Large-Time Asymptotics for the Defocussing NLS Equation

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Acknowledgments

This lecture follows closely the 2019 paper of Momar Dieng, Ken McLaughlin, and Peter Miller in *Nonlinear dispersive partial differential equations* (Fields Institute Communications, vol. 83). Their work draws on previous work of Deift and Zhou on the method of nonlinear steepest descent for Riemann-Hilbert problems. Dieng, McLaughlin, and Miller proposed a modification of Deift-Zhou's method which greatly simplifies the analysis and provides optimal results for NLS.

The Defocussing Cubic NLS

We seek to understand the large-time asymptotic behavior for the Cauchy problem

$$\begin{cases} i\dfrac{\partial q}{\partial t} + \dfrac{\partial^2 q}{\partial x^2} - 2|q|^2 q = 0, \\ q(0,x) = q_0(x) \end{cases}$$

where $q_0 \in H^{1,1}(\mathbb{R})$, so that $r \in H^{1,1}(\mathbb{R})$ with $\|r\|_{L^\infty} < 1$.[1]

To motivate our approach, we will first consider an unusual approach to the linear problem

$$\begin{cases} i\dfrac{\partial q}{\partial t} + \dfrac{\partial^2 q}{\partial x^2} = 0, \\ q(0,x) = q_0(x) \end{cases}$$

following Dieng-McLaughlin-Miller (2019).

[1] Recall that

$$H^{1,1}(\mathbb{R}) = \{u \in L^2 : u', xu \in L^2\}$$

Large-Time Asymptotics

Theorem *For $q_0 \in H^{1,1}(\mathbb{R})$, the Cauchy problem for NLS has a unique weak solution with large-time asymptotics*

$$q(t,x) \underset{t\to\infty}{\sim} t^{-\frac{1}{2}} \alpha(z_0) e^{ix^2/4t - i\nu(z_0)\ln(8t)} + \mathcal{E}(t,x)$$

where $z_0 = -x/4t$,

$$\nu(z) := -\frac{1}{2\pi} \ln(1 - |r(z)|^2),$$

$$|\alpha(z)|^2 = \frac{1}{2}\nu(z),$$

$$\arg(\alpha(z)) = \frac{1}{\pi}\int_{-\infty}^{z} \ln(z-s)\, d\left(\ln(1-|r(s)|^2)\right) + \frac{\pi}{4} + \arg(\Gamma(i\nu(z)) - \arg(r(z))$$

and

$$\mathcal{E}(t,x) = \mathcal{O}\left(t^{-3/4}\right)$$

Linear Schrödinger Equation

We can solve

$$\begin{cases} i\dfrac{\partial q}{\partial t} + \dfrac{\partial^2 q}{\partial x^2} = 0, \\ q(0,x) = q_0(x) \end{cases}$$

by Fourier analysis, setting

$$\widehat{f}(\xi) = \int e^{2i\xi x} f(x)\, dx$$

$$\check{g}(x) = \frac{1}{\pi} \int e^{-2i\xi x} g(\xi)\, d\xi$$

Fourier-transforming in x we have

$$i\frac{\partial}{\partial t}\widehat{q} - 4\xi^2 \widehat{q} = 0$$

or

$$\widehat{q}(t,\xi) = e^{-4it\xi^2}\widehat{q}_0(\xi)$$

Linear Schrödinger Equation: Fourier Analysis

From

$$\widehat{q}(t,\xi) = e^{-4it\xi^2}\widehat{q}_0(\xi)$$

we recover the oscillatory integral

$$q(t,x) = \frac{1}{\pi}\int e^{-2it(2\xi^2+\xi x/t)}\widehat{q}_0(\xi)\,d\xi$$

The phase function

$$\phi(\xi) = 2\xi^2 + \xi x/t$$

has a single critical point at

$$z_0 = -\frac{x}{4t}.$$

We can also recover the solution through a Riemann-Hilbert problem.

Linear Schrödinger Equation: RHP

Riemann-Hilbert Problem *Given* $q_0 \in H^{1,1}(\mathbb{R})$*, find a* 2×2 *matrix-valued function* $\mathbf{M}(z) = \mathbf{M}(x; z, t)$ *on* $\mathbb{C} \setminus \mathbb{R}$ *so that*

(i) $\mathbf{M}_+(z) = \mathbf{M}_-(z)\mathbf{V}(z), \quad \mathbf{V}(z) = \begin{pmatrix} 1 & -\widehat{q}_0(z)e^{-2it\phi(z;z_0)} \\ 0 & 0 \end{pmatrix}$

(ii) $\mathbf{M}(z; x, t) = \mathbb{I} + z^{-1}\mathbf{M}_1(x,t) + o(z^{-1})$

and recover

$$q(t,x) = 2i(\mathbf{M}_1)_{1,2}(x,t)$$

This problem has the explicit solution

$$\mathbf{M}(z; x, t) = \mathbb{I} + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\widehat{q}_0(\zeta)e^{-2it\phi(\zeta;z_0)}}{\zeta - z}\, d\zeta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Linear Schrödinger Equation: RHP

From

$$\mathbf{M}(z;x,t) = \mathbb{I} + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\widehat{q_0}(z)e^{-2it\phi(\zeta;z_0)}}{\zeta - z}\,d\zeta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\mathbf{M}(z;x,t) = \mathbb{I} + z^{-1}\mathbf{M}_1(x,t) + o(z^{-1})$$

we conclude that

$$\mathbf{M}_1(x,t) = \frac{1}{2\pi i}\int_{\mathbb{R}} \widehat{q_0}(\zeta)e^{-2it\phi(\zeta;z_0)}\,d\zeta$$

so

$$q(t,x) = 2i(\mathbf{M}_1)_{1,2}(x,t) = \frac{1}{\pi}\int \widehat{q_0}(\zeta)e^{-2it\phi(\zeta;z_0)}\,d\zeta.$$

