

Cubic Nonlinear Schrödinger Equation

The Lax representation

$$\frac{\partial w}{\partial x} = -i\lambda\sigma_3 w + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} w$$

$$\frac{\partial w}{\partial t} = -i\lambda^2\sigma_3 w + \lambda \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} w + \begin{pmatrix} -\frac{1}{2}iqr & \frac{1}{2}iq_x \\ -\frac{1}{2}ir_x & \frac{1}{2}iqr \end{pmatrix} w$$

gives rise to the coupled system

$$iq_t + \frac{1}{2}q_{xx} - q^2 r = 0$$

$$-ir_t + \frac{1}{2}r_{xx} - r^2 q = 0$$

Taking $r = \pm\overline{q}$ we get the cubic nonlinear Schrödinger equation

$$iq_t + \frac{1}{2}q_{xx} \mp |q|^2 q = 0$$

which is *defocussing* for the $-$ sign and *focussing* for the $+$ sign.

Lax Representation and Inverse Scattering

A Lax representation

$$\frac{\partial w}{\partial x} = \mathbf{U}(q)w$$
$$\frac{\partial w}{\partial t} = \mathbf{V}(q)w$$

defines

(1) A *spectral problem* which maps a given potential q to *scattering data* r

(2) A *time evolution* of scattering solutions which determines how the scattering data evolve in time

(3) A *Riemann-Hilbert problem* which defines a map from scattering data r to the potential q

This leads to a strategy for solving the associated nonlinear equation by *inverse scattering*

NLS Equation: Direct Scattering Map

Spectral Problem:

$$\frac{\partial \Psi}{\partial x} = -iz\sigma_3 \Psi + \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix} \Psi$$

Fix z. One can show that:

(1) $\det \Psi(x)$ is constant for any solution

(2) If ψ_1 and ψ_2 are solutions, $\psi_1(x) = \psi_2(x) M$ for a constant matrix M

(3) The map

$$\psi(x,z) \mapsto \sigma_1 \overline{\psi(x,\overline{z})} \sigma_1^{-1}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

preserves the solution space

For $q = 0$, $z = \lambda \in \mathbb{R}$, $\Psi(x,\lambda) = e^{-i\lambda\sigma_3 x}\Psi(0)$ are exact solutions.

For $q \neq 0$, $z = \lambda \in \mathbb{R}$, look for solutions $\Psi^{\pm}$ satisfying

$$\lim_{x \to \pm\infty} \Psi^{\pm}(x) e^{i\lambda x \sigma_3} = \mathbb{I}$$

NLS Equation: Direct Scattering Map

Spectral Problem:

$$\frac{\partial \Psi}{\partial x} = -iz\sigma_3\Psi + \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix} \Psi$$

For $z = \lambda \in \mathbb{R}$, there exist unique solutions $\Psi^{\pm}$ satisfying $\lim_{x\to\pm\infty} \Psi^{\pm}(x)e^{i\lambda x\sigma_3} = \mathbb{I}$.

From properties (2) and (3), there is a matrix

$$T(\lambda) = \begin{pmatrix} a(\lambda) & \overline{b(\lambda)} \\ b(\lambda) & \overline{a(\lambda)} \end{pmatrix}, \qquad |a(\lambda)|^2 - |b(\lambda)|^2 = 1$$

so that $\Psi^+(x,\lambda) = \Psi^-(x,\lambda)T(\lambda)$.

The functions $a(\lambda)$, $b(\lambda)$ are *scattering data* for q, and are uniquely determined by

$$r(\lambda) = -b(\lambda)/\overline{a(\lambda)}$$

The map $\mathcal{R} : q \mapsto r$ is called the *direct scattering map*

NLS: Direct Scattering Map

Deift and Zhou (2003) proved the following mapping property of the direct scattering map. Let

$$H^{1,1}(\mathbb{R}) = \{u \in L^2(\mathbb{R}) : u', xu \in L^2(\mathbb{R})\}$$

$$H_1^{1,1}(\mathbb{R}) = \{u \in H^{1,1}(\mathbb{R}) : \|u\|_{L^\infty} < 1\}$$

Theorem (Deift-Zhou)

The direct scattering map $\mathcal{R} : q \to r$ *is a Lipschitz continuous map from* $H^{1,1}(\mathbb{R})$ *onto* $H_1^{1,1}(\mathbb{R})$.