We can obtain large-time asymptotics of $q(t,x)$ either

(1) Using stationary phase methods (Stokes-Thompson) or

(2) Evaluating the integral by contour deformation

Linear Schödinger Equation: Steepest Descent

$$q(t,x) = 2i(\mathbf{M}_1)_{1,2}(x,t) = \frac{1}{\pi}\int_{\mathbb{R}} \widehat{q_0}(u) e^{-2it\phi(u;z_0)}\, du. \tag{1}$$

Recall the Cauchy-Pompeiu formula: if $\Omega \subset \mathbb{C}$ is a simply connected domain and $f : \Omega \to \mathbb{C}$ is differentiable,

$$\oint_{\partial\Omega} f(u,v)\, dz = \iint_{\Omega} 2i(\overline{\partial} f)(u,v)\, dA(u,v)$$

where

$$\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

We will evaluate (1) by

(1) Making an almost analytic extension of $\widehat{q_0}(\xi)$ to the complex plane

(2) Deforming the contour in (1) to the line $e^{3i\pi/4}\mathbb{R}$ using the Cauchy-Pompeiu formula

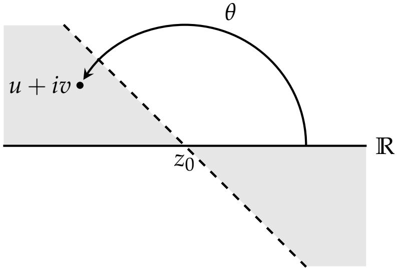
Linear Schrödinger Equation: Steepest Descent

$$q(t,x) = \frac{1}{\pi} \int_{\mathbb{R}} \widehat{q}_0(u) e^{-2it\phi(u;z_0)}\, du$$

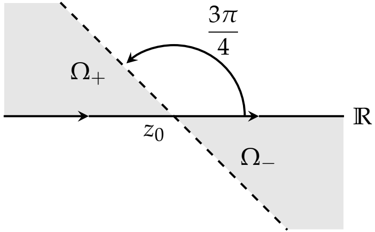
Almost analytic extension of $\widehat{q}_0(u)$:

$$E(u,v) = \cos(2\theta(u,v))\widehat{q}_0(u) + (1 - \cos(2\theta(u,v))\widehat{q}_0(z_0)$$

where $\theta = \arg(u + iv - z_0)$:



Linear Schrödinger Equation: Steepest Descent



$$q(t,x) = \frac{1}{\pi}\int_{\mathbb{R}} \hat{q}_0(u) e^{-2it\phi(u;z_0)}\,du$$

Apply Cauchy-Pompieu formula to $\pm E(u,v)e^{-2it\phi(u+iv;z_0)}$

$$q(t,x) = \frac{1}{\pi}\int_{\infty e^{3\pi i/4}}^{\infty e^{-i\pi/4}} \hat{q}_0(z_0) e^{-2it\phi(z;z_0)}\,dz + \frac{1}{\pi}\int_{\Omega_+ - \Omega_-} 2i\bar{\partial}(E(u,v))e^{-2it\phi(u+iv;z_0)}\,dA$$

The first term is a Gaussian integral that gives the correct leading asymptotics, and the second term is an error term whose size depends on $\bar{\partial}(E(u,v))$

Linear Schrödinger Equation: Steepest Descent

$$q(t,x) = \frac{1}{\pi}\int_{\infty e^{3\pi i/4}}^{\infty e^{-i\pi/4}} \hat{q}_0(z_0) e^{-2it\phi(z;z_0)}\,dz + \frac{1}{\pi}\int_{\Omega_+ - \Omega_-} 2i\bar{\partial}(E(u,v))e^{-2it\phi(u+iv;z_0)}\,dA$$

First term:

$$\frac{1}{\pi}\hat{q}_0(z_0)\int_{\infty e^{3\pi i/4}}^{\infty e^{-i\pi/4}} e^{-4it(z^2-2z_0z)}\,dz = \frac{1}{\sqrt{4\pi t}}e^{-i\pi/4}e^{-ix^2/4t}\hat{q}_0\left(-\frac{x}{4t}\right).$$

Second term: Need to estimate

$$\begin{aligned}(\bar{\partial}E)(u,v) &= \bar{\partial}\left(\cos(2\theta(u,v))\hat{q}_0(u) + (1-\cos(2\theta(u,v))\hat{q}_0(z_0)\right)\\ &= (\bar{\partial}\hat{q}_0(u)\cos(2\theta) + (\hat{q}_0(u) - \hat{q}_0(z_0))\bar{\partial}(\cos(2\theta))\end{aligned}$$

Linear Schrödinger Equation: Steepest Descent

The remainder term is

$$\frac{1}{\pi}\int_{\Omega_+ - \Omega_-} 2i\overline{\partial}(E(u,v))e^{-2it\phi(u+iv;z_0)}\,dA$$

where

$$(\overline{\partial}E)(u,v) = (\overline{\partial}\widehat{q}_0(u))\cos(2\theta) + (\widehat{q}_0(u) - \widehat{q}_0(z_0))\overline{\partial}(\cos(2\theta))$$

and

$$e^{-2it\phi(u+iv;z_0)} = e^{8t(u-z_0)v}$$

If $u = z_0 + \rho\cos\theta$, $v = z_0 + \rho\sin\theta$, then

$$\overline{\partial} = \frac{e^{i\theta}}{2}\left(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\frac{\partial}{\partial\theta}\right)$$

so we may estimate

$$|(\overline{\partial}E)(u,v)| \leq \frac{1}{2}\left|\widehat{q}_0{}'(u)\right| + \frac{|\widehat{q}_0(u) - \widehat{q}_0(z_0)|}{\sqrt{(u-z_0)^2+v^2}}, \quad u+iv \in \Omega_+ \cup \Omega_-$$

Linear Schrödinger Equation: Steepest Descent

We obtain

$$\left| \int_{\Omega_\pm} (2i\bar{\partial} E(u,v) e^{-2it(u+iv;z_0)}\, du\, dv \right| \leq I^\pm(t,x) + 2 \left\| \widehat{q_0}' \right\|_{L^2} J^\pm(t,x).$$

where

$$I_\pm(t,x) = \int_{\Omega_\pm} |\widehat{q_0}'(u)| e^{8t(u-z_0)v}\, dA$$

$$J_\pm(t,x) = \int_{\Omega_\pm} \frac{e^{8t(u-z_0)v}}{[(u-z_0)^2+v^2]^{\frac{1}{4}}}\, dA$$

Using the exponential decay and the fact that $\widehat{q_0}' \in L^2$, we may conclude that

$$|I^\pm| \leq Kt^{-\frac{3}{4}}, \quad |J^\pm| \leq Lt^{-\frac{3}{4}}$$

for constants L and M independent of (t,x).

Summary

To solve the linear Schrödinger equation by $\overline{\partial}$-steepest descent, we formulate a Riemann-Hilbert problem depending on $\widehat{q}_0$, the Fourier transform of the initial data. We then take the following steps.

(1) We make a contour deformation in the solution formula to obtain a reconstruction formula with a "frozen coefficient term" and a remainder term

(2) We compute the "frozen coefficient" term, which is a Gaussian integral involving only $\widehat{q}_0(z_0)$

(3) We estimate the remainder using $\overline{\partial}$ estimates

(4) We obtain large-time asymptotics of the solution $q(t,x)$ as the sum of a Gaussian integral and a correction term of lower order

Next, we will see that a very similar approach to the Riemann-Hilbert problem for NLS can be used to obtain sharp asymptotics for the solution as $t \to \infty$

Preview

To solve the nonlinear Schrödinger equation by $\overline{\partial}$-steepest descent, we formulate a Riemann-Hilbert problem depending on r, the scattering transform of the initial data. The asymptotics of the solution determine $q(t,x)$. We then make the following steps:

(1) We factorize the jump matrix

(2) We deform the contour in the Riemann-Hilbert problem to get exponential decay of the jump matrix away from the critical point

(3) We identify and solve a “model” Riemann-Hilbert problem with “frozen coefficients” (involving $r(z_0)$)

(4) We solve a $\overline{\partial}$ problem which gives the correction to the model Riemann-Hilbert problem

(5) We obtain large-time asymptotics of $q(t,x)$ as the sum of contributions from a model Riemann-Hilbert problem and a correction term estimated by $\overline{\partial}$ methods

Riemann-Hilbert Problem

For $r \in H^{1,1}(\mathbb{R})$, $\|r\|_{L^\infty} < 1$ and parameters x, t, find $\mathbf{M}(z; x, t)$ so that:

(i) $\mathbf{M})(z; x, t)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$,

(ii) $\lim_{z\to\infty} \mathbf{M}(z; x, t) = \mathbb{I}$,

(iii) $\mathbf{M}(z; x, t)$ has continuous boundary values $\mathbf{M}_\pm(\lambda; x, t)$ on $\mathbb{R}$

(iv) The jump relation

$$\mathbf{M}_+(\lambda; x, t) = \mathbf{M}_-(\lambda; x, t)\mathbf{V}(\lambda; x, t)$$

holds, where

$$\mathbf{V}(\lambda; x, t) = \begin{pmatrix} 1 - |r(\lambda)|^2 & -\overline{r(\lambda)}e^{-2it\theta} \\ r(\lambda)e^{2it\theta} & 1 \end{pmatrix}$$

and

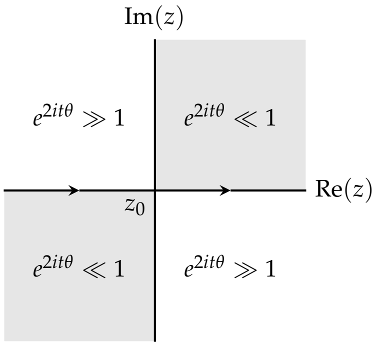
$$\theta(z; z_0) = 2z^2 - 4z_0 z, \quad z_0 = -\frac{x}{4t}.$$

We recover $q(t, x)$ via

$$q(t, x) = \lim_{z\to\infty} 2iz\mathbf{M}_{12}(z; x, t).$$

Step 1: Factorize the Jump Matrix

We will first factorize $\mathbf{V}(z)$ to separate exponential factors $e^{2it\theta}$