A consequence is that, for $q \in H^{1,1}(\mathbb{R})$, the corresponding reflection coefficient satisfies the bound

$$|r(\lambda)| \leq \rho$$

for some $\rho \in (0,1)$.

NLS: Time Evolution of Scattering Data

Suppose now $q = q(x,t)$ and that $\Psi^{\pm}(x,t,\lambda)$ solves

$$\frac{\partial \Psi}{\partial x} = \mathbf{U}\Psi, \quad \mathbf{U} = -iz\sigma_3 + \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix}$$

We seek solutions

$$W^{\pm}(x,t,\lambda) = \Psi^{\pm}(x,t,\lambda)\mathbf{C}(t,\lambda)$$

of

$$\frac{\partial W^{\pm}}{\partial t} = \mathbf{V}W^{\pm}$$

By substitution

$$\frac{\partial \Psi^{\pm}}{\partial t}\mathbf{C}^{\pm} + \Psi^{\pm}\frac{\partial \mathbf{C}^{\pm}}{\partial t} = \mathbf{V}\Psi^{\pm}\mathbf{C}^{\pm}$$

or

$$\frac{\partial \mathbf{C}^{\pm}}{\partial t} = (\Psi^{\pm})^{-1}\mathbf{V}\Psi^{\pm}\mathbf{C}^{\pm} - (\Psi^{\pm})^{-1}\frac{\partial \Psi^{\pm}}{\partial t}\mathbf{C}^{\pm}$$

NLS: Time Evolution of Scattering Data

$$\frac{\partial \mathbf{C}^{\pm}}{\partial t} = (\Psi^{\pm})^{-1}\mathbf{V}\Psi^{\pm}\mathbf{C}^{\pm} - (\Psi^{\pm})^{-1}\frac{\partial \Psi^{\pm}}{\partial t}\mathbf{C}^{\pm}$$

Recall

$$\mathbf{V} = -i\lambda^2\sigma_3 + \lambda\begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}i|q|^2 & \frac{1}{2}iq_x \\ -\frac{1}{2}i\overline{q}_x & \frac{1}{2}i|q|^2 \end{pmatrix}$$

Assume

$$\Psi^{\pm}(x,t,\lambda) = e^{-i\lambda x\sigma_3} + \mathbf{E}^{\pm}(x,\lambda,t)$$

where

$$\mathbf{E}^{\pm}(x,\lambda,t),\ \frac{\partial \mathbf{E}^{\pm}}{\partial t}(x,\lambda,t) \to 0 \text{ as } x \to \pm\infty$$

Assume $q, q_x \to 0$ as $x \to \pm\infty$. Taking $x \to \pm\infty$ we conclude

$$\frac{\partial \mathbf{C}^{\pm}}{\partial t} = -i\lambda^2\sigma_3\mathbf{C}^{\pm}$$

so

$$\mathbf{C}^{\pm}(t,\lambda) = e^{-i\lambda^2 t\sigma_3}$$

NLS: Time Evolution of Scattering Data

We will compute the time evolution of

$$T(\lambda,t) = (\Psi^-(x,t,\lambda))^{-1}\Psi^+(x,t,\lambda)$$

Since $W^\pm = \Psi^\pm e^{-i\lambda^2 t\sigma_3}$ satisfies $\dfrac{\partial W^\pm}{\partial t} = \mathbf{V}W^\pm$, it follows that

$$\frac{\partial \Psi^\pm}{\partial t} = i\lambda^2\Psi^\pm\sigma_3 + \mathbf{V}\Psi^\pm$$

Hence

$$\begin{aligned}\frac{\partial T(\lambda,t)}{\partial t} &= -(\Psi^-)^{-1}\frac{\partial \Psi^-}{\partial t}(\Psi^-)^{-1}\Psi^+ + (\Psi^-)^{-1}\frac{\partial \Psi^+}{\partial t} \\ &= -i\lambda^2\sigma_3 T(\lambda,t) - (\Psi^-)^{-1}\mathbf{V}\Psi^+ + i\lambda^2 T(\lambda,t)\sigma_3 + (\Psi^-)^{-1}\mathbf{V}\Psi^+ \\ &= i\lambda^2[T(\lambda,t),\sigma_3]\end{aligned}$$

NLS: Time Evolution of Scattering Data

$$\frac{\partial T(\lambda,t)}{\partial t} = i\lambda^2 [T(\lambda,t), \sigma_3]$$

or

$$\frac{\partial}{\partial t}\begin{pmatrix} a & \overline{b} \\ b & \overline{a} \end{pmatrix} = i\lambda^2 \begin{pmatrix} 0 & -2\overline{b} \\ 2b & 0 \end{pmatrix}$$

Hence

$$a(\lambda,t) = a(\lambda,0)$$
$$b(\lambda,t) = e^{2i\lambda^2 t} b(\lambda,0)$$

so that

$$r(\lambda,t) = -\frac{b(\lambda,t)}{\overline{a(\lambda,t)}} = e^{2i\lambda^2 t} r(\lambda,0).$$

NLS: Inverse Scattering Map

Recall the spectral problem

$$\frac{\partial \Psi}{\partial x} = -iz\sigma_3 \Psi + \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix} \Psi$$

and solutions $\Psi^{\pm}$ with $\lim_{x \to \pm\infty} e^{i\lambda x \sigma_3} \Psi^{\pm} = \mathbb{I}$. For a solution Ψ, let

$$\Psi(x,z) = \mathbf{M}(x,z) e^{-izx\sigma_3}$$

so that

$$\frac{\partial}{\partial x} \mathbf{M}(x,z) = -iz \,\mathrm{ad}(\sigma_3) \mathbf{M}(x,z) + \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix} \mathbf{M}(x,z)$$

For a matrix A,

$$\mathrm{ad}(\sigma_3) A = [\sigma_3, A]$$

We denote by $\mathbf{M}^{\pm}$ the normalized Jost solutions, i.e.,

$$\Psi^{\pm}(x,z) = \mathbf{M}^{\pm}(x,z) e^{-izx\sigma_3}$$

NLS: Beals-Coifman Solutions

$$\frac{\partial}{\partial x}\mathbf{M}(x,z) = -iz\,\mathrm{ad}(\sigma_3)\mathbf{M}(x,z) + \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix}\mathbf{M}(x,z)$$

Beals and Coifman (1984) showed that this equation admits special solutions, now called the *Beals-Coifman solutions*, with the following properties:

(i) For each $x \in \mathbb{R}$, $\mathbf{M}(x,z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$

(ii) For each x, $\mathbf{M}(x,z)$ has continuous boundary values

$$\mathbf{M}_{\pm}(x,\lambda) = \lim_{\varepsilon \downarrow 0} \mathbf{M}(x, \lambda \pm i\varepsilon)$$

(iii) For each $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\mathbf{M}(x,z) \to \mathbb{I} \text{ as } x \to +\infty, \qquad \mathbf{M}(x,z) \text{ is bounded as } x \to -\infty$$

(iv) The potential $q(x)$ can be recovered from their asymptotic behavior:

$$q(x) = \lim_{z \to \infty} 2iz\,\mathbf{M}_{12}(x,z)$$

NLS: Riemann-Hilbert Problem

Most importantly, the Beals-Coifman solutions satisfy a Riemann-Hilbert Problem:

Riemann-Hilbert Problem 1. For given r and $x \in \mathbb{R}$, find $\mathbf{M}(x,z)$ so that:

(i) $\mathbf{M}(x,z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$ for each $x \in \mathbb{R}$

(ii) $\lim_{z\to\infty} \mathbf{M}(x,z) = \mathbb{I}$

(iii) $\mathbf{M}(x,z)$ has continuous boundary values $\mathbf{M}_{\pm}(x,\lambda)$ on $\mathbb{R}$

(iv) The jump relation

$$\mathbf{M}_+(x,\lambda) = \mathbf{M}_-(x,\lambda)\mathbf{V}(x,\lambda), \quad \mathbf{V}(x,\lambda) = \begin{pmatrix} 1-|r(\lambda)|^2 & -\overline{r(\lambda)}e^{-2i\lambda x} \\ r(\lambda)e^{2i\lambda x} & 1 \end{pmatrix}$$

holds, where r is the scattering data.

In (ii), the limit is uniform in proper subsectors of the upper and lower half-planes

NLS: Inverse Scattering Map

We can now define the *inverse scattering map* $\mathcal{I} : r \mapsto q$ as follows.

Step 1: Given $r \in H_1^{1,1}(\mathbb{R})$, solve the Riemann-Hilbert problem:

Find $\mathbf{M}(x,z)$ analytic in $z \in \mathbb{C} \setminus \mathbb{R}$ for each x so that

(i) $\lim_{z\to\infty} \mathbf{M}(x,z) = \mathbb{I}$

(ii) $\mathbf{M}(x,z)$ has continuous boundary values $\mathbf{M}_\pm(x,\lambda)$ on $\mathbb{R}$

(iii) The jump relation

$$\mathbf{M}_+(x,\lambda) = \mathbf{M}_-(x,\lambda)\mathbf{V}(x,\lambda), \quad \mathbf{V}(x,\lambda) = \begin{pmatrix} 1 - |r(\lambda)|^2 & -\overline{r(\lambda)}e^{-2i\lambda x} \\ r(\lambda)e^{2i\lambda x} & 1 \end{pmatrix}$$

holds

Step 2: Recover $q(x)$ from the reconstruction formula

$$q(x) = \lim_{z\to\infty} 2iz\mathbf{M}_{12}(x,z)$$

NLS: Inverse Scattering Map

Deift and Zhou (2003) proved the following mapping property of the inverse scattering map. Recall

$$H^{1,1}(\mathbb{R}) = \{u \in L^2(\mathbb{R}) : u', xu \in L^2(\mathbb{R})\}$$

$$H_1^{1,1}(\mathbb{R}) = \{u \in H^{1,1}(\mathbb{R}) : \|u\|_{L^\infty} < 1\}$$

Theorem (Deift-Zhou)

The inverse scattering map $\mathcal{I} : r \to q$ is a Lipschitz continuous map from $H_1^{1,1}(\mathbb{R})$ onto $H^{1,1}(\mathbb{R})$.

NLS: Reconstruction Formula

Combining the three results:

(1) The direct scattering map $\mathcal{R} : q \to r$ is a Lipschitz map from $H^{1,1}(\mathbb{R})$ onto $H^{1,1}_1(\mathbb{R})$

(2) The reflection coefficient evolves according to $r(\lambda, t) = e^{2i\lambda^2 t} r(\lambda, 0)$, a continuous curve in $H^{1,1}(\mathbb{R})$

(3) The inverse scattering map $\mathcal{I} : r \to q$ is a Lipschitz map from $H^{1,1}_1(\mathbb{R})$ onto $H^{1,1}(\mathbb{R})$

we obtain the solution formula

$$q(t, x) = \mathcal{I}\left(e^{2i\lambda^2 t}\mathcal{R}(q_0)\right)(x)$$

which defines a continuous map

$$H^{1,1}(\mathbb{R}) \times \mathbb{R} \ni (q_0, t) \mapsto H^{1,1}(\mathbb{R})$$