Recall

$$\mathbf{V}(z) = \begin{pmatrix} 1 - |r(\lambda)|^2 & -\overline{r(\lambda)}e^{-2it\theta} \\ r(\lambda)e^{2it\theta} & 1 \end{pmatrix}$$

and

$$\theta(z; z_0) = 2(z - z_0)^2 + 2z_0^2$$

where

$$z_0 = -\frac{x}{4t}$$

Step 1: Factorize the Jump Matrix

We can make the factorization

$$\mathbf{V}(z) = \begin{cases} \begin{pmatrix} 1 & -\overline{r(z)}e^{-2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(z)e^{2it\theta} & 1 \end{pmatrix}, & z > z_0 \\ \begin{pmatrix} 1 & 0 \\ \dfrac{r(z)e^{2it\theta}}{1-|r(z)|^2} & 1 \end{pmatrix} (1-|r(z)|^2)^{\sigma_3} \begin{pmatrix} 1 & \dfrac{\overline{r(z)}e^{-2it\theta}}{1-|r(z)|^2} \\ 0 & 1 \end{pmatrix}, & z < z_0 \end{cases}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$a^{\sigma_3} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

Step 1: Modify the Diagonal Jump

Introduce the function

$$\delta(z;z_0) = \exp\left(\frac{1}{2\pi i}\int_{-\infty}^{z_0} \frac{\ln(1-|r(s)|^2)}{s-z}\,ds\right)$$

on $\mathbb{C} \setminus (-\infty, z_0)$, which satisfies

$$\delta_+(z;z_0) = \begin{cases} \delta_-(z;z_0)(1-|r(z)|^2), & z < z_0 \\ \delta_-(z;z_0), & z > z_0 \end{cases}$$

$$\delta(z;z_0) = K(z-z_0)^{i\nu(z_0)}(1+o(1)), \quad z \to z_0$$

where $K = K(z_0)$ is a constant and $\nu(z) = -\dfrac{1}{2\pi}\ln(1-|r(z)|^2)$. To remove the singularity at $z = z_0$ we use

$$f(z;z_0) = c(z_0)\delta(z;z_0)(z-z_0)^{-i\nu(z_0)}$$

Step 1: Modify the Diagonal Jump

$$f(z;z_0) = c(z_0)\delta(z;z_0)(z-z_0)$$

where

$$c(z_0) = \exp\left(\frac{1}{2\pi i}\int_{-\infty}^{z_0} \ln(z_0 - s)\, d\ln(1-|r(s)|^2)\right)$$

The function $f_\pm(z;z_0)$ has continuous boundary values on $(-\infty, z_0)$ with

$$f_+(z;z_0) = f_-(z;z_0)\frac{1-|r(z)|^2}{1-|r(z_0)|^2}$$

We use it to define a new unknown

$$\mathbf{N}(z) = g(z;z_0)\mathbf{M}(z)g(z;z_0)^{-1}$$

where

$$g(z;z_0) = e^{i\omega(z_0)\sigma_3/2}e^{it\theta(z_0;z_0)}c(z_0)^{\sigma_3}.$$

and

$$\omega(z_0) = \arg(r(z_0)).$$

Step 1: New Riemann-Hilbert Problem

Given $(x,t) \in \mathbb{R}^2$, find a 2×2 matrix-valued function $\mathbf{N}(z;x,t)$ with

(i) $\mathbf{N}(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$ with continuous boundary values $\mathbf{N}_\pm(z)$ for $z \in \mathbb{R}$

(ii) $\mathbf{N}_+(z) = \mathbf{N}_-(z)\mathbf{V}_\mathbf{N}(z)$

(iii) There is a matrix $\mathbf{N}_1(x,t)$ so that

$$\mathbf{N}(z)(z-z_0)^{-i\nu(z_0)\sigma_3} = \mathbb{I} + z^{-1}\mathbf{N}_1(x,t) + o(z^{-1}), \quad z \to \infty$$

The solution $q(t,x)$ is recovered as follows:

$$q(t,x) = 2ie^{-i\omega(z_0)}e^{-2it\theta(z_0;z_0)}c(z_0)^{-2}N_{1,12}(x,t)$$

What's the Jump Matrix?

The new jump matrix $\mathbf{V_N}(z)$ is given by

$$\mathbf{V_N}(z) = \begin{cases} \begin{pmatrix} 1 & F_6(z)e^{2it\varphi} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ F_1(z)e^{-2it\varphi} & 1 \end{pmatrix}, & z > z_0, \\ \begin{pmatrix} 1 & 0 \\ F_4(z)e^{2it\varphi} & 1 \end{pmatrix} (1 - |r(z_0)|^2)^{\sigma_3} \begin{pmatrix} 1 & F_3(z)e^{-2it\varphi} \\ 0 & 1 \end{pmatrix}, & z < z_0 \end{cases}$$

where

$$F_6(z) = -f(z;z_0)^2\overline{r(z)}e^{i\omega(z_0)}, \qquad F_1(z) = f(z;z_0)^{-2}r(z)e^{-i\omega(z_0)},$$

$$F_4(z) = \frac{f_-(z;z_0)^{-2}r(z)e^{-i\omega(z_0)}}{1-|r(z)|^2} \qquad F_3(z) = \frac{f_+(z;z_0)^2\overline{r(z)}e^{i\omega(z_0)}}{1-|r(z)|^2}$$

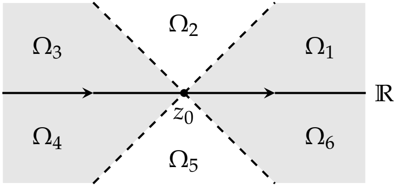
and

$$\varphi = \varphi(z;z_0) = \theta(z;z_0) - \theta(z_0;z_0)$$

(the reason for the numbering will become clear in a moment)

Step 2: Contour Deformation

We will move the oscillatory factors $e^{\pm 2i\theta}$ off the real axis and into appropriate contours where they decay exponentially as $t \to +\infty$.