Lipschitz Continuity of $\mathcal{R}$

We'll give key ideas of the proof that

$$\mathcal{R} : H^{1,1}(\mathbb{R}) \ni q \to r \in H_1^{1,1}(\mathbb{R})$$

is Lipschitz continuous. Recall that $\Psi^{\pm}(x, \lambda)$ solve

$$\frac{\partial \Psi^{+}}{\partial x} = i\lambda\sigma_3\Psi + \mathbf{Q}\Psi, \quad \mathbf{Q} = \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix}$$

and

$$\Psi^{+}(x, \lambda) = \Psi^{-}(x, \lambda) \begin{pmatrix} a(\lambda) & \overline{b(\lambda)} \\ b(\lambda) & \overline{a(\lambda)} \end{pmatrix}$$

with $r(\lambda) = -b(\lambda)/\overline{a(\lambda)}$.

Lipschitz Continuity of $\mathcal{R}$

To study $\mathcal{R} : q \to r$, set

$$\Psi^+(x,\lambda) = e^{-i\lambda x \sigma_3}\mathbf{N}(x,\lambda)$$

which satisfies

$$\begin{cases} \dfrac{\partial}{\partial x}\mathbf{N}(x,\lambda) = \begin{pmatrix} 0 & e^{2i\lambda x}q(x) \\ e^{-2i\lambda x}\overline{q(x)} & 0 \end{pmatrix}\mathbf{N}(x,\lambda), \\ \lim\limits_{x\to+\infty} \mathbf{N}(x,\lambda) = \mathbb{I}. \end{cases} \tag{3}$$

and note that

$$\lim_{x\to-\infty} \mathbf{N}(x,\lambda) = T(\lambda).$$

Lipschitz Continuity of $\mathcal{R}$

The function $\mathbf{N}$ obeys the integral equation

$$\mathbf{N}(x,\lambda) = \mathbb{I} - \int_x^\infty \begin{pmatrix} 0 & e^{2i\lambda y}q(y) \\ e^{-2i\lambda y}\overline{q(y)} & 0 \end{pmatrix} \mathbf{N}(y,\lambda)\, dy$$

and

$$\lim_{x\to-\infty} \mathbf{N}(x,\lambda) = \begin{pmatrix} a(\lambda) & \overline{b(\lambda)} \\ b(\lambda) & \overline{a(\lambda)} \end{pmatrix}$$

Focus on

$$\begin{pmatrix} N_{11}(x,\lambda) \\ N_{21}(x,\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \int_x^\infty \begin{pmatrix} e^{2i\lambda y}q(y)N_{21}(y,\lambda) \\ e^{-2i\lambda y}\overline{q(y)}N_{11}(y,\lambda) \end{pmatrix} dy$$

which is a Volterra-type integral equation solvable by a Volterra series

Lipschitz Continuity of $\mathcal{R}$

Renaming $N_{11}(x,\lambda) = a(x,\lambda)$, $N_{21}(x,\lambda) = b(x,\lambda)$, we have

$$a(x,\lambda) = 1 + \sum_{n-1}^{\infty} A_{2n}(x,\lambda), \qquad b(x,\lambda) = -\sum_{n=0}^{\infty} A_{2n+1}(x,\lambda)$$

and on taking limits

$$a(\lambda) = 1 + \sum_{n-1}^{\infty} A_{2n}(\lambda), \qquad b(\lambda) = -\sum_{n=0}^{\infty} A_{2n+1}(\lambda)$$

where

$$A_n(\lambda) = \int_{y_1<y_2<\ldots<y_n} Q_n(y_1, \ldots y_n) e^{2i\lambda\phi_n(y_1,\ldots,y_n)}\, dy_n \ \ldots\ dy_1,$$