We will make extensions of F_1, F_3, F_4, F_6:

$$E_1(u,v) = \cos(2\phi)F_1(u) + (1-\cos(2\phi))|r(z_0)|$$

$$E_3(u,v) = \cos(2\phi)F_3(u) + (1-\cos(2\phi))\frac{|r(z_0)|}{1-|r(z_0)|^2}$$

$$E_4(u,v) = \cos(2\phi)F_4(u) + (1-\cos(2\phi))\frac{|r(z_0)|}{1-|r(z_0)|^2}$$

$$E_6(u,v) = \cos(2\phi)F_6(u,v) + (1-\cos(2\phi))|r(z_0)|$$

Step 2: Change of Variable

Let $\mathbf{O}(z) = \mathbf{N}(z)$ in sectors Ω_2 and Ω_5 but

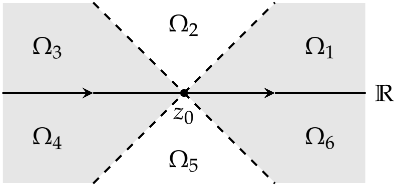
$$\mathbf{O}(z) = \begin{cases} \mathbf{N}(z) \begin{pmatrix} 1 & 0 \\ E_1(u,v)e^{2it\varphi} & 1 \end{pmatrix}, & z = u + iv \in \Omega_1 \\ \mathbf{N}(z) \begin{pmatrix} 1 & E_3(u,v)e^{-2it\varphi} \\ 0 & 1 \end{pmatrix}, & z = u + iv \in \Omega_3 \\ \mathbf{N}(z) \begin{pmatrix} 1 & 0 \\ E_4(u,v)e^{2it\varphi} & 1 \end{pmatrix}, & z = u + iv \in \Omega_4, \\ \mathbf{N}(z) \begin{pmatrix} 1 & E_6(u,v)e^{-2it\varphi} \\ 0 & 1 \end{pmatrix}, & z = u + iv \in \Omega_6 \end{cases}$$

It remains to:

(1) Solve the "frozen coefficient" Riemann-Hilbert problem

(2) Estimate the correction due to non-analyticity

Step 3: "Frozen Coefficient" RHP



Along the diagonal boundaries of $\Omega_1, \Omega_3, \Omega_4, \Omega_6$ we have, for $m = |r(z_0)|$,

$$E_1(u - z_0, u) = m, \qquad E_6(u - z_0, -u) = -m, \qquad z > z_0$$
$$E_3(u - z_0, u) = -\frac{m}{1 - m^2}, \qquad E_4(u - z_0, u) = \frac{m}{1 - m^2}, \qquad z < z_0.$$

If we rescale $\zeta = 2t^{\frac{1}{2}}(z - z_0)$ we obtain the following model Riemann-Hilbert problem (the "isomonodromy problems of Its")

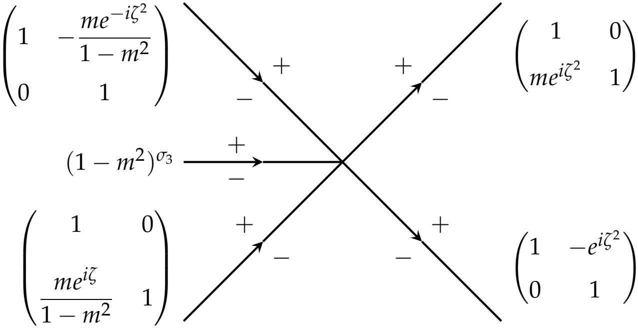
Step 3: "Frozen Coefficent" RHP

Given m with $0 < m < 1$, find a 2×2 matrix-valued function $\mathbf{P}(\zeta)$ so that:

(i) $\mathbf{P}(\zeta)$ is analytic in the sectors $|\arg(\zeta)| < \pi/4$, $\pi/4 < \pm\arg(\zeta) < \frac{3}{4}\pi$, and $\frac{3}{4}\pi < \pm\arg(\zeta) < \pi$

(ii) $\mathbf{P}(\zeta)$ has continuous boundary values $\mathbf{P}_\pm$ with jumps as shown below

(iii) $\mathbf{P}(\zeta; m)\zeta^{-ln(1-m^2)\sigma_3/2\pi i} \to \mathbb{I}$ as $\zeta \to \infty$



Step 3: "Frozen Coefficient" RHP

The Riemann-Hilbert problem has an explicit solution in terms of parabolic cylinder functions. It has the following properties:

(i) Fix $0 < \rho < 1$. Then

$$\sup_{\zeta \in \mathbb{C} \setminus \Sigma_P;\, |m| \leq \rho} \left(\|\mathbf{P}(\zeta; m)\| + \left\| \mathbf{P}(\zeta; m)^{-1} \right\| \right) \leq C$$

for $C > 0$ depending on ρ

(ii) The asymptotics

$$\mathbf{P}(\zeta; m) = \left(\mathbb{I} + \zeta^{-1} \mathbf{P}_1(m) + \mathcal{O}\left(\zeta^{-2} \right) \right) \zeta^{\ln(1-m^2)\sigma_3/(2\pi i)}, \quad \zeta \to \infty$$

hold.

The function

$$\mathbf{P}(2t^{\frac{1}{2}}(u + iv - z_0); |r(z_0)|)$$

will serve as a parametrix for

$$(2t^{\frac{1}{2}})^{i\nu(z_0)\sigma_3} \mathbf{O}(u, v; x, t)$$

Step 4: The $\overline{\partial}$ Problem

Let

$$\mathbf{E}(u,v;x,t) = (2t^{\frac{1}{2}})^{iv(z_0)\sigma_3}\mathbf{O}(u,v;x,t)\mathbf{P}(2t^{\frac{1}{2}}(u+iv-z_0);|r(z_0)|)^{-1}$$

Then $\mathbf{E}$ satisfies the following $\overline{\partial}$ problem. Find a 2×2 matrix-valued function $\mathbf{E}(u,v;x,t)$, $(x,t)\in\mathbb{R}^2$ with the following properties:

1. E a continuous function of $(u,v)\in\mathbb{R}^2$
2. E is a weak solution of the partial differential equation

   $$\overline{\partial}\mathbf{E}(u,v) = \mathbf{E}(u,v)\mathbf{W}(u,v)$$

   where

   $$\mathbf{W}(u,v) = \mathbf{P}(2t^{\frac{1}{2}}(u+iv-z_0);|r(z_0)|)\Delta(u,v;x,t)\mathbf{P}(2t^{\frac{1}{2}}(u+iv-z_0);|r(z_0)|)^{-1}$$

3. $E(u,v;x,t)\to\mathbb{I}$ as $(u,v)\to\infty$

Step 4: The $\bar{\partial}$ Problem

$$\bar{\partial}\mathbf{E}(u,v) = \mathbf{E}(u,v)\mathbf{W}(u,v)$$

where

$$\mathbf{W}(u,v) = \mathbf{P}(2t^{\frac{1}{2}}(u+iv-z_0);|r(z_0)|)\Delta(u,v;x,t)\mathbf{P}(2t^{\frac{1}{2}}(u+iv-z_0);|r(z_0)|)^{-1}$$

Here $\Delta(u,v;x,t) = 0$ for $u+iv \in \Omega_2 \cup \Omega_5$ and

$$\Delta(u,v;x,t) = \begin{cases} \begin{pmatrix} 0 & 0 \\ -(\bar{\partial}E_1)(u,v)e^{2it\theta} & 0 \end{pmatrix}, & u+iv \in \Omega_1, \\ \begin{pmatrix} 0 & -(\bar{\partial}E_3)(u,v)e^{-2it\theta} \\ 0 & 0 \end{pmatrix}, & u+iv \in \Omega_3 \\ \begin{pmatrix} 0 & 0 \\ (\bar{\partial}E_4)(u,v)e^{2it\theta} & 0 \end{pmatrix}, & u+iv \in \Omega_4, \\ \begin{pmatrix} 0 & (\bar{\partial}E_6)(u,v)e^{-2it\theta} \\ 0 & 0 \end{pmatrix}, & u+iv \in \Omega_6 \end{cases}$$

How to Solve a $\bar\partial$ Problem

To solve

$$\begin{cases} (\bar\partial h)(z) = f(z) \text{ on } \mathbb{C} \\ h(z) \to \mathbb{I} \text{ as } z \to \infty \end{cases}$$

use the fact that $-1/(\pi z)$ is a fundamental solution for $\bar\partial$:

$$h(z) = \mathbb{I} - \frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{\zeta - z} f(\zeta) \, dA(\zeta)$$

To solve

$$\begin{cases} (\bar\partial h)(z) = h(z)w(z) \\ h(z) \to \mathbb{I} \text{ as } z \to \infty \end{cases}$$

solve the integral equation

$$h(z) = \mathbb{I} - \frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{\zeta - z} h(\zeta)w(\zeta) \, dA(\zeta)$$

The second term defines an integral operator

$$h \mapsto -\frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{\zeta - z} h(\zeta)w(\zeta) \, dA(\zeta)$$

How to Solve a $\bar{\partial}$ Problem

$$h(z) = \mathbb{I} + (Th)(z)$$

where

$$(Th)(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{1}{\zeta - z} h(\zeta) w(\zeta)\, dA(\zeta)$$

Formal solution

$$h(z) = \mathbb{I} + (I - T)^{-1} T(\mathbb{I})$$

We can solve this equation for $h \in L^\infty(\mathbb{C})$ provided $\|T\|_{L^\infty \to L^\infty} < 1$. But

$$\|Tg\|_{L^\infty} \leq \|g\|_{L^\infty} \int_{\mathbb{C}} \frac{\|w(u,v)\|}{((u-u')^2 + (v-v')^2)^{\frac{1}{2}}}\, du\, dv$$

so we need show that the right-hand integral is small.

In the application,

$$\|w(u,v)\| = \|\mathbf{W}(u,v)\|$$

Step 4: The $\bar\partial$ Problem

In our case

$$\|\mathbf{W}(u,v)\| \leq C^2 \begin{cases} e^{-8t(u-z_0)v}|(\bar\partial E_1)(u,v)|, & z = u+iv \in \Omega_1, \\ e^{8t(u-z_0)v}|(\bar\partial E_3)(u,v)|, & z = u+iv \in \Omega_3, \\ e^{-8t(u-z_0)v}|(\bar\partial E_4)(u,v)|, & z = u+iv \in \Omega 4, \\ e^{8t(u-z_0)v}|(\bar\partial E_6)(u,v)|, & z = u+iv \in \Omega_6 \end{cases}$$

where C bounds $\sup_\zeta \|\mathbf{P}(\zeta;m)\|$ and $\sup_\zeta \left\|(\mathbf{P}(\zeta;m))^{-1}\right\|$. Notice that the exponentials vanish as $t \to \infty$. Using the estimates

$$|(\bar\partial E_j)(u,v)| \leq L|r'(u)| + \frac{M}{((u-z_0)^2+v^2)^{\frac{1}{2}}}$$

where L and M depend on $\rho < 1$, $\|r\|_{L^2}$, and $\|r'\|_{L^2}$, one can show that

$$\|T\|_{L^\infty \to L^\infty} \leq Dt^{-1/4}, \quad D = D(\rho, \|r\|_{L^2}, \|r'\|_{L^2}).$$