φ_n is a real phase function, and

$$Q_n(y_1,\ldots,y_n) = \begin{cases} \prod_{j=1}^m q(y_{2j-1})\overline{q(y_{2j})}, & n = 2m, \\ \overline{q(y_1)} \prod_{j=1}^m q(y_{2j})\overline{q(y_{2j+1})}, & n = 2m+1. \end{cases}$$

Lipschitz Continuity of $\mathcal{R}$

We have the multilinear series

$$a(\lambda) = 1 + \sum_{n-1}^{\infty} A_{2n}(\lambda), \qquad b(\lambda) = -\sum_{n=0}^{\infty} A_{2n+1}(\lambda)$$

where A_m is a multilinear integral with phase function and amplitude Q_m:

$$Q_n(y_1, \ldots, y_n) = \begin{cases} \prod_{j=1}^{m} q(y_{2j-1})\overline{q(y_{2j})}, & n = 2m, \\ \overline{q(y_1)} \prod_{j=1}^{m} q(y_{2j})\overline{q(y_{2j+1})}, & n = 2m+1. \end{cases}$$

From these series one can analyze the map $q \mapsto r$ in four stages:

(1) Show that $\mathcal{R} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$

(2) Show that $\mathcal{R} : L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \to L^2(\mathbb{R})$

(3) Show that $\mathcal{R} : H^{1,1}(\mathbb{R}) \to H^{0,1}(\mathbb{R})$

(4) Show that $\mathcal{R} : H^{1,1}(\mathbb{R}) \to H^{1,1}(\mathbb{R})$

Lipschitz Continuity of $\mathcal{I}$

To prove Lipschitz continuity of $\mathcal{I} : r \to q$, we need to:

1. Reduce the Riemann-Hilbert problem to an integral equation, the *Beals-Coifman integral equation*
2. Show that the Beals-Coifman integral equation admits a unique solution μ for $r \in H_1^{1,1}(\mathbb{R})$
3. Find an explicit reconstruction formula in terms of r and μ and use it to establish Lipschitz continuity of $\mathcal{I}$

Lipschitz Continuity of $\mathcal{I}$

Remark: Recall the $\overline{\partial}$ operator for $z = x + iy$:

$$\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and that $\overline{\partial} F = 0$ if F is analytic.

Our Riemann-Hilbert problem can be viewed as a $\overline{\partial}$ problem with boundary conditions:

$$\begin{aligned} (\overline{\partial}\mathbf{M})(z) &= 0 & z \in \mathbb{C} \setminus \mathbb{R}, \\ \mathbf{M}_+(z) &= \mathbf{M}_-(z) \begin{pmatrix} 1 - |r(\lambda)|^2 & \overline{r(\lambda)} e^{-i\lambda x} \\ -r(\lambda) e^{i\lambda x} & 1 \end{pmatrix}, & z \in \mathbb{R} \end{aligned}$$

It is natural that a PDE boundary value problem can be reduced to a boundary integral equation

Beals-Coifman Integral Equation

We can write the jump relation for **M** in the form

$$\mathbf{M}_+(z) = (I - w_x^-(\lambda))^{-1}(I + w_x^+(\lambda))\mathbf{M}_-(z)$$

where

$$w_x^-(\lambda) = \begin{pmatrix} 0 & 0 \\ e^{2i\lambda x} r(\lambda) & 0 \end{pmatrix}, \quad w_x^+(\lambda) = \begin{pmatrix} 0 & -e^{2i\lambda x}\overline{r(\lambda)} \\ 0 & 0 \end{pmatrix}$$

Now let

$$\mu(x,\lambda) = \mathbf{M}_+(x,\lambda)(I + w_x^+(\lambda))^{-1} = \mathbf{M}_-(x,\lambda)(I - w_x^-(\lambda))^{-1}$$

Then

$$\mathbf{M}_+(x,\lambda) - \mathbf{M}_-(x,\lambda) = \mu(x,\lambda)(w_x^+(\lambda) + w_x^-(\lambda))$$