Step 4: The $\bar{\partial}$ Problem

From $\|T\|_{L^\infty \to L^\infty} \leq Dt^{-1/4}$, we obtain

$$\|T\mathbb{I}\|_{L^\infty} \leq Dt^{-\frac{1}{4}}$$

We can then solve the equation

$$\mathbf{E}(u,v) = \mathbb{I} + (T\mathbf{E})(u,v)$$

for large t by the formula

$$\mathbf{E} = \mathbb{I} + (I - T)^{-1} T\mathbb{I}$$

where the resolvent exists because $\|T\|_{L^\infty \to L^\infty} < 1$ for large t. From the solution formula we recover

$$\begin{aligned} \|\mathbf{E} - \mathbb{I}\|_{L^\infty} &= \left\|(I - T)^{-1} T\mathbb{I}\right\|_{L^\infty} \\ &\lesssim \|T\mathbb{I}\|_{L^\infty} \\ &\lesssim t^{-\frac{1}{4}} \end{aligned}$$

uniformly in (x,t).

Step 4: The $\overline{\partial}$ Problem

To compute the large-time asymptotics of $q(t,x)$, we will need some further results on the large-t behavior of $\mathbf{E}(u,v;x,t)$ as $v \to \infty$ along the line $u = 0$.

We compute

$$(u+iv)\left(\mathbf{E}(u,v;x,t) - \mathbb{I}\right) = T_1 + T_2$$

where

$$T_1 = \frac{1}{\pi}\int_{\mathbb{C}} \mathbf{E}(u',v';x,t)\mathbf{W}(u',v';x,t)\,du'\,dv'$$

$$T_2 = \frac{1}{\pi}\int_{\mathbb{C}} \frac{u'+iv'}{(u-u')+i(v-v')}\mathbf{E}(u',v';x,t)\mathbf{W}(u',v';x,t)\,du'\,dv'.$$

We may estimate

$$|T_1| \leq \frac{1}{\pi}\,\|\mathbf{E}\|_{L^\infty}\,\|\mathbf{W}(\,\cdot\,,\,\cdot\,;x,t)\|_{L^1(\mathbb{R}^2)}$$

and it can be shown that $T_2 \to 0$ as $v \to \infty$ along $u = 0$. Moreover it can be shown that

$$\|\mathbf{W}(\,\cdot\,,\,\cdot\,;x,t)\|_{L^1(\mathbb{R}^2)} \leq Ct^{-\frac{3}{4}}$$

Step 5: Reconstruction

We show that the paramatrix $\mathbf{P}((2t^{\frac{1}{2}})(u+iv-z_0);|r(z_0)|)$ gives the correct long-time asymptotics for $q(t,x)$ up to an $\mathcal{O}\left(t^{-\frac{3}{4}}\right)$ error. Recall that

$$q(t,x) = 2ie^{-i\omega(z_0)}e^{-2it\theta(z_0;z_0)}c(z_0)^{-2}N_{1,12}(x,t)$$

where

$$\mathbf{N}(z;x,t)(z-z_0)^{-i\nu(z_0)\sigma_3} = \mathbb{I} + z^{-1}\mathbf{N}_1(x,t) + o(z^{-1})$$

and

$$\mathbf{N}(u+iv;x,t) = \mathbf{O}(u,v;x,t) \text{ in } \Omega_2 \cup \Omega_5.$$

Using this fact, we can compute

$$\mathbf{N}(iv;x,t) = (2t^{\frac{1}{2}})^{-i\nu(z_0)\sigma_3}\mathbf{E}(0,v;x,t)\mathbf{P}((2t^{\frac{1}{2}})(iv-z_0);|r(z_0)|)$$

and find $\mathbf{N}_1(x,t)$.

Step 5: Reconstruction

To compute

$$q(t,x) = 2ie^{-i\omega(z_0)}e^{-2it\theta(z_0;z_0)}c(z_0)^{-2}N_{1,12}(x,t)$$

we have

$$\begin{aligned}\mathbf{N}(z;x,t)(z-z_0)^{-i\nu(z_0)\sigma_3} &= \mathbb{I} + z^{-1}\mathbf{N}_1(x,t) + o(z^{-1}) \\ \mathbf{N}(u+iv;x,t) &= (2t^{\frac{1}{2}})^{-i\nu(z_0)\sigma_3}\mathbf{E}(u,v;x,t)\mathbf{P}((2t^{\frac{1}{2}})(u+iv-z_0);|r(z_0)|)\end{aligned}$$

so that

$$\begin{aligned}\mathbf{N}_1(x,t) &= \lim_{v\to\infty,u=0}(u+iv)\left(\mathbf{N}(u+iv;x,t)(u+iv-z_0)^{-i\nu(z_0)\sigma_3} - \mathbb{I}\right) \\ &= (2t^{\frac{1}{2}})^{-i\nu(z_0)\sigma_3}\mathbf{Q}(x,t)(2t^{\frac{1}{2}})^{i\nu(z_0)\sigma_3}\end{aligned}$$

where

$$\mathbf{Q}(x,t) = (2t^{\frac{1}{2}})^{i\nu(z_0)\sigma_3}\left\{\lim_{v\to\infty,u=0}(u+iv)\left[\mathbf{N}(u+iv;x,t)(z-z_0)^{-i\nu(z_0)\sigma_3} - \mathbb{I}\right]\right\}(2t^{\frac{1}{2}})^{-i\nu(z_0)\sigma_3}$$

Step 5: Reconstruction

$$q(t,x) = 2ie^{-i\omega(z_0)}e^{-2it\theta(z_0;z_0)}c(z_0)^{-2}N_{1,12}(x,t)$$

$$\mathbf{N}_1(x,t) = (2t^{\frac{1}{2}})^{-i\nu(z_0)\sigma_3}\mathbf{Q}(x,t)(2t^{\frac{1}{2}})^{i\nu(z_0)\sigma_3}$$

$$\mathbf{Q}(x,t) = (2t^{\frac{1}{2}})^{i\nu(z_0)\sigma_3}\left\{\lim_{v\to\infty,u=0}(u+iv)\left[\mathbf{N}(u+iv;x,t)(z-z_0)^{-i\nu(z_0)\sigma_3} - \mathbb{I}\right]\right\}(2t^{\frac{1}{2}})^{-i\nu(z_0)\sigma_3}$$

$$= \lim_{v\to\infty,u=0}(u+iv)\left[\mathbf{E}(u,v;x,t)\mathbf{P}(2t^{\frac{1}{2}})(u+iv-z_0);|r(z_0)|)(u+iv-z_0)^{-i\nu(z_0)\sigma_3} - \mathbb{I}\right]$$

Since

$$\mathbf{E}(u,v;x,t) = \mathbb{I} + \frac{1}{u+iv}\mathbf{E}_1(x,t) + o\left(\frac{1}{u+iv}\right)$$

and

$$\mathbf{P}(\zeta;m) = \left(\mathbb{I} + \zeta^{-1}\mathbf{P}_1(m) + \mathcal{O}\left(\zeta^{-2}\right)\right)\zeta^{\ln(1-m^2)\sigma_3/2\pi i}$$

where $\zeta = (2t^{\frac{1}{2}})(u+iv-z_0)$ and $m = |r(z_0)|$, we have

$$\mathbf{Q}(x,t) = \mathbf{E}_1(x,t) + \frac{1}{2}t^{-\frac{1}{2}}\mathbf{P}_1(|r(z_0)|)$$

Step 5: Reconstruction

$$\mathbf{N}_1(x,t) = (2t^{\frac{1}{2}})^{-i\nu(z_0)\sigma_3}\mathbf{Q}(x,t)(2t^{\frac{1}{2}})^{i\nu(z_0)\sigma_3}$$

$$\mathbf{Q}(x,t) = \mathbf{E}_1(x,t) + \frac{1}{2}t^{-\frac{1}{2}}\mathbf{P}_1(|r(z_0)|)$$

As $\|\mathbf{E}_1(x,t)\| = \mathcal{O}\left(t^{-\frac{3}{4}}\right)$, we have

$$\begin{aligned}
q(t,x) &= 2ie^{-i\omega(z_0)}e^{-2it\theta(z_0;z_0)}c(z_0)^{-2}N_{1,12}(x,t) \\
&= 2ie^{-i\omega(z_0)}e^{-2it\theta(z_0;z_0)}c(z_0)^{-2}(2t^{\frac{1}{2}})^{-2i\nu(z_0)}\left[\mathbf{E}_1(x,t) + \frac{1}{2}t^{-\frac{1}{2}}\mathbf{P}_1(|r(z_0)|)\right]_{12} \\
&= e^{-i\omega(z_0)}e^{-2it\theta(z_0;z_0)}c(z_0)^{-2}(2t^{\frac{1}{2}})^{-2i\nu(z_0)}\left[2iE_{1,12}(x,t) + \frac{1}{2}t^{-\frac{1}{2}}2iP_{1,12}(|r(z_0)|)\right] \\
&= e^{-i\omega(z_0)}e^{-2it\theta(z_0;z_0)}c(z_0)^{-2}(2t^{\frac{1}{2}})^{-2i\nu(z_0)}\left[2iE_{1,12}(x,t) + \frac{1}{2}t^{-\frac{1}{2}}\beta(|r(z_0)|)\right]
\end{aligned}$$

where $|\beta(m)|^2 = -\frac{1}{\pi}\ln(1-m^2)$ and

$$\arg(\beta(m)) = \frac{\pi}{4} + \frac{1}{2\pi}\ln(2)\ln(1-m^2) - \arg\left(\Gamma\left(\frac{i}{2\pi}\ln(1-m^2)\right)\right)$$

Step 5: Reconstruction

$$q(t,x) = e^{-i\omega(z_0)} e^{-2it\theta(z_0;z_0)} c(z_0)^{-2} (2t^{\frac{1}{2}})^{-2i\nu(z_0)} \left[2iE_{1,12}(x,t) + \frac{1}{2} t^{-\frac{1}{2}} \beta(|r(z_0)|) \right]$$

It now follows that

$$q(t,x) \underset{t\to\infty}{\sim} \frac{1}{2t^{\frac{1}{2}}} e^{ix^2/4t} c(z_0)^{-2} \exp\left(-i\nu(z_0)\ln(4t)\right) \beta(|r(z_0)|) e^{-i\arg(r(z_0))} + \mathcal{O}\left(t^{-\frac{3}{4}}\right).$$