Beals-Coifman Integral Equation

Recall

$$\mu(x,\lambda) = \mathbf{M}_+(x,\lambda)(I + w_x^+(\lambda))^{-1} = \mathbf{M}_-(x,\lambda)(I - w_x^-(\lambda))^{-1} \tag{4}$$

and

$$\mathbf{M}_+(x,\lambda) - \mathbf{M}_-(x,\lambda) = \mu(x,\lambda)(w_x^+(\lambda) + w_x^-(\lambda))$$

From the formula

$$\mathbf{M}(x,z) = \mathbb{I} + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\mathbf{M}_+(x,\lambda) - \mathbf{M}_-(x,\lambda)}{\lambda - z}\, d\lambda$$

we get

$$\mathbf{M}(x,z) = \mathbb{I} + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\mu(x,\lambda)(w_x^+(\lambda) + w_x^-(\lambda))}{\lambda - z}\, d\lambda \tag{5}$$

We will use (4) and (5) to derive the Beals-Coifman integral equation.

Beals-Coifman Integral Equation

Recall that $(I + w_x^-(\lambda))\mu(x,\lambda) = \mathbf{M}_+(x,\lambda)$.

For $f \in H^1(\mathbb{R})$, define the *Cauchy projectors* $C_\pm$ by

$$(C_\pm f)(\lambda) = \lim_{\varepsilon \downarrow 0} \int \frac{f(s)}{s - (\lambda \pm i\varepsilon)}\, ds,$$

and recall that $\|C_\pm\|_{L^2 \to L^2} = 1$ and $C_+ - C_- = I$. Using

$$\mathbf{M}(x,z) = \mathbb{I} + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\mu(x,\lambda)(w_x^+(\lambda) + w_x^-(\lambda))}{\lambda - z}\, d\lambda$$

we take boundary values to recover

$$\mathbf{M}_+(x,\lambda) = \mathbb{I} + C_+(\mu w_x^+ + \mu w_x^-).$$

Hence

$$(\mathbb{I} + w_x^-(\lambda))\mu = \mathbb{I} + C_+(\mu w_x^+ + \mu w_x^-)$$

or

$$\mu = \mathbb{I} + C_+(\mu w_x^-) + C_-(\mu w_x^+)$$

Beals-Coifman Integral Equations

Let

$$\mathcal{C}_w(h) = C_+(hw_x^-) + C_-(hw_x^+)$$

The equation

$$\mu = \mathbb{I} + \mathcal{C}_w(\mu) \tag{6}$$

is the *Beals-Coifman Integral Equation*.

Theorem

Suppose that $r \in H^{1,1}(\mathbb{R})$ *with* $\|r\|_{L^\infty} = \rho < 1$. *There exists a unique solution* μ *of* (6) *with* $\mu(\lambda) - 1 \in L^2(\mathbb{R})$.

This is a consequence of the operator estimate

$$\|\mathcal{C}_w\|_{L^2 \to L^2} = \|r\|_{L^\infty}$$

and the solution formula

$$\mu - \mathbb{I} = (I - \mathcal{C}_w)^{-1}\mathcal{C}_w(\mathbb{I})$$

.

Vanishing Theorem

Using the reduction of the RHP to an integral equation, we can also prove a uniqueness theorem for the RHP.

Theorem (Vanishing Theorem)

Fix $x \in \mathbb{R}$*, suppose* $r \in H^{1,0}(\mathbb{R})$ *with* $\|r\|_{L^\infty} < 1$ *and suppose that* $\mathbf{n}(x,z) : \mathbb{C} \setminus \mathbb{R} \to M_2(\mathbb{C})$ *solves the Riemann-Hilbert problem with boundary values* $\mathbf{n}_\pm(x,z) \in L^2(\mathbb{R})$*. Then* $\mathbf{n}(x,z) \equiv 0$*.*

Idea of proof: Repeat the reduction to a Beals-Coifman integral equation with

$$\nu(x,\lambda) = \mathbf{n}_+(x,\lambda)(I + w_x^+(\lambda))^{-1} = \mathbf{n}_-(x,\lambda)(I - w_x^-(\lambda))^{-1}$$

and arrive at the integral equation

$$\nu = \mathcal{C}_w \nu$$

which has only the zero solution.

A vector $\mathbf{n}$ satisfying the hypothesis of the vanishing theorem is called a *null vector* for the RHP.

Reconstruction Formula

The solution μ of the Beals-Coifman integral equation determines the solution $\mathbf{M}$ of the Riemann-Hilbert problem via

$$\mathbf{M}(x,z) = \mathbb{I} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mu(x,\lambda)(w_x^+(\lambda) + w_x^-(\lambda))}{\lambda - z}\, d\lambda$$

Theorem

Suppose that $r \in H_1^{1,1}(\mathbb{R})$, let $\mathbf{M}(x,z)$ be the unique solution of the Riemann-Hilbert problem. Then

$$\frac{d}{dx}\mathbf{M}(x,z) = -iz\,\mathrm{ad}\,\sigma_3(\mathbf{M}) + \mathbf{Q}(x)\mathbf{M}(x,z)$$

where

$$\mathbf{Q}(x) = \begin{pmatrix} 0 & q(x) \\ \overline{q(x)} & 0 \end{pmatrix}, \qquad q(x) = -\frac{1}{\pi}\int \overline{r(s)}e^{-2ixs}\mu_{11}(x,s)\, ds$$

Idea of Proof

To prove that

$$\frac{d}{dx}\mathbf{M}(x,z) = -iz\,\mathrm{ad}\,\sigma_3(\mathbf{M}) + \mathbf{Q}(x)\mathbf{M}(x,z)$$

and identify $\mathbf{Q}$, differentiate the jump relation for $\mathbf{M}$ to obtain

$$\left(\frac{\partial \mathbf{M}_+}{\partial x} + i\lambda\,\mathrm{ad}\,\sigma_3(\mathbf{M}_+)\right) = \left(\frac{\partial \mathbf{M}_-}{\partial x} + i\lambda\,\mathrm{ad}\,\sigma_3(\mathbf{M}_-)\right)\mathbf{V}(x,\lambda)$$

and show that

$$i\lambda\,\mathrm{ad}\,\sigma_3(\mathbf{M}_\pm) - \mathbf{Q}(x) \in L^2(\mathbb{R})$$

to conclude that

$$\mathbf{n}(x,z) = \frac{\partial}{\partial x}\mathbf{M}(x,z) + i\lambda\,\mathrm{ad}\,\sigma_3(\mathbf{M}(x,z)) - \mathbf{Q}(x)\mathbf{M}(x,z)$$

is a null vector for the Riemann-Hilbert problem, hence $\mathbf{n}(x,z) \equiv 0$.

Lipschitz Continuity of $\mathcal{I}$

From

$$q(x) = -\frac{1}{\pi} \int \overline{r(s)} e^{-2ixs} \mu_{11}(x,s)\, ds$$

write $q(x) = q_0(x) + q_1(x)$ where

$$q_0(x) = -\frac{1}{\pi} \int \overline{r(s)} e^{-2ixs}\, ds$$

$$q_1(x) = -\frac{1}{\pi} \int \overline{r(s)} e^{-2ixs} (\mu_{11}(x,s) - 1)\, ds$$

The map $r \mapsto q_0$ is a Fourier transform with the required properties.

To analyze $r \mapsto q_1$ we use the identity

$$q_1'(x) = -q(x) \left(\frac{1}{\pi} \int \overline{r(s)} e^{-2ixs} \mu_{21}(x,s)\, ds \right)$$

and use Lipschitz continuity properties of $r \mapsto \mu_{11} - 1$ and $r \mapsto \mu_{12}$.